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Soft Union Tri-bi-ideals of Semigroups

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ABSTRACT. The concept of tri-quasi ideal was presented as a generalization of quasi-ideal, interior ideal, and bi-ideal. In this paper, we transfer this concept to soft set theory and semigroups,and introduce a novel type of soft union (S-uni) ideal form called "soft union (S-uni) tri-bi-ideal". The main aim of this study is to obtain the relations between S-uni tri-bi-ideals and other certain types of S-uni ideals of a semigroup. Our results show that every S-uni tri-bi-ideal of a band is an S-uni subsemigroup. Moreover, an S-uni tri-bi-ideal is a generalization of an S-uni ideal, interior ideal, bi-ideal, quasi-ideal, weak-interior ideal, bi-interior ideal and bi-quasi ideal, however in order to satisfy the converses, the semigroup should have specific conditions. We also demonstrate that the S-uni quasi-interior ideal of a left or right simple semigroup is an S-uni tri-bi-ideal, nevertheless the converse holds for the zero semigroup. Furthermore, the S-uni bi-quasi-interior ideal of a commutative semigroup is an S-uni tri-bi-ideal, however, for the converse to hold the semigroup must be a band. We have shown that an S-uni tri-ideal coincides with an S-uni tri-bi-ideal of a band, and every S-uni tri-bi-ideal of a group is an S-uni tri-ideal. We also obtain a relation between tri-bi-ideal and its soft characteristic function, enabling us to get the relation between semigroup and soft set theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of the soft set operations, the soft (anti/inverse) image, supporting our assertions with illuminating examples.

1. Introduction

In many areas of mathematics, semigroups are essential as they provide the abstract algebraic foundation for "memoryless" systems, which restart with each iteration. Since finite

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automata and finite semigroups are closely related, studying them is crucial to theoretical computer science. Furthermore, in probability theory, semigroups and Markov processes are related. Many mathematicians have concentrated their research on generalizing ideals in algebraic structures since the idea of ideals is essential to comprehending mathematical structures together with their applications. Specifically, the generalization of algebraic structure ideals is necessary for further algebraic structure studies. By employing the notion and characteristics of the generalization of ideals in algebraic structures, several mathematicians demonstrated significant findings and characterizations of algebraic structures. Dedekind established the idea of ideals for the theory of algebraic numbers, and Noether expanded it to include associative rings. The idea of a one-sided ideal of any algebraic structure is an extension of the idea of an ideal, and the one-and two-sided ideals are still fundamental ideas in ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals for semigroups. The concept of quasi-ideals was first introduced by Steinfeld [2] for semigroups and later extended to rings. Bi-ideals are generalizations of quasi-ideals, while quasi-ideals are generalizations of left and right ideals. The concept of interior ideal was first demonstrated by Lajos [3] and further studied by Szasz [4,5]. Interior ideals are generalization of ideals. Rao [6-10] recently introduced several new types of ideals of semigroups. Furthermore, the idea of essential ideals in semigroups was proposed by Baupradist et al. [11] and the idea of tri-quasi ideals in semigroups introduced by Rao et al [12]. As a more generalized concept of various types of ideals, the notion of "almost" ideals was proposed, their characteristics and the relationships between the related ideals were thoroughly examined. Grosek and Satko [13] established the idea of almost ideals of semigroups in this context. Bogdanovic [14] proposed almost bi-ideals in semigroups. In 2018, Wattanatripop et al. [15] defined almost quasi-ideals and in 2020, Kaopusek et al. [16] proposed the concepts of (weakly) almost interior ideals of semigroups. Different types of almost ideals in semigroups were proposed by the authors in [17-20]. Furthermore, in [15, 17-22], several fuzzy almost ideal types for semigroups were investigated.

The "Soft Set Theory" [23] was first presented in order to comprehend and provide appropriate solutions for problems involving uncertainty. Since then, a great deal of important research has been done on soft set notion, particularly soft set operations. Maji et al. [24] described some operations on soft sets and provided some concepts pertaining to soft sets. Various operations of soft sets were introduced by Pei and Miao [25] and Ali et al. [26]. We refer [27–41] for additional information on soft set operations, which has been popular since their inception. The concept and operations of soft sets were modified by Çağman and Enginoğlu [42]. Several kinds of soft algebraic systems were investigated since Çağman et al. [43] introduced the idea of soft-int groups. Sezgin [44], using soft sets in the application of semigroup theory, defined soft union (S-uni) semigroups, ideals, and bi-ideals of semigroups; Sezgin et al. [45] defined S-uni interior ideals, quasi-ideals, and generalized bi-ideals of semigroups, and thoroughly examined their fundamental properties. In terms of the S-uni substructures of semigroups, Sezer et al. [46] defined and classified certain kinds of semigroups. In [47] certain kinds of regularities of semigroups are characterized by soft union quasi-ideals, soft union (generalized) bi-ideals, and soft union semiprime ideals of a semigroup. Recently, the different types of S-uni ideals of semigroups such as S-uni weak-interior ideals, S-uni bi-interior ideals, S-uni bi-quasi ideals, S-uni quasi-interior ideals, S-uni bi-quasi-interior ideals and S-uni tri-ideals defined and thoroughly examined their essential traits by various researchers [48-53]. Soft intersection almost ideals were introduced and examined in [54–65] as a generalization of several soft intersection ideal types. The soft forms of various algebraic structures have been studied in [66-80].

As a generalization of the quasi-ideal, interior ideal, and bi-ideal of a semigroup, Rao et al. [12] developed the concept of tri-quasi ideals and investigated their characteristics. Moreover, Rao [81,82] has examined the notion of tri-quasi ideals for Γ-semirings and semirings, respectively. In this study, we introduce "soft union (S-uni) tri-bi-ideals of semigroups" to apply this approach to semigroups and soft set theory. We obtain the relations between S-uni tri-bi-ideals and other types of S-uni ideals of a semigroup. Our results show every S-uni tri-bi-ideal of a band is an S-uni subsemigroup. Moreover, the concept of S-uni tri-bi-ideal is a generalization of S-uni left (right/two-sided) ideal, interior ideal, bi-ideal, quasi-ideal, (left/right) weak-interior ideal, bi-interior ideal and (left/right) bi-quasi ideal, however, the converses are not valid with counterexamples. We show that to satisfy the converses, the semigroup should have specific conditions such as band, left (right) simple band or idempotent group. We also demonstrate that the S-uni left (resp. right) quasi-interior ideal of a left (resp. right) simple semigroup is an S-uni tri-bi-ideal, while the converse holds for the right (resp. left) zero semigroup. As a result, we showed that for an S-uni quasi-interior ideal to be an S-uni tri-bi-ideal, the semigroup must be left or right simple semigroup, and for the converse to be valid, the semigroup must be the zero semigroup. Furthermore, we have shown that the S-uni bi-quasi-interior ideal of a commutative semigroup is an S-uni tri-bi-ideal, but for the converse to hold the semigroup must be a regular idempotent. We have shown that an S-uni (left/right) tri-ideal of a band is equivalent with an S-uni tri-bi-ideal, and that an S-uni tri-bi-ideal of a right (resp. left) simple semigroup is an S-uni left (resp. right) tri-ideal. As a result, we showed that for an S-uni tri-bi-ideal of group is an S-uni tri-ideal. We also obtain a relation between tri-bi-ideal and its soft characteristic function, demonstrating how this idea connects classical semigroup theory and soft set theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of soft set operations, soft anti image, and soft inverse image, supporting our assertions with particular, illuminating examples. The paper consists of four sections. Section 1 provides an overview of the subject, while Section 2 delves into the basic concept of semigroup and soft set ideals, as well as their relevant definitions and consequences. In Section 3, we propose the concept of S-uni tri-bi-ideals and analyze their properties as well as their relationships with other types of S-uni ideals using concrete examples. Section 4 summarizes our findings and discusses the potential future research.

2. Preliminaries

Throughout this paper, S denotes a semigroup. $\emptyset \neq Ѧ \subseteq S$ is called a subsemigroup ($\mathcal{SS}$) of S if $ѦѦ \subseteq Ѧ$, is called a bi-ideal of S if $ѦѦ \subseteq Ѧ$ and $ѦSѦ \subseteq Ѧ$, is called an interior ideal of S if $ѦѦ \subseteq Ѧ$ and $SѦS \subseteq Ѧ$, and is called a quasi-ideal of S if $ѦS \cup SѦ \subseteq Ѧ$. If there exists an element $ḏ \in S$ such that $ɱ = ɱḏɱ$ for all $ɱ \in S$, then S is regular. If S is band (idempotent semigroup), then for all $\varphi \in S$, $\varphi = \varphi\varphi$, that is, every element in S is idempotent. S is called a left (right) simple if it contains no proper left (right) ideal of S and is called a simple if it contains no proper ideal.

Definition 2.1 [10]. $\emptyset \neq Ѧ \subseteq S$ is called a left (right) tri-ideal of S if $Ѧ$ is an $\mathcal{SS}$ and $ѦSѦѦ \subseteq Ѧ$ ($ѦѦSѦ \subseteq Ѧ$), is called a tri-ideal of S if $Ѧ$ is an $\mathcal{SS}$, $ѦSѦѦ \subseteq Ѧ$ and $ѦѦSѦ \subseteq Ѧ$.

Definition 2.2 [12]. $\emptyset \neq Ѧ \subseteq S$ is called a tri-quasi ideal of S if $Ѧ$ is an $\mathcal{SS}$ and $ѦѦSѦѦ \subseteq Ѧ$.

Note 2.3. Rao et al. [12] terms "tri-bi-ideal (ŦɃ-ideal)" as "tri-quasi ideal"; however, when we look at the definition of left tri-ideal, we see that "$SѦ$", representing the left ideal, is in the middle, and the other two "$Ѧ$" are placed on the left and right; and when we look at the definition of right tri-ideal, "$ѦS$", representing the right ideal, is in the middle and the other two "$Ѧ$" are placed on the left and right. As for the the tri-quasi ideal, we see that "$ѦSѦ$", which is placed in the middle, evokes in the mind as the bi-ideal, and again the other two "$Ѧ$" are placed on the left and right. Therefore, we prefer to name "tri-bi-ideal (ŦɃ-ideal)" for "tri-quasi ideal" throughout the paper in order to be consistent with the definition of left (right) tri-ideal.

Theorem 2.4 [83,84]. For S, we have the following:

i) S is left (right) simple iff $S\alpha = S$ ($\alpha S = S$) for all $\alpha \in S$. That is, for every $\alpha, ȸ \in S$, there exist $ɔ \in S$ such that $ȸ = ɔ\alpha$ ($ȸ = \alpha ɔ$).

ii) S is group iff S is both a left simple and a right simple.

iii) S is left (right) zero if $\mathrm{s}\mathrm{v} = \mathrm{s}$ ($\mathrm{s}\mathrm{v} = \mathrm{v}$) for all $\mathrm{s}, \mathrm{v} \in S$.

iv) S is called a zero if $ǎƅ = 0$ for all $ǎ, ƅ \in S$. There, $0 \in S$ is a zero element of S.

Definition 2.5 [23, 42]. Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and $\Upsilon \subseteq E$. The soft set f_Υ over U is a function such that $f_\Upsilon: E \to P(U)$, where for all $x \notin \Upsilon$, $f_\Upsilon(x) = \emptyset$. That is,

$$f_\Upsilon = \{(x, f_\Upsilon(x)): x \in E, f_\Upsilon(x) \in P(U)\}$$

The set of all soft sets over U is designated by $S_E(U)$ throughout this paper.

Definition 2.6 [42]. Let $f_{\mathcal{H}} \in S_E(U)$. If $f_{\mathcal{H}}(x) = \emptyset$ for all $x \in E$, then $f_{\mathcal{H}}$ is called a null soft set and indicated by $\emptyset_E$.

Definition 2.7 [42]. Let $f_{\mathcal{H}}, f_{\aleph} \in S_E(U)$. If $f_{\mathcal{H}}(x) \subseteq f_{\aleph}(x)$, for all $x \in E$, then $f_{\mathcal{H}}$ is a soft subset of $f_{\aleph}$ and indicated by $f_{\mathcal{H}} \tilde{\subseteq} f_{\aleph}$. If $f_{\mathcal{H}}(x) = f_{\aleph}(x)$, for all $x \in E$, then $f_{\mathcal{H}}$ is called soft equal to $f_{\aleph}$ and denoted by $f_{\mathcal{H}} = f_{\aleph}$.

Definition 2.8 [42]. Let $f_{\mathcal{H}}, f_{\aleph} \in S_E(U)$. The union (intersection) of $f_{\mathcal{H}}$ and $f_{\aleph}$ is the soft set $f_{\mathcal{H}} \tilde{\cup} f_{\aleph}$ $(f_{\mathcal{H}} \tilde{\cap} f_{\aleph})$, where $(f_{\mathcal{H}} \tilde{\cup} f_{\aleph})(w) = f_{\mathcal{H}}(w) \cup f_{\aleph}(w)$ $\big((f_{\mathcal{H}} \tilde{\cap} f_{\aleph})(w) = f_{\mathcal{H}}(w) \cap f_{\aleph}(w)\big)$, for all $w \in E$, respectively.

Definition 2.9 [42]. Let $f_{\mathcal{H}}, f_{\aleph} \in S_E(U)$. Then, ∨-product (∧-product) of $f_{\mathcal{H}}$ and $f_{\aleph}$, denoted by $f_{\mathcal{H}} \vee f_{\aleph}$ $(f_{\mathcal{H}} \wedge f_{\aleph})$ is defined by $(f_{\mathcal{H}} \vee f_{\aleph})(\eta, \mathrm{v}) = f_{\mathcal{H}}(\eta) \cup f_{\aleph}(\mathrm{v})$ $\big((f_{\mathcal{H}} \wedge f_{\aleph})(\eta, \mathrm{v}) = f_{\mathcal{H}}(\eta) \cap f_{\aleph}(\mathrm{v})\big)$ for all $(\eta, \mathrm{v}) \in E \times E$, respectively.

Definition 2.10 [43]. Let $f_{\mathcal{H}}, f_{\aleph} \in S_E(U)$ and ψ be a function from $\mathcal{H}$ to $\aleph$. Then, soft anti image of $f_{\mathcal{H}}$ under ψ, and soft pre-image (or soft inverse image) of $f_{\aleph}$ under ψ are the soft sets $\psi^*(f_{\mathcal{H}})$ and $\psi^{-1}(f_{\aleph})$ such that

$$\big(\psi^*(f_{\mathcal{H}})\big)(\mathrm{v}) = \begin{cases} \bigcap \{f_{\mathcal{H}}(\mathrm{e}) \mid \mathrm{e} \in \mathcal{H} \text{ and } \psi(\mathrm{e}) = \mathrm{v}\}, & if\ \psi^{-1}(\mathrm{v}) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $\mathrm{v} \in \aleph$ and $\left(\psi^{-1}(f_{\aleph})\right)(\mathrm{e}) = f_{\aleph}(\psi(\mathrm{e}))$ for all $\mathrm{e} \in \mathcal{H}$.

Definition 2.11 [66]. Let $f_{\mathcal{H}} \in S_E(U)$ and $\omega \subseteq U$. Then, lower ω-inclusion of $f_{\mathcal{H}}$, denoted by $\mathcal{S}(f_{\mathcal{H}}; \omega)$, is defined as

$$\mathcal{S}(f_{\mathcal{H}}; \omega) = \{x \in \mathcal{H} \mid f_{\mathcal{H}}(x) \subseteq \omega\}$$

Definition 2.12 [44]. Let $\hbar_S, \mathrm{б}_S \in S_S(U)$. S-uni product $\hbar_S * \mathrm{б}_S$ is defined by

$$(\hbar_S * \mathrm{б}_S)(\eta) = \begin{cases} \bigcap\limits_{\eta = \mathrm{ɯ}\mathrm{d}} \{\hbar_S(\mathrm{ɯ}) \cup \mathrm{б}_S(\mathrm{d})\}, & if\ \exists \mathrm{ɯ}, \mathrm{d} \in S \ such\ that\ \eta = \mathrm{ɯ}\mathrm{d} \\ U, & otherwise \end{cases}$$

Theorem 2.13 [44]. Let $p_S, \omega_S, \mu_S \in S_S(U)$. Then,

i. $(p_S * \omega_S) * \mu_S = p_S * (\omega_S * \mu_S)$
ii. $p_S * \omega_S \neq p_S * \omega_S$, generally.
iii. $p_S * (\omega_S \tilde{\cup} \mu_S) = (p_S * \omega_S) \tilde{\cup} (p_S * \mu_S)$ and $(p_S \tilde{\cup} \omega_S) * \mu_S = (p_S * \mu_S) \tilde{\cup} (\omega_S * \mu_S)$
iv. $p_S * (\omega_S \tilde{\cap} \mu_S) = (p_S * \omega_S) \tilde{\cap} (p_S * \mu_S)$ and $(p_S \tilde{\cap} \omega_S) * \mu_S = (p_S * \mu_S) \tilde{\cap} (\omega_S * \mu_S)$
v. If $p_S \tilde{\subseteq} \omega_S$, then $p_S * \mu_S \tilde{\subseteq} \omega_S * \mu_S$ and $\mu_S * p_S \tilde{\subseteq} \mu_S * \omega_S$
vi. If $x_S, y_S \in S_S(U)$ such that $x_S \tilde{\subseteq} p_S$ and $y_S \tilde{\subseteq} \omega_S$, then $x_S * y_S \tilde{\subseteq} p_S * \omega_S$.

Definition 2.14 [44]. Let $\mathrm{Ƀ} \subseteq S$. We denote by $\zeta_{\mathrm{Ƀ}^c}$ the soft characteristic function of the complement Ƀ and it is defined as

$$\zeta_{\mathrm{Ƀ}^c}(\upsilon) = \begin{cases} U, & if\ \upsilon \in S \setminus \mathrm{Ƀ} \\ \emptyset, & if\ \upsilon \in \mathrm{Ƀ} \end{cases}$$

Theorem 2.15 [44]. Let $\emptyset \neq \mathcal{H}, \mathcal{M} \subseteq \mathrm{S}$. Then,

i. If $\mathcal{H} \subseteq \mathcal{M}$, then $\zeta_{\mathcal{H}^c} \tilde{\supseteq} \zeta_{\mathcal{M}^c}$.
ii. $\zeta_{\mathcal{H}^c} \tilde{\cap} \zeta_{\mathcal{M}^c} = \zeta_{\mathcal{H}^c \cap \mathcal{M}^c}$ and $\zeta_{\mathcal{H}^c} \tilde{\cup} \zeta_{\mathcal{M}^c} = \zeta_{\mathcal{H}^c \cup \mathcal{M}^c}$.

Definition 2.16 [44,45,48,51-53]. $\eta_S \in S_S(U)$ is called

1. an S-uni subsemigroup of S over U (S-uni $\mathcal{SS}$) if $\eta_S(\ell g) \subseteq \eta_S(\ell) \cup \eta_S(g)$ for all $\ell, g \in S$,
2. an S-uni left (right) ideal of S over U (S-uni Ł($\mathcal{R}$)-ideal) if $\eta_S(eꝛ) \subseteq \eta_S(ꝛ)$ $\left(\eta_S(eꝛ) \subseteq \eta_S(e)\right)$ for all $e, ꝛ \in S$, and is called an S-uni two-sided ideal of S over U (S-uni ideal) if it is both S-uni Ł-ideal and S-uni $\mathcal{R}$-ideal,
3. an S-uni bi-ideal of S over U (S-uni $\mathcal{B}$-ideal) if η_S is an S-uni $\mathcal{SS}$ and $\eta_S(ǎƀr) \subseteq \eta_S(ǎ) \cup \eta_S(r)$ for all $ǎ, ƀ, r \in S$,
4. an S-uni interior ideal of S over U (S-uni I-ideal) if $\eta_S(\upsilon \theta \eta) \subseteq \eta_S(\theta)$ for all $\upsilon, \theta, \eta \in S$,
5. an S-uni left (right) weak-interior ideal of S over U (S-uni Ł($\mathcal{R}$)-WI-ideal) if $\eta_S(\text{ɚ}ƀ\sigma) \subseteq \eta_S(ƀ) \cup \eta_S(\sigma)$ $\left(\eta_S(\text{ɚ}ƀ\sigma) \subseteq \eta_S(\text{ɚ}) \cup \eta_S(ƀ)\right)$ for all ɚ, $ƀ, \sigma \in S$, and is called an S-uni weak-interior ideal of S over U (S-uni WI-ideal) if it is both S-uni Ł-WI-ideal and S-uni $\mathcal{R}$-WI-ideal,
6. an S-uni left (right) quasi-interior ideal of S over U (S-uni Ł($\mathcal{R}$)-$\mathcal{QI}$-ideal) if $\eta_S(ǎ\text{б}v\rho) \subseteq \eta_S(\text{б}) \cup \eta_S(\rho)$ $\left(\eta_S(ǎ\text{б}v\rho) \subseteq \eta_S(ǎ) \cup \eta_S(v)\right)$ for all $ǎ$, б, $v, \rho \in S$, and is called an S-uni quasi-interior ideal of S over U (S-uni $\mathcal{QI}$-ideal) if it is both S-uni Ł-$\mathcal{QI}$-ideal and S-uni right $\mathcal{R}$-$\mathcal{QI}$-ideal,
7. an S-uni bi-quasi-interior ideal of S over U (S-uni $\mathcal{BQI}$-ideal) if $\eta_S(\text{ɯ}ꝛ\text{ꞩ}v\text{ɉ}) \subseteq \eta_S(\text{ɯ}) \cup \eta_S(\text{ꞩ}) \cup \eta_S(\text{ɉ})$ for all ɯ, ꝛ, ꞩ, v, ɉ $\in S$,
8. an S-uni left (right) tri-ideal of S over U (S-uni Ł(Ꞧ)-Ţ-ideal) if $\eta_S(\text{ꞩ}\text{ƅ}\sigma\text{ɱ}) \subseteq \eta_S(\text{ꞩ}) \cup \eta_S(\sigma) \cup \eta_S(\text{ɱ})$ $\left(\eta_S(\text{ꞩ}\text{ƅ}\sigma\text{ɱ}) \subseteq \eta_S(\text{ꞩ}) \cup \eta_S(\text{ƅ}) \cup \eta_S(\text{ɱ})\right)$ for all ꞩ, ƅ, σ, ɱ $\in S$, and is called an S-uni tri-ideal of S over U (S-uni Ţ-ideal) if it is both S-uni Ł-Ţ-ideal and S-uni right $\mathcal{R}$-Ţ-ideal.

Here note that in [44], the definition of "S-uni $\mathcal{SS}$" is given as "S-uni semigroup of S"; however in this paper, we prefer to use "S-uni $\mathcal{SS}$".
if $\eta_S(\wp) = \emptyset$ for all $\wp \in S$, then η_S is an S-uni $\mathcal{SS}$ (Ł-ideal, Ꞧ-ideal, ideal, $\mathcal{B}$-ideal, I-ideal, WI-ideal, $\mathcal{QI}$-ideal, $\mathcal{BQI}$-ideal, Ţ-ideal). We denote such a kind of S-uni $\mathcal{SS}$ (Ł-ideal, Ꞧ-ideal, ideal, $\mathcal{B}$-ideal, I-ideal, WI-ideal, $\mathcal{QI}$-ideal, $\mathcal{BQI}$-ideal, Ţ-ideal) by $\widetilde{\Theta}$. It is clear that $\widetilde{\Theta} = \zeta_{S^C}$, that is, $\widetilde{\Theta}(\wp) = \emptyset$ for all $\wp \in S$ [44,45,48,51-53].

Definition 2.17 [45,49,50]. $\eta_S \in S_S(U)$ is called

1. an S-uni quasi-ideal of S over U (S-uni $\mathcal{Q}$-ideal) if $\left(\widetilde{\Theta} * \eta_S\right) \tilde{\cup} \left(\eta_S * \widetilde{\Theta}\right) \tilde{\supseteq} \eta_S$,
2. an S-uni bi-interior ideal of S over U ($\mathcal{BI}$-ideal) if $\left(\widetilde{\Theta} * \eta_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\eta_S * \widetilde{\Theta} * \eta_S\right) \tilde{\supseteq} \eta_S$,
3. an S-uni left (right) bi-quasi ideal of S over U (S-uni Ł(Ꞧ)-$\mathcal{BQ}$-ideal) if $\left(\widetilde{\Theta} * \eta_S\right) \tilde{\cup} \left(\eta_S * \widetilde{\Theta} * \eta_S\right) \tilde{\supseteq} \eta_S$ $\left(\left(\eta_S * \widetilde{\Theta}\right) \tilde{\cup} \left(\eta_S * \widetilde{\Theta} * \eta_S\right) \tilde{\supseteq} \eta_S\right)$, and is called an S-uni bi-quasi ideal of S over U (S-uni $\mathcal{BQ}$-ideal) if it is both S-uni Ł-$\mathcal{BQ}$-ideal and S-uni Ꞧ-$\mathcal{BQ}$-ideal.

Theorem 2.18 [44]. Let $\varkappa_S \in S_S(U)$. Then,

i) $\widetilde{\Theta} * \widetilde{\Theta} \widetilde{\supseteq} \widetilde{\Theta}$

ii) $\widetilde{\Theta} * \varkappa_S \widetilde{\supseteq} \widetilde{\Theta}$ and $\varkappa_S * \widetilde{\Theta} \widetilde{\supseteq} \widetilde{\Theta}$

iii) $\varkappa_S \widetilde{\cap} \widetilde{\Theta} = \widetilde{\Theta}$ and $\varkappa_S \widetilde{\cup} \widetilde{\Theta} = \varkappa_S$

Theorem 2.19 [44,45,48,51-53]. Let $\eta_S \in S_S(U)$. Then,

(1) η_S is an S-uni $\mathcal{SS}$ iff $(\eta_S * \eta_S) \widetilde{\supseteq} \eta_S$,

(2) η_S is an S-uni Ⱡ(Ɍ)-ideal iff $(\widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$ $\left((\eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S\right)$, η_S is an S-uni ideal iff $(\widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$ and $(\eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S$,

(3) η_S is an S-uni Ƀ-ideal iff $(\eta_S * \eta_S) \widetilde{\supseteq} \eta_S$ and $(\eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$,

(4) η_S is an S-uni *I*-ideal iff $(\widetilde{\Theta} * \eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S$,

(5) η_S is an S-uni Ⱡ(Ɍ)-WI-ideal iff $(\widetilde{\Theta} * \eta_S * \eta_S) \widetilde{\supseteq} \eta_S$ $\left((\eta_S * \eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S\right)$, η_S is an S-uni WI-ideal iff $(\widetilde{\Theta} * \eta_S * \eta_S) \widetilde{\supseteq} \eta_S$ and $(\eta_S * \eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S$,

(6) η_S is an S-uni Ⱡ(Ɍ)-*ꝖI*-ideal iff $(\widetilde{\Theta} * \eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$ $\left((\eta_S * \widetilde{\Theta} * \eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S\right)$, η_S is an S-uni *ꝖI*-ideal iff $(\widetilde{\Theta} * \eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$ and $(\eta_S * \widetilde{\Theta} * \eta_S * \widetilde{\Theta}) \widetilde{\supseteq} \eta_S$,

(7) η_S is an S-uni *ɃꝖI*-ideal iff $(\eta_S * \widetilde{\Theta} * \eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$,

(8) η_S is an S-uni Ⱡ(Ɍ)-Ʈ-ideal iff $(\eta_S * \widetilde{\Theta} * \eta_S * \eta_S) \widetilde{\supseteq} \eta_S$ $\left((\eta_S * \eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S\right)$, η_S is an S-uni Ʈ-ideal iff $(\eta_S * \widetilde{\Theta} * \eta_S * \eta_S) \widetilde{\supseteq} \eta_S$ and $(\eta_S * \eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$.

Theorem 2.20 [44,45,48-50,52]. For certain S-uni ideals, we have the following:

(1) Every S-uni Ⱡ(Ɍ/two-sided)-ideal is an S-uni $\mathcal{SS}$.

(2) Every S-uni ideal is an S-uni *I*-ideal.

(3) Every S-uni *Ꝗ*-ideal is an S-uni Ƀ-ideal.

(4) Every S-uni (Ⱡ/ Ɍ)-ideal is an S-uni (Ⱡ/ Ɍ)-WI-ideal.

(5) Every S-uni *Ꝗ*-ideal is an S-uni *ɃꝖI*-ideal.

(6) Every S-uni Ƀ-ideal is an S-uni *ɃI*-ideal.

(7) Every S-uni Ƀ-ideal is an S-uni (Ⱡ/ Ɍ)-*ɃꝖ*-ideal.

Theorem 2.21 [45]. Every S-uni Ƀ-ideal is an S-uni *Ꝗ*-ideal of a regular semigroup.

Theorem 2.22 [51]. Every S-uni Ⱡ(Ɍ)-*ꝖI*-ideal is an S-uni $\mathcal{SS}$ of a left (right) simple semigroup.

Theorem 2.23 [53]. Every S-uni (Ⱡ/ Ɍ)-Ʈ-ideal is an S-uni $\mathcal{SS}$ of band.

Proposition 2.24 [44]. Let $\mathfrak{f}_S \in S_S(U)$, $\omega \subseteq U$, $Im(\mathfrak{f}_S)$ be the image of $\mathfrak{f}_S$ such that $\omega \in Im(\mathfrak{f}_S)$. If $\mathfrak{f}_S$ is an S-uni $\mathcal{SS}$, then $\mathcal{S}(\mathfrak{f}_S;\omega)$ is an $\mathcal{SS}$.

Theorem 2.25 [44]. $\emptyset \neq R \subseteq S$ *is an* $\mathcal{SS}$ *iff the soft set* $\mathfrak{f}_S$ *defined by*

$$\mathfrak{f}_S(m) = \begin{cases} \alpha, & if\ m \in S \setminus R \\ \beta, & if\ m \in R \end{cases}$$

is an S-uni $\mathcal{SS}$*, where* $\alpha, \beta \subseteq U$ *such that* $\alpha \supseteq \beta$.

3. Soft Union Tri-bi-ideals of Semigroups

We propose the concept of soft union (S-uni) tri-bi-ideal of semigroups, examine in detail their relations with other S-uni ideals, and analyze in terms of some soft set concepts and operations in this section.

Definition 3.1. A soft set $\mathfrak{f}_S$ over U is called a soft union (S-uni) tri-bi-ideal of S over U if

$$\mathfrak{f}_S(\mathrm{b}\eta\mathrm{e}\mathrm{t}\upsilon) \subseteq \mathfrak{f}_S(\mathrm{b}) \cup \mathfrak{f}_S(\eta) \cup \mathfrak{f}_S(\mathrm{t}) \cup \mathfrak{f}_S(\upsilon)$$

for all $\mathrm{b}, \eta, \mathrm{e}, \mathrm{t}, \upsilon \in S$.

For brevity, soft union (S-uni) tri-bi-ideal of S over U is abbreviated by S-uni ŦɃ-ideal.

Example 3.2. Let $S = \{Ꞓ, ⋔, ꝗ\}$ be:

Table 1. *Cayley Table of "·" binary operation.*

·	Ꞓ	⋔	ꝗ
Ꞓ	ꝗ	ꝗ	ꝗ
⋔	Ꞓ	⋔	ꝗ
ꝗ	ꝗ	ꝗ	ꝗ

Let $\mathfrak{f}_S$ and $\mathfrak{u}_S$ over $U = \mathbb{Z}_8^*$ be as follows:

$$\mathfrak{f}_S = \{(Ꞓ, \{\bar{1}, \bar{5}, \bar{7}\}), (⋔, \{\bar{1}, \bar{3}\}), (ꝗ, \{\bar{1}\})\}$$

$$\mathfrak{u}_S = \{(Ꞓ, \{\bar{1}, \bar{3}, \bar{5}\}), (⋔, \{\bar{3}\}), (ꝗ, \{\bar{1}, \bar{5}, \bar{7}\})\}$$

Then, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$$\mathfrak{f}_S(Ꞓ⋔ꝗꞒ⋔) = \mathfrak{f}_S(ꝗ) = \{\bar{1}\} \subseteq \mathfrak{f}_S(Ꞓ) \cup \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(Ꞓ) \cup \mathfrak{f}_S(⋔) = U$$

$$\mathfrak{f}_S(Ꞓ⋔Ꞓ⋔Ꞓ) = \mathfrak{f}_S(ꝗ) = \{\bar{1}\} \subseteq \mathfrak{f}_S(Ꞓ) \cup \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(Ꞓ) = U$$

$$\mathfrak{f}_S(⋔⋔⋔⋔Ꞓ) = \mathfrak{f}_S(Ꞓ) = \{\bar{1}, \bar{5}, \bar{7}\} \subseteq \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(⋔) \cup \mathfrak{f}_S(Ꞓ) = U$$

It can be easily shown that the soft set $\mathfrak{f}_S$ satisfies the S-uni ŦɃ-ideal condition for all other element combinations of the set S. However, since

$$\mathfrak{u}_S(Ꞓ⋔ꝗꞒ⋔) = \mathfrak{u}_S(ꝗ) = \{\bar{1}, \bar{5}, \bar{7}\} \nsubseteq \mathfrak{u}_S(Ꞓ) \cup \mathfrak{u}_S(⋔) \cup \mathfrak{u}_S(Ꞓ) \cup \mathfrak{u}_S(⋔) = \{\bar{1}, \bar{3}, \bar{5}\}$$

$\mathfrak{u}_S$ is not an S-uni ŦɃ-ideal.

Theorem 3.3. Let $\mathfrak{f}_S \in S_S(U)$. Then, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal iff $\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S \,\widetilde{\supseteq}\, \mathfrak{f}_S$.

Proof: Suppose that $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal and $\mathfrak{m} \in S$. If $(\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S)(\mathfrak{m}) = U$, then $\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S \,\widetilde{\supseteq}\, \mathfrak{f}_S$. Otherwise, there exist elements $\nu, \hbar, \mathrm{d}, \mathrm{o}, \omega, \mathrm{j}, \varrho, \mathrm{t} \in S$ such that $\mathfrak{m} = \nu\hbar$, $\nu = \mathrm{d}\mathrm{o}$, $\mathrm{d} = \omega\mathrm{j}$ and $\omega = \varrho\mathrm{t}$ for $\mathfrak{m} \in S$. Since $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal,

$$\mathfrak{f}_S(\mathfrak{m}) = \mathfrak{f}_S(\nu\hbar) = \mathfrak{f}_S((\mathrm{d}\mathrm{o})\hbar) = \mathfrak{f}_S((\omega\mathrm{j})\mathrm{o}\hbar)$$

$$= \mathfrak{f}_S((\varrho\mathrm{t})n\mathrm{o}\hbar) \subseteq \mathfrak{f}_S(\varrho) \cup \mathfrak{f}_S(\mathrm{t}) \cup \mathfrak{f}_S(\mathrm{o}) \cup \mathfrak{f}_S(\hbar).$$

Therefore,

$$(\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S)(\mathfrak{m}) = [(\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S) * \mathfrak{f}_S](\mathfrak{m})$$

$$= \bigcap_{ɱ=\nu\hbar}\{(ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S)(\nu) \cup ƒ_S(\hbar)\}$$

$$= \bigcap_{ɱ=\nu\hbar}\left\{\bigcap_{\nu=đ̱ɵ}\{(ƒ_S * ƒ_S * \widetilde{\Theta})(đ̱) \cup ƒ_S(ɵ)\} \cup ƒ_S(\hbar)\right\}$$

$$= \bigcap_{ɱ=\nu\hbar}\left\{\bigcap_{\nu=đ̱ɵ}\left\{\bigcap_{đ̱=\omega ȷ̣}\{(ƒ_S * ƒ_S)(\omega) \cup \widetilde{\Theta}(ȷ̣)\} \cup ƒ_S(ɵ)\right\} \cup ƒ_S(\hbar)\right\}$$

$$= \bigcap_{ɱ=\nu\hbar}\left\{\bigcap_{\nu=đ̱ɵ}\left\{\bigcap_{đ̱=\omega ȷ̣}\left\{\bigcap_{\omega=\varrho ŧ}\{ƒ_S(\varrho) \cup ƒ_S(ŧ)\} \cup \widetilde{\Theta}(ȷ̣)\right\} \cup ƒ_S(ɵ)\right\} \cup ƒ_S(\hbar)\right\}$$

$$= \bigcap_{ɱ=\varrho ŧ ȷ̣ ɵ \hbar}\{ƒ_S(\varrho) \cup ƒ_S(ŧ) \cup ƒ_S(ɵ) \cup ƒ_S(\hbar)\}$$

$$\supseteq \bigcap_{ɱ=\varrho ŧ ȷ̣ ɵ \hbar}\{ƒ_S(\varrho ŧ ȷ̣ ɵ \hbar)\} = ƒ_S(ɱ)$$

Thus, we have $ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S \,\widetilde{\supseteq}\, ƒ_S$. Moreover, in the case where $ɱ = \nu\hbar$ and $\nu \neq đ̱ɵ$ for $ɱ \in S$, since $(ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S)(\nu) = U$, $ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S \,\widetilde{\supseteq}\, ƒ_S$ is satisfied.

Conversely, assume that $ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S \,\widetilde{\supseteq}\, ƒ_S$. Let $ɱ = \nu\hbar đ̱ ɵ ɼ$ for $ɱ, \nu, \hbar, đ̱, ɵ, ɼ \in S$. Then, we have

$$ƒ_S(\nu\hbar đ̱ ɵ ɼ) = ƒ_S(ɱ) \subseteq (ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S)(ɱ) = [(ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S) \circ ƒ_S](ɱ)$$

$$= \bigcap_{ɱ=ƀ\ell}\{(ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S)(ƀ) \cup ƒ_S(\ell)\} \subseteq (ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S)(\nu\hbar đ̱ ɵ) \cup ƒ_S(ɼ)$$

$$= \bigcap_{\nu\hbar đ̱ ɵ = ʐɕ}\{(ƒ_S * ƒ_S * \widetilde{\Theta})(ʐ) \cup ƒ_S(ɕ)\} \cup ƒ_S(ɼ) \subseteq [(ƒ_S * ƒ_S * \widetilde{\Theta})(\nu\hbar đ̱) \cup ƒ_S(ɵ)] \cup ƒ_S(ɼ)$$

$$= \left[\bigcap_{\nu\hbar đ̱ = ɯȷ̣}\{(ƒ_S * ƒ_S)(ɯ) \cup \widetilde{\Theta}(ȷ̣)\} \cup ƒ_S(ɵ)\right] \cup ƒ_S(ɼ) \subseteq \left[\left((ƒ_S * ƒ_S)(\nu\hbar) \cup \widetilde{\Theta}(đ̱)\right) \cup ƒ_S(ɵ)\right] \cup ƒ_S(ɼ)$$

$$= \left[\left\{\bigcap_{\nu\hbar = ɡʞ}\{ƒ_S(ɡ) \cup ƒ_S(ʞ)\} \cup \widetilde{\Theta}(đ̱)\right\} \cup ƒ_S(ɵ)\right] \cup ƒ_S(ɼ)$$

$$\subseteq \left[\left((ƒ_S(\nu) \cup ƒ_S(\hbar)) \cup \widetilde{\Theta}(đ̱)\right) ƒ_S(ɵ)\right] \cup ƒ_S(ɼ)$$

$$= [ƒ_S(\nu) \cup ƒ_S(\hbar) \cup U \cup ƒ_S(ɵ)] \cup ƒ_S(ɼ) = ƒ_S(\nu) \cup ƒ_S(\hbar) \cup ƒ_S(ɵ) \cup ƒ_S(ɼ)$$

Hence, $ƒ_S(\nu\hbar đ̱ ɵ ɼ) \subseteq ƒ_S(\nu) \cup ƒ_S(\hbar) \cup ƒ_S(ɵ) \cup ƒ_S(ɼ)$ implying that $ƒ_S$ is an S-uni ŦɃ-ideal.

Corollary 3.4. $\widetilde{\Theta}$ is an S-uni ŦɃ-ideal.

Proposition 3.5. $\emptyset \neq Ꞧ \subseteq S$ is a ŦɃ-ideal iff the S-uni SS $ƒ_S$ defined by

$$ƒ_S(m) = \begin{cases} \alpha, & if\ m \in S \setminus Ꞧ \\ \beta, & if\ m \in Ꞧ \end{cases}$$

is an S-uni ŦɃ-ideal, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof: Suppose $Ɍ$ is a ŦɃ-ideal and $c,d,x,a,b \in S$. If $a,b,c,d \in Ɍ$, then $cdxab \in Ɍ$. Hence, $\mathfrak{f}_S(cdxab) = \mathfrak{f}_S(a) = \mathfrak{f}_S(b) = \mathfrak{f}_S(c) = \mathfrak{f}_S(d) = \beta$ and so $\mathfrak{f}_S(cdxab) \subseteq \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) \cup \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b)$. If $a,b,c,d \notin Ɍ$, then, $cdxab \in Ɍ$ or $cdxab \notin Ɍ$. In this case, if $cdxab \in Ɍ$, then $\beta = \mathfrak{f}_S(cdxab) \subseteq \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) \cup \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \alpha$. If $cdxab \notin Ɍ$, then $\alpha = \mathfrak{f}_S(cdxab) \subseteq \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) \cup \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) = \alpha$. If $a \in Ɍ$, $b \in Ɍ$, $c \in Ɍ$ or $d \in Ɍ$, then $cdxab \in Ɍ$ or $cdxab \notin Ɍ$. Here, firstly note that, if $a \in Ɍ$, $b \in Ɍ$, $c \in Ɍ$ or $d \in Ɍ$, then either $\mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) \cup \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) = \beta$ (the case where $a \in Ɍ$, $b \in Ɍ$, $c \in Ɍ$ and $d \in Ɍ$) or $\mathfrak{f}_S(a) \cup \mathfrak{f}_S(b) \cup \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) = \alpha$ (the case where $a \in Ɍ$ and either $b \notin Ɍ$, $c \notin Ɍ$ or $d \notin Ɍ$ (or $b \in Ɍ$ and either $a \notin Ɍ$, $c \notin Ɍ$ or $d \notin Ɍ$; or $c \in Ɍ$ and either $a \notin Ɍ$, $b \notin Ɍ$ or $d \notin Ɍ$; or $d \in Ɍ$ and either $a \notin Ɍ$, $b \notin Ɍ$ or $c \notin Ɍ$)). Thus, either $cdxab \in Ɍ$ or $cdxab \notin Ɍ$, in any case $\mathfrak{f}_S(cdxab) \supseteq \mathfrak{f}_S(c) \cup \mathfrak{f}_S(d) \cup \mathfrak{f}_S(a) \cup \mathfrak{f}_S(b)$, since $\alpha \supseteq \beta$. Hence, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Conversely assume that S-uni $\mathcal{SS}$ $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal. Let $a,b,c,d \in Ɍ$, and $x \in S$. Then, $\mathfrak{f}_S(cdxab) \subseteq \mathfrak{f}_S(c) = \mathfrak{f}_S(d) = \mathfrak{f}_S(a) = \mathfrak{f}_S(b) = \beta$. Since $\beta \subseteq \alpha$ and the function is a two-valued function, $\mathfrak{f}_S(cdxab) \neq \alpha$, implying that $\mathfrak{f}_S(cdxab) = \beta$. Hence, $cdxab \in Ɍ$. By Theorem 2.25, $Ɍ$ is an $\mathcal{SS}$. Thus, $Ɍ$ is a ŦɃ-ideal.

Theorem 3.6. Let H be an $\mathcal{SS}$. Then, H is a ŦɃ-ideal iff ζ_{H^c} is an S-uni ŦɃ-ideal.

Proof: Since

$$\zeta_{H^c}(\upsilon) = \begin{cases} U, & if\ \upsilon \in S\setminus H \\ \emptyset, & if\ \upsilon \in H \end{cases}$$

and $U \supseteq \emptyset$, the remainder of the proof is completed based on Proposition 3.5.

Example 3.7. Let the semigroup in Example 3.2. $\Upsilon = \{Ꞇ, Ꞁ\}$ is a ŦɃ-ideal. By the definition of soft characteristic function, $\zeta_{\Upsilon^c} = \{(Ꞇ,\emptyset),(⋔,U),(Ꞁ,\emptyset)\}$. Then, ζ_{Υ^c} is an S-uni ŦɃ-ideal. Conversely, by choosing the S-uni ŦɃ-ideal as $\mathfrak{f}_S = \{(Ꞇ,U),(⋔,\emptyset),(Ꞁ,\emptyset)\}$, which is the soft characteristic function of $\mathcal{R} = \{⋔, Ꞁ\}$, one can show that $\mathcal{R}$ is a ŦɃ-ideal.

Now, we continue with the relationships between S-uni ŦɃ-ideals and other types of S-uni ideals of S.

Theorem 3.8. Every S-uni ŦɃ-ideal is an S-uni $\mathcal{SS}$ of a band.

Proof: Let $\mathfrak{f}_S$ be an S-uni ŦɃ-ideal of a band S and $ꝟ, ℯ \in S$. By assumption, for all $ꝟ \in S$, $ꝟ = ꝟꝟ$. Thus,

$$\mathfrak{f}_S(ꝟℯ) = \mathfrak{f}_S((ꝟꝟ)ℯ) = \mathfrak{f}_S((ꝟꝟ)ꝟℯ) = \mathfrak{f}_S((ꝟꝟ)ꝟꝟℯ) \subseteq \mathfrak{f}_S(ꝟ) \cup \mathfrak{f}_S(ꝟ) \cup \mathfrak{f}_S(ꝟ) \cup \mathfrak{f}_S(ℯ) = \mathfrak{f}_S(ꝟ) \cup \mathfrak{f}_S(ℯ)$$

Hence, $\mathfrak{f}_S$ is an S-uni $\mathcal{SS}$.

Theorem 3.9. Every S-uni Ⱡ-ideal is an S-uni ŦɃ-ideal.

Proof: Let $\hbar_S$ be an S-uni Ⱡ-ideal. Then, $\widetilde{\Theta} * \hbar_S \,\widetilde{\supseteq}\, \hbar_S$ and $\hbar_S * \hbar_S \,\widetilde{\supseteq}\, \hbar_S$. Thus,

$$\hbar_S * \hbar_S * \widetilde{\Theta} * \hbar_S * \hbar_S \,\widetilde{\supseteq}\, \hbar_S * \hbar_S * \hbar_S * \hbar_S \,\widetilde{\supseteq}\, \hbar_S * \hbar_S \,\widetilde{\supseteq}\, \hbar_S$$

Hence, $\hbar_S$ is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.9 does not hold.

Example 3.10. Let $S = \{Ɔ, Ꭻb, ϟ\}$ be:

Table 2. *Cayley Table of "⋄" binary operation.*

⋄	Ↄ	Ꮭ	ϟ
Ↄ	ϟ	Ↄ	ϟ
Ꮭ	ϟ	Ꮭ	ϟ
ϟ	ϟ	ϟ	ϟ

Let $ɥ_S$ be a soft set over $U = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ as follows:

$$ɥ_S = \{(Ↄ, \{y, yx, yx^2\}), (Ꮭ, \{e, x, x^2\}), (ϟ, \emptyset)\}$$

Then, $ɥ_S$ is an S-uni ŦɃ-ideal. In fact;

$(ɥ_S * ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ↄ) = ɥ_S(Ↄ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ꮭ) = ɥ_S(Ↄ) \cup [(ɥ_S * \widetilde{\Theta})(Ꮭ) \cup (ɥ_S * ɥ_S)(Ꮭ)] = ɥ_S(Ↄ) \cup [(ɥ_S(Ꮭ) \cup \widetilde{\Theta}(Ꮭ)) \cup (ɥ_S(Ꮭ) \cup ɥ_S(Ꮭ))] = ɥ_S(Ↄ) \cup ɥ_S(Ꮭ) \supseteq ɥ_S(Ↄ)$

$(ɥ_S * ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ꮭ) = ɥ_S(Ꮭ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ꮭ) = ɥ_S(Ꮭ) \cup [(ɥ_S * \widetilde{\Theta})(Ꮭ) \cup (ɥ_S * ɥ_S)(Ꮭ)] = ɥ_S(Ꮭ) \cup [(ɥ_S(Ꮭ) \cup \widetilde{\Theta}(Ꮭ)) \cup (ɥ_S(Ꮭ) \cup ɥ_S(Ꮭ))] = ɥ_S(Ꮭ) \supseteq ɥ_S(Ꮭ)$

$(ɥ_S * ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(ϟ) = (ɥ_S(Ↄ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ↄ)) \cap (ɥ_S(Ꮭ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ↄ)) \cap (ɥ_S(ϟ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ↄ)) \cap (ɥ_S(ϟ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(Ꮭ)) \cap (ɥ_S(Ↄ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(ϟ)) \cap (ɥ_S(Ꮭ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(ϟ)) \cap (ɥ_S(ϟ) \cup (ɥ_S * \widetilde{\Theta} * ɥ_S * ɥ_S)(ϟ)) = (ɥ_S(Ↄ) \cup ɥ_S(Ꮭ)) \cap (ɥ_S(Ꮭ) \cup ɥ_S(Ↄ)) \cap (ɥ_S(ϟ) \cup ɥ_S(Ↄ) \cup ɥ_S(Ꮭ)) \cap (ɥ_S(ϟ) \cup ɥ_S(Ꮭ)) \cap ɥ_S(Ↄ) \cap ɥ_S(Ꮭ) \cap ɥ_S(ϟ) = ɥ_S(Ↄ) \cap ɥ_S(Ꮭ) \cap ɥ_S(ϟ) \supseteq ɥ_S(ϟ)$

thus, $ɥ_S$ is an S-uni ŦɃ-ideal. However, since

$$ɥ_S(ↃᏝ) = ɥ_S(Ↄ) \nsubseteq ɥ_S(Ꮭ)$$

$ɥ_S$ is not an S-uni Ⱡ-ideal.

Theorem 3.11 illustrates that the converse of Theorem 3.9 is valid for the right simple bands.

Theorem 3.11. Let $ʄ_S \in S_S(U)$ and S be a right simple band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ⱡ-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.9. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of a right simple band S and ȷ, ʉ $\in S$. By assumption, for all ȷ, ʉ $\in S$, there exist ρ $\in S$ such that ȷ = ʉρ and ʉ = ʉʉ. Thus,

$$ʄ_S(ȷʉ) = ʄ_S((ʉρ)ʉ) = ʄ_S((ʉʉ)ρ(ʉʉ)) \subseteq ʄ_S(ʉ) \cup ʄ_S(ʉ) \cup ʄ_S(ʉ) \cup ʄ_S(ʉ) = ʄ_S(ʉ)$$

Hence, $ʄ_S$ is an S-uni Ⱡ-ideal.

Theorem 3.12. Every S-uni Ꞧ-ideal is an S-uni ŦɃ-ideal.

Proof: Let $ħ_S$ be an S-uni *Ꞧ*-ideal. Then, $ħ_S * \widetilde{\Theta} \mathrel{\widetilde{\supseteq}} ħ_S$ and $ħ_S * ħ_S \mathrel{\widetilde{\supseteq}} ħ_S$. Thus,

$$ħ_S * ħ_S * \widetilde{\Theta} * ħ_S * ħ_S \mathrel{\widetilde{\supseteq}} ħ_S * ħ_S * ħ_S * ħ_S \mathrel{\widetilde{\supseteq}} ħ_S * ħ_S \mathrel{\widetilde{\supseteq}} ħ_S.$$

Therefore, $ћ_S$ is an S-uni ₮Ꞵ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.12 does not hold:

Example 3.13. Let $ʄ_S$ in Example 3.2. As seen in Example 3.2, $ʄ_S$ is an S-uni ₮Ꞵ-ideal. Since

$$ʄ_S(⫛C) = ʄ_S(C) \nsubseteq ʄ_S(⫛)$$

$ʄ_S$ is not an S-uni Ꞧ-ideal.

Theorem 3.14 illustrates that the converse of Theorem 3.12 is valid for the left simple bands.

Theorem 3.14. Let $ʄ_S \in S_S(U)$ and S be a left simple band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ꞧ-ideal.
2. $ʄ_S$ is an S-uni ₮Ꞵ-ideal.

Proof: (1) implies (2) is by Theorem 3.12. Suppose that $ʄ_S$ is an S-uni ₮Ꞵ-ideal of a left simple band S and $ᶑ, ℏ \in S$. By assumption, for all $ᶑ, ℏ \in S$, there exist $ə \in S$ such that $ℏ = əᶑ$ and $ᶑ = ᶑᶑ$. Thus,

$$ʄ_S(ᶑℏ) = ʄ_S\big(ᶑ(əᶑ)\big) = ʄ_S\big((ᶑᶑ)ə(ᶑᶑ)\big) \subseteq ʄ_S(ᶑ) \cup ʄ_S(ᶑ) \cup ʄ_S(ᶑ) \cup ʄ_S(ᶑ) = ʄ_S(ᶑ)$$

Hence, $ʄ_S$ is an S-uni Ꞧ-ideal.

Theorem 3.15. Every S-uni ideal is an S-uni ₮Ꞵ-ideal.

Proof: It follows by Theorem 3.9 and Theorem 3.12.

Here note that the converse of Theorem 3.15 does not hold follows from Example 3.10 and Example 3.13.

Theorem 3.16 illustrates that the converse of Theorem 3.15 is valid for the idempotent groups.

Theorem 3.16. Let $ʄ_S \in S_S(U)$ and S be an idempotent group. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni ideal.
2. $ʄ_S$ is an S-uni ₮Ꞵ-ideal.

Proof: (1) implies (2) is by Theorem 3.15. Assume that $ʄ_S$ is an S-uni ₮Ꞵ-ideal of an idempotent group S. Then, by Theorem 2.4 (ii), S is both a left simple and a right simple. The remainder of the proof is completed based on Theorem 3.11 and Theorem 3.14.

Theorem 3.17. Every S-uni I-ideal is an S-uni ₮Ꞵ-ideal.

Proof: Let $ɯ_S$ be an S-uni I-ideal. Then, $\widetilde{\Theta} * ɯ_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, ɯ_S$. Thus,

$$ɯ_S * ɯ_S * \widetilde{\Theta} * ɯ_S * ɯ_S \,\widetilde{\supseteq}\, \big(\widetilde{\Theta} * ɯ_S * \widetilde{\Theta}\big) * ɯ_S * ɯ_S \,\widetilde{\supseteq}\, ɯ_S * ɯ_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, \widetilde{\Theta} * ɯ_S * \widetilde{\Theta} \,\widetilde{\supseteq}\, ɯ_S$$

Hence, $ɯ_S$ is an S-uni ₮Ꞵ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.17 does not hold:

Example 3.18. Consider $ʄ_S$ in Example 3.2. As seen in Example 3.2, $ʄ_S$ is an S-uni ₮Ꞵ-ideal. Since

$$ʄ_S(⫛⫛C) = ʄ_S(C) \nsubseteq ʄ_S(⫛)$$

$ʄ_S$ is not an S-uni I-ideal.

Theorem 3.19 illustrates that the converse of Theorem 3.17 is valid for the idempotent groups.

Theorem 3.19. Let $ʄ_S \in S_S(U)$ and S be an idempotent group. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni I-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) by Theorem 3.17. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of an idempotent group. Then, by Theorem 3.16, $ʄ_S$ is an S-uni ideal. The remainder of the proof is completed based on Theorem 2.20 (ii).

Theorem 3.20. Every S-uni Ƀ-ideal is an S-uni ŦɃ-ideal.

Proof: Let $ɖ_S$ be an S-uni Ƀ-ideal. Then, $ɖ_S\widetilde{\Theta} * ɖ_S \,\widetilde{\supseteq}\, ɖ_S$ and $ɖ_S * ɖ_S \,\widetilde{\supseteq}\, ɖ_S$. Thus,

$$ɖ_S * ɖ_S * \widetilde{\Theta} * ɖ_S * ɖ_S \,\widetilde{\supseteq}\, ɖ_S * ɖ_S * ɖ_S \,\widetilde{\supseteq}\, ɖ_S * ɖ_S \,\widetilde{\supseteq}\, ɖ_S$$

Hence, $ɖ_S$ is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.20 does not hold:

Example 3.21. Let $S = \{Ʊ, Ϫ, Э, Ѡ\}$ be:

Table 3. *Cayley Table of "⊚" binary operation.*

⊚	Ʊ	Ϫ	Э	Ѡ
Ʊ	Ʊ	Ʊ	Ʊ	Ʊ
Ϫ	Ʊ	Ʊ	Ʊ	Ʊ
Э	Ʊ	Ʊ	Ʊ	Ϫ
Ѡ	Ʊ	Ʊ	Ϫ	Э

Let $ƃ_S$ be a soft set over $U = S_3$ as follows:

$$ƃ_S = \{(Ʊ, \{(1)\}), (Ϫ, \{(123)\}), (Э, \{(1), (12)\}), (Ѡ, \{(1), (132)\})\}$$

Then, $ƃ_S$ is an S-uni ŦɃ-ideal. In fact;

$$\left(ƃ_S * ƃ_S * \widetilde{\Theta} * ƃ_S * ƃ_S\right)(Ʊ) = \{(1)\} \supseteq ƃ_S(Ʊ)$$

$$\left(ƃ_S * ƃ_S * \widetilde{\Theta} * ƃ_S * ƃ_S\right)(Ϫ) = U \supseteq ƃ_S(Ϫ)$$

$$\left(ƃ_S * ƃ_S * \widetilde{\Theta} * ƃ_S * ƃ_S\right)(Э) = U \supseteq ƃ_S(Э)$$

$$\left(ƃ_S * ƃ_S * \widetilde{\Theta} * ƃ_S * ƃ_S\right)(Ѡ) = U \supseteq ƃ_S(Ѡ)$$

thus, $ƃ_S$ is an S-uni ŦɃ-ideal. However, since

$$ƃ_S(ЭѠ) = ƃ_S(Ϫ) \nsubseteq ƃ_S(Э) \cup ƃ_S(Ѡ)$$

$ƃ_S$ is not an S-uni SS. Hence, $ƃ_S$ is not an S-uni Ƀ-ideal.

Theorem 3.22 illustrates that the converse of Theorem 3.20 is valid for the bands.

Theorem 3.22. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ƀ-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.20. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of a band S and $ɯ, ƀ, ɼ \in S$. By assumption, for all $ɯ, ɼ \in S$, $ɯ = ɯɯ$ and $ɼ = ɼɼ$. Thus,

$$\mathfrak{f}_S(\text{ɯƀɼ}) = \mathfrak{f}_S\big((\text{ɯɯ})\text{ƀ}(\text{ɼɼ})\big) \subseteq \mathfrak{f}_S(\text{ɯ}) \cup \mathfrak{f}_S(\text{ɯ}) \cup \mathfrak{f}_S(\text{ɼ}) \cup \mathfrak{f}_S(\text{ɼ}) = \mathfrak{f}_S(\text{ɯ}) \cup \mathfrak{f}_S(\text{ɼ})$$

Moreover, $\mathfrak{f}_S$ is an S-uni SS by Theorem 3.8. Hence, $\mathfrak{f}_S$ is an S-uni Ƀ-ideal.

Theorem 3.23. Every S-uni Q-ideal is an S-uni ŦɃ-ideal.

Proof: Let $\mathfrak{f}_S$ be an S-uni Q-ideal. Then, by Theorem 2.20 (iii), $\mathfrak{f}_S$ is an S-uni Ƀ-ideal. Hence, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal by Theorem 3.20.

We demonstrate with a counterexample that the converse of Theorem 3.23 does not hold:

Example 3.24. Consider ƅ_S in Example 3.21. As seen in Example 3.21, ƅ_S is an S-uni ŦɃ-ideal, however is not an S-uni Ƀ-ideal. Since ƅ_S is not an S-uni Ƀ-ideal, ƅ_S is not an S-uni Q-ideal.

Theorem 3.25 illustrates that the converse of Theorem 3.23 is valid for the bands.

Theorem 3.25. Let $\mathfrak{f}_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an S-uni Q-ideal.
2. $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.23. Suppose that $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal of a band S. Since S is a band, $\mathfrak{f}_S$ is an S-uni Ƀ-ideal by Theorem 3.22. Then, since every band is regular, S is regular, and hence $\mathfrak{f}_S$ is an S-uni quasi-ideal by Theorem 2.21.

Theorem 3.26. Every S-uni Ł-WI-ideal is an S-uni ŦɃ-ideal.

Proof: Let ƍ_S be an S-uni Ł-WI-ideal. Then, $\widetilde{\Theta} * \text{ƍ}_S * \text{ƍ}_S \supseteq \text{ƍ}_S$. Thus,

$$\text{ƍ}_S * \text{ƍ}_S * \widetilde{\Theta} * \text{ƍ}_S * \text{ƍ}_S \supseteq \text{ƍ}_S * \text{ƍ}_S * \text{ƍ}_S \supseteq \widetilde{\Theta} * \text{ƍ}_S * \text{ƍ}_S \supseteq \text{ƍ}_S$$

Hence, ƍ_S is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.26 does not hold:

Example 3.27. Consider ɥ_S in Example 3.10. As seen in Example 3.10, ɥ_S is an S-uni ŦɃ-ideal. Since

$$\text{ɥ}_S(\text{ϽЉЉ}) = \text{ɥ}_S(\text{Ͻ}) \nsubseteq \text{ɥ}_S(\text{Љ}) \cup \text{ɥ}_S(\text{Љ})$$

ɥ_S is not an S-uni Ł-WI-ideal.

Theorem 3.28 illustrates that the converse of Theorem 3.26 is valid for the right simple bands.

Theorem 3.28. Let $\mathfrak{f}_S \in S_S(U)$ and S be a right simple band. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an S-uni Ł-WI-ideal.
2. $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.26. Assume that $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal of a right simple band. Then, $\mathfrak{f}_S$ is an S-uni Ł-ideal by Theorem 3.11. Thus, $\mathfrak{f}_S$ is an S-uni Ł-WI-ideal by Theorem 2.20 (iv).

Theorem 3.29. Every S-uni Ꞧ-WI-ideal is an S-uni ŦɃ-ideal.

Proof: Let ƍ_S be an S-uni right Ꞧ-WI-ideal. Then, $\text{ƍ}_S * \text{ƍ}_S * \widetilde{\Theta} \supseteq \text{ƍ}_S$. Thus,

$$\text{ƍ}_S * \text{ƍ}_S * \widetilde{\Theta} * \text{ƍ}_S * \text{ƍ}_S \supseteq \text{ƍ}_S * \text{ƍ}_S * \text{ƍ}_S \supseteq \text{ƍ}_S * \text{ƍ}_S * \widetilde{\Theta} \supseteq \text{ƍ}_S$$

Hence, ƍ_S is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.29 does not hold:

Example 3.30. Consider the soft set $ƒ_S$ in Example 3.2. As seen in Example 3.2, $ƒ_S$ is an S-uni Ŧ₿-ideal. Since

$$ƒ_S(⋔⋔Ꞓ) = ƒ_S(Ꞓ) \nsubseteq ƒ_S(⋔) \cup ƒ_S(⋔)$$

$ƒ_S$ is not an S-uni Ꞧ-WI-ideal.

Theorem 3.31 illustrates that the converse of Theorem 3.29 is valid for the left simple bands.

Theorem 3.31. Let $ƒ_S \in S_S(U)$ and S be a simple left band. Then, the following conditions are equivalent:

1. $ƒ_S$ is an S-uni Ꞧ-WI-ideal.
2. $ƒ_S$ is an S-uni Ŧ₿-ideal.

Proof: (1) implies (2) is by Theorem 3.29. Assume that $ƒ_S$ is an S-uni Ŧ₿-ideal of a left simple band. Then, $ƒ_S$ is an S-uni *Ꞧ*-ideal by Theorem 3.14. Thus, $ƒ_S$ is an S-uni Ꞧ-WI-ideal by Theorem 2.20 (iv).

Theorem 3.32. Every S-uni WI-ideal is an S-uni Ŧ₿-ideal.

Proof: It follows by Theorem 3.26 and Theorem 3.29.

Here note that the converse of Theorem 3.32 is not true follows from Example 3.27 and Example 3.30.

Theorem 3.33 illustrates that the converse of Theorem 3.32 is valid for the idempotent groups.

Theorem 3.33. Let $ƒ_S \in S_S(U)$ and S be an idempotent group. Then, the following conditions are equivalent:

1. $ƒ_S$ is an S-uni WI-ideal.
2. $ƒ_S$ is an S-uni Ŧ₿-ideal.

Proof: (1) implies (2) is by Theorem 3.32. Assume that $ƒ_S$ is an S-uni Ŧ₿-ideal of an idempotent group. Then, by Theorem 3.16, $ƒ_S$ is an S-uni ideal. Thus, $ƒ_S$ is an S-uni WI-ideal by Theorem 2.20 (iv).

Theorem 3.34. Every S-uni *ɃI*-ideal is an S-uni Ŧ₿-ideal.

Proof: Let $ƒ_S$ be an S-uni *ɃI*-ideal. Then, $(\widetilde{\Theta} * ƒ_S * \widetilde{\Theta}) \,\widetilde{\cup}\, (ƒ_S * \widetilde{\Theta} * ƒ_S) \,\widetilde{\supseteq}\, ƒ_S$. Thus,

$$\begin{aligned} ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S &= (ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S) \,\widetilde{\cup}\, (ƒ_S * ƒ_S * \widetilde{\Theta} * ƒ_S * ƒ_S) \\ &\widetilde{\supseteq} (\widetilde{\Theta} * \widetilde{\Theta} * \widetilde{\Theta} * ƒ_S * \widetilde{\Theta}) \,\widetilde{\cup}\, (ƒ_S * \widetilde{\Theta} * \widetilde{\Theta} * \widetilde{\Theta} * ƒ_S) \\ &\widetilde{\supseteq} (\widetilde{\Theta} * \widetilde{\Theta} * ƒ_S * \widetilde{\Theta}) \,\widetilde{\cup}\, (ƒ_S * \widetilde{\Theta} * \widetilde{\Theta} * ƒ_S) \\ &\widetilde{\supseteq} (\widetilde{\Theta} * ƒ_S * \widetilde{\Theta}) \,\widetilde{\cup}\, (ƒ_S * \widetilde{\Theta} * ƒ_S) \\ &\widetilde{\supseteq} ƒ_S \end{aligned}$$

Hence, $ƒ_S$ is an S-uni Ŧ₿-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.34 does not hold:

Example 3.35. Consider $ƅ_S$ in Example 3.21. as seen in example 3.21, $ƅ_S$ is an S-uni Ŧ₿-ideal. since

$$[(\widetilde{\Theta} * ƅ_S * \widetilde{\Theta}) \,\widetilde{\cup}\, (ƅ_S * \widetilde{\Theta} * ƅ_S)](Ӿ) = (\widetilde{\Theta} * ƅ_S * \widetilde{\Theta})(Ӿ) \cup (ƅ_S * \widetilde{\Theta} * ƅ_S)(Ӿ) = ƅ_S(Ѡ) \nsupseteq ƅ_S(Ӿ)$$

$ƅ_S$ is not an S-uni *ɃI*-ideal.

Theorem 3.36 illustrates that the converse of Theorem 3.34 is valid for the bands.

Theorem 3.36. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni *ɃI*-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.34. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of a band. Then, $ʄ_S$ is an S-uni *Ƀ*-ideal by Theorem 3.22. Thus, $ʄ_S$ is an S-uni *ɃI*-ideal by Theorem 2.20 (vi).

Theorem 3.37. Every S-uni Ⱡ-*ɃQ*-ideal is an S-uni ŦɃ-ideal.

Proof: Let $ʄ_S$ be an S-uni Ⱡ-*ɃQ*-ideal. Then, $(\widetilde{Ɵ} * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * ʄ_S) \,\widetilde{\supseteq}\, ʄ_S$. Thus,

$$\begin{aligned} ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S &= (ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S) \\ &\widetilde{\supseteq} (\widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} (\widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} (\widetilde{Ɵ} * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} ʄ_S \end{aligned}$$

Hence, $ʄ_S$ is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.37 does not hold:

Example 3.38. Consider $ƃ_S$ in Example 3.21. As seen in Example 3.21, $ƃ_S$ is an S-uni ŦɃ-ideal. Since

$$[(\widetilde{Ɵ} * ƃ_S) \,\widetilde{\cup}\, (ƃ_S * \widetilde{Ɵ} * ƃ_S)](Ꭓ) = (\widetilde{Ɵ} * ƃ_S)(Ꭓ) \cup (ƃ_S * \widetilde{Ɵ} * ƃ_S)(Ꭓ) = ƃ_S(\omega) \not\supseteq ƃ_S(Ꭓ)$$

$ƃ_S$ is not an S-uni Ⱡ-*ɃQ*-ideal.

Theorem 3.39 illustrates that the converse of Theorem 3.37 is valid for the bands.

Theorem 3.39. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ⱡ-*ɃQ*-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.37. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of a band. Then, $ʄ_S$ is an S-uni *Ƀ*-ideal by Theorem 3.22. Thus, $ʄ_S$ is an S-uni Ⱡ-*ɃQ*-ideal by Theorem 2.20 (vii).

Theorem 3.40. Every S-uni Ꞧ-*ɃQ*-ideal is an S-uni ŦɃ-ideal.

Proof: Let $ʄ_S$ be an S-uni Ꞧ-*ɃQ*-ideal. Then, $(ʄ_S * \widetilde{Ɵ}) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * ʄ_S) \,\widetilde{\supseteq}\, ʄ_S$. Thus,

$$\begin{aligned} ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S &= (ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S) \,\widetilde{\cup}\, (ʄ_S * ʄ_S * \widetilde{Ɵ} * ʄ_S * ʄ_S) \\ &\widetilde{\supseteq} (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ}) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ}) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} (ʄ_S * \widetilde{Ɵ}) \,\widetilde{\cup}\, (ʄ_S * \widetilde{Ɵ} * ʄ_S) \\ &\widetilde{\supseteq} ʄ_S \end{aligned}$$

Hence, $ʄ_S$ is an S-uni ŦɃ-ideal.

We demonstrate with a counterexample that the converse of Theorem 3.40 does not hold:

Example 3.41. Consider the soft set $ƃ_S$ in Example 3.21. As seen in Example 3.21, $ƃ_S$ is an S-uni ŦɃ-ideal. Since

$$[(ƃ_S * \widetilde{Ɵ}) \,\widetilde{\cup}\, (ƃ_S * \widetilde{Ɵ} * ƃ_S)](Ꭓ) = (ƃ_S * \widetilde{Ɵ})(Ꭓ) \cup (ƃ_S * \widetilde{Ɵ} * ƃ_S)(Ꭓ) = ƃ_S(\omega) \not\supseteq ƃ_S(Ꭓ)$$

$ƃ_S$ is not an S-uni Ꞧ-*ɃQ*-ideal.

Theorem 3.42 illustrates that the converse of Theorem 3.40 is valid for the bands.

Theorem 3.42. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ɽ-ɃQ-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.40. Suppose that $ʄ_S$ is an S-uni ŦɃ-ideal of a band. Then, $ʄ_S$ is an S-uni *Ƀ*-ideal by Theorem 3.22. Thus, $ʄ_S$ is an S-uni Ɽ-*ɃQ*-ideal by Theorem 2.20 (vii).

Theorem 3.43. Every S-uni *ɃQ*-ideal is an S-uni ŦɃ-ideal.

Proof: It follows Theorem 3.37 and Theorem 3.40.

Here note that the converse of Theorem 3.43 is not true follows from Example 3.38 and Example 3.41.

Theorem 3.44 illustrates that the converse of Theorem 3.43 is valid for the bands.

Theorem 3.44. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni *ɃQ*-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: (1) implies (2) is by Theorem 3.43. Assume that $ʄ_S$ is an S-uni ŦɃ-ideal of a band. The remainder of the proof is completed based on Theorem 3.39 and Theorem 3.40.

Theorem 3.45. Every S-uni Ƚ-*QI*-ideal is an S-uni ŦɃ-ideal of a left simple semigroup.

Proof: Let $đ_S$ be an S-uni Ƚ-*QI*-ideal of a left simple semigroup. Then, $\widetilde{Ɵ} * đ_S * \widetilde{Ɵ} * đ_S \,\widetilde{\supseteq}\, đ_S$ and by Theorem 2.23, $đ_S * đ_S \,\widetilde{\supseteq}\, đ_S$. Thus,

$$đ_S * đ_S * \widetilde{Ɵ} * đ_S * đ_S \,\widetilde{\supseteq}\, \widetilde{Ɵ} * đ_S * \widetilde{Ɵ} * đ_S * đ_S \,\widetilde{\supseteq}\, đ_S * đ_S \,\widetilde{\supseteq}\, đ_S$$

Hence, $đ_S$ is an S-uni ŦɃ-ideal.

Theorem 3.46. Every S-uni ŦɃ-ideal is an S-uni Ƚ-*QI*-ideal of a right zero semigroup.

Proof: Let $ʄ_S$ be an S-uni ŦɃ-ideal of a right zero S and ꟺ, ɳ, ȷ, ɼ $\in S$. By assumption, ꟺɳ = ɳ, ɳɳ = ɳ and ɼ = ɼɼ for all ꟺ, ɳ, ɼ $\in S$. Thus,

$$ʄ_S(ꟺɳȷɼ) = ʄ_S(ɳȷɼ) = ʄ_S(ɳɳȷɼɼ) \subseteq ʄ_S(ɳ) \cup ʄ_S(ɳ) \cup ʄ_S(ɼ) \cup ʄ_S(ɼ) = ʄ_S(ɳ) \cup ʄ_S(ɼ)$$

Hence, $ʄ_S$ is an S-uni Ƚ-*QI*-ideal.

Theorem 3.47. Every S-uni Ɽ-*QI*-ideal is an S-uni ŦɃ-ideal of right simple semigroup.

Proof: Let $đ_S$ be an S-uni Ɽ-*QI*-ideal of right simple S. Then, $đ_S * \widetilde{Ɵ} * đ_S * \widetilde{Ɵ} \,\widetilde{\supseteq}\, đ_S$ and by Theorem 2.23, $đ_S * đ_S \,\widetilde{\supseteq}\, đ_S$. Thus,

$$đ_S * đ_S * \widetilde{Ɵ} * đ_S * đ_S \,\widetilde{\supseteq}\, đ_S * đ_S * \widetilde{Ɵ} * đ_S * \widetilde{Ɵ} \,\widetilde{\supseteq}\, đ_S * đ_S \,\widetilde{\supseteq}\, đ_S$$

Hence, $đ_S$ is an S-uni ŦɃ-ideal.

Theorem 3.48. Every S-uni ŦɃ-ideal is an S-uni Ɽ-*QI*-ideal of a left zero semigroup.

Proof: Let $ʄ_S$ be an S-uni ŦɃ-ideal of a left zero S and ꝑ, ɍ, ɑ, ĭ $\in S$. By assumption, ɑĭ = ɑ, ɑ = ɑɑ and ꝑ = ꝑꝑ for all ɑ, ĭ, ꝑ $\in S$. Thus,

$$ʄ_S(ꝑɍɑĭ) = ʄ_S(ꝑɍɑ) = ʄ_S(ꝑꝑɍɑɑ) \subseteq ʄ_S(ꝑ) \cup ʄ_S(ꝑ) \cup ʄ_S(ɑ) \cup ʄ_S(ɑ) = ʄ_S(ꝑ) \cup ʄ_S(ɑ)$$

Hence, $ʄ_S$ is an S-uni Ɽ-*QI*-ideal.

Theorem 3.49. Every S-uni QI-ideal is an S-uni ŦɃ-ideal of a left simple or right simple semigroup.

Proof: The proof is presented only for left simple semigroup, as the proof for right simple semigroup can be shown similarly. Let $\mathfrak{f}_S$ be an S-uni QI-ideal of a left simple semigroup. Then, $\mathfrak{f}_S$ is an S-uni Ł-QI-ideal and the remainder of the proof is obvious from Theorem 3.45.

Theorem 3.50. Every S-uni ŦɃ-ideal is an S-uni QI-ideal of a zero semigroup.

Proof: Let $\mathfrak{f}_S$ be an S-uni ŦɃ-ideal of a zero S and let zero element as $0 \in S$. By assumption, $\gamma\xi = u0 = 0r = \gamma 0 = 0\xi = 0$ for all $u, \gamma, \xi, r \in S$. Since

$$\mathfrak{f}_S(u(\gamma\xi)r) = \mathfrak{f}_S\big((u0)r\big) = \mathfrak{f}_S(0r) = \mathfrak{f}_S\big((0r)r\big) = \mathfrak{f}_S\big((\gamma 0)rr\big) = \mathfrak{f}_S(\gamma(\gamma 0)rr)$$
$$\subseteq \mathfrak{f}_S(\gamma) \cup \mathfrak{f}_S(\gamma) \cup \mathfrak{f}_S(r) \cup \mathfrak{f}_S(r) = \mathfrak{f}_S(\gamma) \cup \mathfrak{f}_S(r)$$

$\mathfrak{f}_S$ is an S-uni Ł-QI-ideal and since

$$\mathfrak{f}_S(u(\gamma\xi)r) = \mathfrak{f}_S\big(u(0r)\big) = \mathfrak{f}_S(u0) = \mathfrak{f}_S\big(u(u0)\big) = \mathfrak{f}_S\big(uu(0\xi)\big) = \mathfrak{f}_S(uu(0\xi)\xi)$$
$$\subseteq \mathfrak{f}_S(u) \cup \mathfrak{f}_S(u) \cup \mathfrak{f}_S(\xi) \cup \mathfrak{f}_S(\xi) = \mathfrak{f}_S(u) \cup \mathfrak{f}_S(\xi)$$

$\mathfrak{f}_S$ is an S-uni Ꝧ-QI-ideal. Thus, $\mathfrak{f}_S$ is an S-uni QI-ideal.

Theorem 3.51. Every S-uni ɃQI-ideal is an S-uni ŦɃ-ideal of a commutative semigroup.

Proof: Let φ_S be an S-uni ɃQI-ideal of a commutative S and $\nu, u, r, \mathrm{h}, \mathrm{t} \in S$. By assumption, for all $u, r \in S$, $ur = ru$. Thus,

$$\varphi_S(\nu(ur)\mathrm{h}\mathrm{t}) = \varphi_S(\nu(ru)\mathrm{h}\mathrm{t}) \subseteq \varphi_S(\nu) \cup \varphi_S(u) \cup \varphi_S(\mathrm{t}) \subseteq \varphi_S(\nu) \cup \varphi_S(u) \cup \varphi_S(\mathrm{h}) \cup \varphi_S(\mathrm{t})$$

Hence, φ_S is an S-uni ŦɃ-ideal.

Theorem 3.52. Every S-uni ŦɃ-ideal is an S-uni ɃQI-ideal of a band.

Proof: Let $\mathfrak{f}_S$ be an S-uni ŦɃ-ideal of a band. Then, by Theorem 3.25, $\mathfrak{f}_S$ is an S-uni Q-ideal. The remainder of the proof is obvious from Theorem 2.20 (v).

Theorem 3.53. Let $\mathfrak{f}_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an S-uni Ł-Ţ-ideal.
2. $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Proof: Let (1) hold and f_S be an S-uni Ł-Ţ-ideal of a band. Then, $\mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S * \mathrm{f}_S \supseteq \mathrm{f}_S$ and by Theorem 2.23, $\mathrm{f}_S * \mathrm{f}_S \supseteq \mathrm{f}_S$. Thus,

$$\mathrm{f}_S * \mathrm{f}_S * \widetilde{\Theta} * \mathrm{f}_S * \mathrm{f}_S \supseteq \mathrm{f}_S * \mathrm{f}_S \supseteq \mathrm{f}_S$$

Hence, f_S is an S-uni ŦɃ-ideal which means that (1) implies (2).

Conversely let (2) hold and $\mathfrak{f}_S$ be an S-uni ŦɃ-ideal of a band S and ƀ, ɥ, ҳ, ɋ $\in S$. By assumption, for all ƀ $\in S$, ƀ = ƀƀ. Thus,

$$\mathfrak{f}_S(\text{ƀɥҳɋ}) = \mathfrak{f}_S(\text{ƀƀɥҳɋ}) \subseteq \mathfrak{f}_S(\text{ƀ}) \cup \mathfrak{f}_S(\text{ƀ}) \cup \mathfrak{f}_S(\text{ҳ}) \cup \mathfrak{f}_S(\text{ɋ}) = \mathfrak{f}_S(\text{ƀ}) \cup \mathfrak{f}_S(\text{ҳ}) \cup \mathfrak{f}_S(\text{ɋ})$$

Hence, $\mathfrak{f}_S$ is an S-uni Ł-Ţ-ideal, so (2) implies (1).

Theorem 3.54. Every S-uni ŦɃ-ideal is an S-uni Ł-Ţ-ideal of a right simple semigroup.

Proof: Let $\mathfrak{f}_S$ be an S-uni ŦɃ-ideal of a right simple S and ɯ, ɳ, ᵽ, ɋ $\in S$. By assumption, for all ɳ, ᵽ $\in S$, there exist ɦ $\in S$ such that ɳ = ᵽɦ. Thus,

$$\mathfrak{f}_S(\text{ɯɳᵽɋ}) = \mathfrak{f}_S(\text{ɯ(ᵽɦ)ᵽɋ}) \subseteq \mathfrak{f}_S(\text{ɯ}) \cup \mathfrak{f}_S(\text{ᵽ}) \cup \mathfrak{f}_S(\text{ᵽ}) \cup \mathfrak{f}_S(\text{ɋ}) = f_S(\text{ɯ}) \cup f_S(\text{ᵽ}) \cup f_S(\text{ɋ})$$

Hence, $ʄ_S$ is an S-uni Ⱡ-Ţ-ideal.

Theorem 3.55. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ꞧ-Ţ-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: Let (1) hold and $ʄ_S$ be an S-uni Ⱡ-Ţ-ideal of a band. Then, $ʄ_S * ʄ_S * \widetilde{\Theta} * ʄ_S \mathrel{\widetilde{\supseteq}} ʄ_S$ and by Theorem 2.23, $ʄ_S * ʄ_S \mathrel{\widetilde{\supseteq}} ʄ_S$. Thus,

$$ʄ_S * ʄ_S * \widetilde{\Theta} * ʄ_S * ʄ_S \mathrel{\widetilde{\supseteq}} ʄ_S * ʄ_S \mathrel{\widetilde{\supseteq}} ʄ_S$$

Hence, $ʄ_S$ is an S-uni ŦɃ-ideal which means that (1) implies (2).

Conversely, let (2) hold and $ʄ_S$ be an S-uni ŦɃ-ideal of a band S and $ŧ, ϼ, ɥ, ă \in S$. By assumption, for all $ă \in S$, $ă = ăă$. Thus,

$$ʄ_S(ŧϼɥă) = ʄ_S(ŧϼɥăă) \subseteq ʄ_S(ŧ) \cup ʄ_S(ϼ) \cup ʄ_S(ă) \cup ʄ_S(ă) = ʄ_S(ŧ) \cup ʄ_S(ϼ) \cup ʄ_S(ă)$$

Hence, $ʄ_S$ is an S-uni Ꞧ-Ţ-ideal, so (2) implies (1).

Theorem 3.56. Every S-uni ŦɃ-ideal is an S-uni Ꞧ-Ţ-ideal of a left simple semigroup.

Proof: Let $ʄ_S$ be an S-uni ŦɃ-ideal of a left simple S and $ɯ, ŋ, ϼ, r \in S$. By assumption, for all $ŋ, ϼ \in S$, there exist $ă \in S$ such that $ϼ = ăŋ$. Thus,

$$ʄ_S(ɯŋϼr) = ʄ_S(ɯŋ(ăŋ)r) \subseteq ʄ_S(ɯ) \cup ʄ_S(ŋ) \cup ʄ_S(ŋ) \cup ʄ_S(r) = ʄ_S(ɯ) \cup ʄ_S(ŋ) \cup ʄ_S(r)$$

Hence, $ʄ_S$ is an S-uni Ꞧ-Ţ-ideal.

Theorem 5.57. Let $ʄ_S \in S_S(U)$ and S be a band. Then, the following conditions are equivalent:

1. $ʄ_S$ is an S-uni Ţ-ideal.
2. $ʄ_S$ is an S-uni ŦɃ-ideal.

Proof: It follows by Theorem 3.53 and Theorem 3.55.

Theorem 3.58. Every S-uni ŦɃ-ideal is an S-uni Ţ-ideal of a group.

Proof: Let $ʄ_S$ be an S-uni ŦɃ-ideal of a group S. By Theorem 2.4 (ii), S is both a left simple and a right simple. The remainder of the proof is completed based on Theorem 3.54 and Theorem 3.56.

Proposition 3.59. Let $ʄ_S$ and $ʄ_T$ be S-uni ŦɃ-ideals of S and T, respectively. Then, $ʄ_S \vee ʄ_T$ is an S-uni ŦɃ-ideal of $S \times T$.

Proof: Let $(ă_1, ƚ_1), (ă_2, ƚ_2), (ă_3, ƚ_3), (ă_4, ƚ_4), (ă_5, ƚ_5) \in S \times T$. Then,

$$\begin{aligned} ʄ_{S\vee T}\big((ă_1, ƚ_1)(ă_2, ƚ_2)(ă_3, ƚ_3)(ă_4, ƚ_4)(ă_5, ƚ_5)\big) &= ʄ_{S\vee T}(ă_1ă_2ă_3ă_4ă_5, ƚ_1ƚ_2ƚ_3ƚ_4ƚ_5) \\ &= ʄ_S(ă_1ă_2ă_3ă_4ă_5) \cup ʄ_T(ƚ_1ƚ_2ƚ_3ƚ_4ƚ_5) \\ &\supseteq \big(ʄ_S(ă_1) \cup ʄ_S(ă_2) \cup ʄ_S(ă_4) \cup ʄ_S(ă_5)\big) \\ &\quad \cup \big(ʄ_T(ƚ_1) \cup ʄ_T(ƚ_2) \cup ʄ_T(ƚ_4) \cup ʄ_T(ƚ_5)\big) \\ &= \big(ʄ_S(ă_1) \cup ʄ_T(ƚ_1)\big) \cup \big(ʄ_S(ă_2) \cup ʄ_T(ƚ_2)\big) \\ &\quad \cup \big(ʄ_S(ă_4) \cup ʄ_T(ƚ_4)\big) \cup \big(ʄ_S(ă_5) \cup ʄ_T(ƚ_5)\big) \\ &= ʄ_{S\vee T}(ă_1, ƚ_1) \cup ʄ_{S\vee T}(ă_2, ƚ_2) \cup ʄ_{S\vee T}(ă_4, ƚ_4) \cup ʄ_{S\vee T}(ă_5, ƚ_5) \end{aligned}$$

Thus, $ʄ_S \vee ʄ_T$ is an S-uni ŦɃ-ideal of $S \times T$.

Proposition 3.60. Let $ʄ_S$ and $ɥ_S$ be S-uni ŦɃ-ideals. Then, $ʄ_S \mathbin{\widetilde{\cup}} ɥ_S$ is an S-uni ŦɃ-ideal.

Proof: Let $\mathfrak{f}_S$ and $\mathfrak{h}_S$ be S-uni ŦɃ-ideals. Then, $\mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S \,\widetilde{\supseteq}\, \mathfrak{f}_S$ and $\mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S \,\widetilde{\supseteq}\, \mathfrak{h}_S$. Thus,

$$(\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * \widetilde{\Theta} * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) \,\widetilde{\supseteq}\, \mathfrak{f}_S * \mathfrak{f}_S * \widetilde{\Theta} * \mathfrak{f}_S * \mathfrak{f}_S \,\widetilde{\supseteq}\, \mathfrak{f}_S$$

and

$$(\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * \widetilde{\Theta} * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) \,\widetilde{\supseteq}\, \mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S \,\widetilde{\supseteq}\, \mathfrak{h}_S$$

Hence,

$$(\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * \widetilde{\Theta} * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) * (\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S) \,\widetilde{\supseteq}\, \mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S$$

Thus, $\mathfrak{f}_S \,\widetilde{\cup}\, \mathfrak{h}_S$ is an S-uni ŦɃ-ideal.

Corollary 3.61. The finite union of S-uni ŦɃ-ideals is an S-uni ŦɃ-ideal.

Proposition 3.62. Let $\mathfrak{f}_S$ be an S-uni SS over U, $\omega \subseteq U$, $Im(\mathfrak{f}_S)$ be the image of $\mathfrak{f}_S$ such that $\omega \in Im(\mathfrak{f}_S)$. If $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal, then $\mathcal{L}(\mathfrak{f}_S;\omega)$ is a ŦɃ-ideal.

Proof: Since, $\mathfrak{f}_S(r) = \omega$ for some $r \in S$, $\emptyset \neq \mathcal{L}(\mathfrak{f}_S;\omega) \subseteq S$. Let $\eth, \flat, \mathfrak{c}, \mathrm{d} \in \mathcal{L}(\mathfrak{f}_S;\omega)$ and $\varphi \in S$. Then, $\mathfrak{f}_S(\eth) \subseteq \omega$, $\mathfrak{f}_S(\flat) \subseteq \omega$, $\mathfrak{f}_S(\mathfrak{c}) \subseteq \omega$ and $\mathfrak{f}_S(\mathrm{d}) \subseteq \omega$. It is needed to show that $\eth\flat\varphi\mathfrak{c}\mathrm{d} \in \mathcal{L}(\mathfrak{f}_S;\omega)$ for all $\eth, \flat, \mathfrak{c}, \mathrm{d} \in \mathcal{L}(\mathfrak{f}_S;\omega)$ and $\varphi \in S$. Since $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal, it follows that

$$\mathfrak{f}_S(\eth\flat\varphi\mathfrak{c}\mathrm{d}) \subseteq \mathfrak{f}_S(\eth) \cup \mathfrak{f}_S(\flat) \cup \mathfrak{f}_S(\mathfrak{c}) \cup \mathfrak{f}_S(\mathrm{d}) \subseteq \omega \cup \omega \cup \omega \cup \omega = \omega$$

implying that $\eth\flat\varphi\mathfrak{c}\mathrm{d} \in \mathcal{L}(\mathfrak{f}_S;\omega)$. Moreover, since $\mathfrak{f}_S$ is an S-uni SS over U, by Proposition 2.24, $\mathcal{L}(\mathfrak{f}_S;\omega)$ is an SS. Therefore, $\mathcal{L}(\mathfrak{f}_S;\omega)$ is a ŦɃ-ideal.

We illustrate Proposition 3.62 with Example 3.63.

Example 3.63. Consider $\mathfrak{f}_S$ in Example 3.2. As seen in Example 3.2, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal. One can easily show that $\mathfrak{f}_S$ is an S-uni SS. In fact,

$$(\mathfrak{f}_S * \mathfrak{f}_S)(\mathcal{C}) = U \supseteq \mathfrak{f}_S(\mathcal{C}) = \{\bar{1}, \bar{5}, \bar{7}\}$$

$$(\mathfrak{f}_S * \mathfrak{f}_S)(\pitchfork) = \{\bar{1}, \bar{3}\} \supseteq \mathfrak{f}_S(\pitchfork) = \{\bar{1}, \bar{3}\}$$

$$(\mathfrak{f}_S * \mathfrak{f}_S)(\mathfrak{q}) = \{\bar{1}\} \supseteq \mathfrak{f}_S(\mathfrak{q}) = \{\bar{1}\}$$

thus, $\mathfrak{f}_S$ is an S-uni SS. By considering the image set of $\mathfrak{f}_S$, that is,

$$Im(\mathfrak{f}_S) = \left\{\{\bar{1}\}\{\bar{1}, \bar{3}\}\{\bar{1}, \bar{5}, \bar{7}\}\right\}$$

we obtain the following:

$$\mathcal{L}(\mathfrak{f}_S;\omega) = \begin{cases} \{\mathfrak{q}\}, & \omega = \{\bar{1}\} \\ \{\pitchfork, \mathfrak{q}\}, & \omega = \{\bar{1}, \bar{3}\} \\ \{\mathcal{C}, \mathfrak{q}\}, & \omega = \{\bar{1}, \bar{5}, \bar{7}\} \end{cases}$$

Here, $\{\mathcal{C}, \mathfrak{q}\}$, $\{\pitchfork, \mathfrak{q}\}$ and $\{\mathfrak{q}\}$ are all ŦɃ-ideals. In fact, since

$$\{\mathcal{C}, \mathfrak{q}\} \cdot \{\mathcal{C}, \mathfrak{q}\} = \{\mathfrak{q}\} \supseteq \{\mathcal{C}, \mathfrak{q}\}$$

$$\{\pitchfork, \mathfrak{q}\} \cdot \{\pitchfork, \mathfrak{q}\} = \{\pitchfork, \mathfrak{q}\} \supseteq \{\pitchfork, \mathfrak{q}\}$$

$$\{\mathfrak{q}\} \cdot \{\mathfrak{q}\} = \{\mathfrak{q}\} \supseteq \{\mathfrak{q}\}$$

each $\mathcal{L}(\mathfrak{f}_S;\omega)$ is an SS. Similarly, since

$$\{\mathcal{C}, \mathfrak{q}\} \cdot \{\mathcal{C}, \mathfrak{q}\} \cdot S \cdot \{\mathcal{C}, \mathfrak{q}\} \cdot \{\mathcal{C}, \mathfrak{q}\} = \{\mathfrak{q}\} \supseteq \{\mathcal{C}, \mathfrak{q}\}$$

$$\{\pitchfork, \mathfrak{q}\} \cdot \{\pitchfork, \mathfrak{q}\} \cdot S \cdot \{\pitchfork, \mathfrak{q}\} \cdot \{\pitchfork, \mathfrak{q}\} = \{\pitchfork, \mathfrak{q}\} \supseteq \{\pitchfork, \mathfrak{q}\}$$

$$\{\mathfrak{q}\} \cdot \{\mathfrak{q}\} \cdot S \cdot \{\mathfrak{q}\} \cdot \{\mathfrak{q}\} = \{\mathfrak{q}\} \supseteq \{\mathfrak{q}\}$$

each $\mathcal{S}(\mathfrak{f}_S;\omega)$ is a ŦɃ-ideal.

Now, consider $\mathfrak{u}_S$ in Example 3.2. by taking into account

$$Im(\mathfrak{u}_S)=\{\{\bar{1},\bar{3},\bar{5}\},\{\bar{1},\bar{5},\bar{7}\},\{\bar{3}\}\}$$

we obtain the following:

$$\mathcal{S}(\mathfrak{u}_S;\omega)=\begin{cases}\{\pitchfork\}, & \omega=\{\bar{3}\}\\ \{ꟸ\}, & \omega=\{\bar{1},\bar{5},\bar{7}\}\\ \{\mathcal{C},\pitchfork\}, & \omega=\{\bar{1},\bar{3},\bar{5}\}\end{cases}$$

Here, $\{\mathcal{C},\pitchfork\}$ is not a ŦɃ-ideal. In fact, since

$$\{\mathcal{C},\pitchfork\}\cdot\{\mathcal{C},\pitchfork\}=\{\mathcal{C},\pitchfork,ꟸ\}\nsubseteq\{\mathcal{C},\pitchfork\}$$

one of the $\mathcal{S}(\mathfrak{u}_S;\omega)$ is not an $\mathcal{SS}$, hence it is not a ŦɃ-ideal. It is seen that each of $\mathcal{S}(\mathfrak{u}_S;\omega)$ is not a ŦɃ-ideal. On the other hand, as seen in Example 3.2, $\mathfrak{u}_S$ is not an S-uni ŦɃ-ideal.

Definition 3.64. Let $\mathfrak{f}_S$ be an S-uni $\mathcal{SS}$ and S-uni ŦɃ-ideal. Then, the ŦɃ-ideals $\mathcal{S}(\mathfrak{f}_S;\omega)$ are called lower ω-ŦɃ-ideals of $\mathfrak{f}_S$.

Proposition 3.65. Let $\mathfrak{f}_S\in S_S(U)$, $\mathcal{S}(\mathfrak{f}_S;\omega)$ be the lower ω-ŦɃ-ideal of $\mathfrak{f}_S$ for each $\omega\subseteq U$ and $Im(\mathfrak{f}_S)$ be an ordered set by inclusion. Then, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Proof: Let $\hbar,\mathrm{u},\rho,ɨ,ɱ\in S$ and $\mathfrak{f}_S(\mathrm{u})=\omega_1,\mathfrak{f}_S(\rho)=\omega_2,\mathfrak{f}_S(ɨ)=\omega_3$ and $\mathfrak{f}_S(ɱ)=\omega_4$. Suppose that $\omega_1\subseteq\omega_2\subseteq\omega_3\subseteq\omega_4$. It is obvious that $\mathrm{u}\in\mathcal{S}(\mathfrak{f}_S;\omega_1)$, $\rho\in\mathcal{S}(\mathfrak{f}_S;\omega_2)$, $ɨ\in\mathcal{S}(\mathfrak{f}_S;\omega_3)$ and $ɱ\in\mathcal{S}(\mathfrak{f}_S;\omega_4)$. Since $\omega_1\subseteq\omega_2\subseteq\omega_3\subseteq\omega_4$, $\mathrm{u},\rho,ɨ,ɱ\in\mathcal{S}(\mathfrak{f}_S;\omega_4)$ and since $\mathcal{S}(\mathfrak{f}_S;\omega)$ is a ŦɃ-ideal for all $\omega\subseteq U$, it follows that $\mathrm{u}\rho\hbarɨɱ\in\mathcal{S}(\mathfrak{f}_S;\omega_4)$. Hence, $\mathfrak{f}_S(\mathrm{u}\rho\hbarɨɱ)\subseteq\omega_4=\omega_1\cup\omega_2\cup\omega_3\cup\omega_4=\mathfrak{f}_S(\mathrm{u})\cup\mathfrak{f}_S(\rho)\cup\mathfrak{f}_S(ɨ)\cup\mathfrak{f}_S(ɱ)$. Thus, $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal.

Proposition 3.66. Let $\mathfrak{f}_S,\mathfrak{f}_T\in S_E(U)$ and $\not{V}$ be a semigroup isomorphism from S to T. If $\mathfrak{f}_S$ is an S-uni ŦɃ-ideal of S, then $\not{V}^*(\mathfrak{f}_S)$ is an S-uni ŦɃ-ideal of T.

Proof: Let $\mathrm{j}_1,\mathrm{j}_2,\mathrm{j}_3,\mathrm{j}_4,\mathrm{j}_5\in T$. Since $\not{V}$ is surjective, there exist $s_1,s_2,s_3,s_4,s_5\in S$ such that $\not{V}(s_1)=\mathrm{j}_1$, $\not{V}(s_2)=\mathrm{j}_2$, $\not{V}(s_3)=\mathrm{j}_3$, $\not{V}(s_4)=\mathrm{j}_4$ and $\not{V}(s_5)=\mathrm{j}_5$. Then,

$$\begin{aligned}\left(\not{V}^*(\mathfrak{f}_S)\right)(\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3\mathrm{j}_4\mathrm{j}_5)&=\bigcap\{\mathfrak{f}_S(s):s\in S,\not{V}(s)=\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3\mathrm{j}_4\mathrm{j}_5\}\\&=\bigcap\{\mathfrak{f}_S(s):s\in S,s=\not{V}^{-1}(\mathrm{j}_1\mathrm{j}_2\mathrm{j}_3\mathrm{j}_4\mathrm{j}_5)\}\\&=\bigcap\{\mathfrak{f}_S(s):s\in S,s=\not{V}^{-1}\left(\psi(s_1s_2s_3s_4s_5)\right)=s_1s_2s_3s_4s_5\}\\&=\bigcap\{\mathfrak{f}_S(s_1s_2s_3s_4s_5):s_i\in S,\not{V}(s_i)=\mathrm{j}_i,i=1,2,\dots,5\}\\&\subseteq\bigcap\{\mathfrak{f}_S(s_1)\cup\mathfrak{f}_S(s_2)\cup\mathfrak{f}_S(s_4)\cup\mathfrak{f}_S(s_5):s_1,s_2,s_4,s_5\in S,\not{V}(s_1)=\mathrm{j}_1,\not{V}(s_2)\\&=\mathrm{j}_2,\not{V}(s_4)=\mathrm{j}_4\ and\ \not{V}(s_5)=\mathrm{j}_5\}\\&=\left(\not{V}^*(\mathfrak{f}_S)\right)(\mathrm{j}_1)\cup\left(\not{V}^*(\mathfrak{f}_S)\right)(\mathrm{j}_2)\cup\left(\not{V}^*(\mathfrak{f}_S)\right)(\mathrm{j}_4)\cup\left(\not{V}^*(\mathfrak{f}_S)\right)(\mathrm{j}_5)\end{aligned}$$

Hence, $\not{V}^*(\mathfrak{f}_S)$ is an S-uni ŦɃ-ideal of T.

Proposition 3.67. Let $\mathfrak{f}_S,\mathfrak{f}_T\in S_E(U)$ and $\not{V}$ be a semigroup isomorphism from S to T. If $\mathfrak{f}_T$ is an S-uni ŦɃ-ideal of T, then $\not{V}^{-1}(\mathfrak{f}_T)$ is an S-uni ŦɃ-ideal of S.

Proof: Let $s_1, s_2, s_3, s_4, s_5 \in S$. Then,

$$\begin{aligned}\left(\digamma^{-1}(\mathfrak{f}_T)\right)(s_1s_2s_3s_4s_5) &= \mathfrak{f}_T\left(\digamma(s_1s_2s_3s_4s_5)\right)\\ &= \mathfrak{f}_T\left(\digamma(s_1)\digamma(s_2)\digamma(s_3)\digamma(s_4)\digamma(s_5)\right)\\ &\subseteq \mathfrak{f}_T\left(\digamma(s_1)\right) \cup \mathfrak{f}_T\left(\digamma(s_2)\right) \cup \mathfrak{f}_T\left(\digamma(s_4)\right) \cup \mathfrak{f}_T\left(\digamma(s_5)\right)\\ &= \left(\digamma^{-1}(\mathfrak{f}_T)\right)(s_1) \cup \left(\digamma^{-1}(\mathfrak{f}_T)\right)(s_2) \cup \left(\digamma^{-1}(\mathfrak{f}_T)\right)(s_4) \cup \left(\digamma^{-1}(\mathfrak{f}_T)\right)(s_5)\end{aligned}$$

Thus, $\digamma^{-1}(\mathfrak{f}_T)$ is an S-uni ꞱɃ-ideal of S.

Theorem 3.68. For S, the following conditions are equivalent:

1. S is regular.
2. $\mathfrak{h}_S = \mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S$ for every idempotent S-uni ꞱɃ-ideal $\mathfrak{h}_S$.

Proof: Let (1) hold. Let S be regular, $\mathfrak{h}_S$ be an idempotent S-uni ꞱɃ-ideal and $m \in S$. Then, $\mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S \,\widetilde{\supseteq}\, \mathfrak{h}_S$, $\mathfrak{h}_S * \mathfrak{h}_S = \mathfrak{h}_S$ and there exists an element $y \in S$ such that $m = mym$. Thus,

$$\begin{aligned}\left(\mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S\right)(m) &= \left(\mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S\right)(m)\\ &= \bigcap_{m=\eth\rho}\left\{\left(\mathfrak{h}_S * \widetilde{\Theta}\right)(\eth) \cup \mathfrak{h}_S(\rho)\right\}\\ &\subseteq \left(\mathfrak{h}_S * \widetilde{\Theta}\right)(my) \cup \mathfrak{h}_S(m)\\ &= \bigcap_{my=r k}\left\{\mathfrak{h}_S(r) \cup \widetilde{\Theta}(k)\right\} \cup \mathfrak{h}_S(m)\\ &\subseteq \mathfrak{h}_S(m) \cup \widetilde{\Theta}(y) \cup \mathfrak{h}_S(m)\\ &= \mathfrak{h}_S(m) \cup U \cup \mathfrak{h}_S(m) = \mathfrak{h}_S(m)\end{aligned}$$

Therefore, $\mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S \,\widetilde{\subseteq}\, \mathfrak{h}_S$ implying that $\mathfrak{h}_S = \mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S$.

Conversely, let (2) hold. Let $\mathfrak{h}_S * \mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S * \mathfrak{h}_S = \mathfrak{h}_S$, where $\mathfrak{h}_S$ is an S-uni ꞱɃ-ideal. To prove that S is regular, we need to show that $V = VVSVV$ for every ꞱɃ-ideal V. It is clear that $VVSVV \subseteq V$. Thus, it suffices to prove that $V \subseteq VVSVV$. On the contrary, let there exist $m \in V$ such that $m \notin VVSVV$. By Theorem 3.6, ζ_{V^c} is an S-uni ꞱɃ-ideal. Since $m \in V$, thus, $\zeta_{V^c}(m) = \emptyset$. Moreover, $m \notin VVSVV$, and this implies that there do not exist $x, y, z, t \in V$ and $a \in S$ such that $m = xyazt$. Thus,

$$\left(\zeta_{V^c} * \zeta_{V^c} * \zeta_{S^c} * \zeta_{V^c} * \zeta_{V^c}\right)(m) = \left(\zeta_{V^c} * \zeta_{V^c} * \tilde{\theta} * \zeta_{V^c} * \zeta_{V^c}\right)(m) = U$$

However, this conflicts with our hypothesis. Thus, $V \subseteq VVSVV$ and so $V = VVSVV$. Therefore, S is regular.

4. Conclusion

As a generalization of the quasi-ideal, interior ideal, and bi-ideal of a semigroup, Rao et al. [12] developed the concept of tri-quasi ideals and investigated their characteristics. In this study, we introduce "S-uni tri-bi-ideals (abbreviated by S-uni ꞱɃ-ideal throughout the text) of semigroups" to apply this approach to semigroups and soft set theory. We obtain the relations between S-uni tri-bi-ideals and other types of S-uni ideals of a semigroup. Our results show every S-uni tri-bi-ideal of a band is an S-uni subsemigroup. Moreover, S-uni tri-bi-ideal is a generalization of S-uni left (right/two-sided) ideal, interior ideal, bi-ideal, quasi-ideal, (left/right) weak-interior ideal, bi-interior ideal and (left/right) bi-quasi ideal, however, the converses are not true with counterexamples. We show that in order to satisfy the converses, the semigroup should have specific conditions, such as band, left (right) simple band or idempotent group. We also demonstrate that the S-uni left (resp. right) quasi-interior ideal of a left (resp. right) simple semigroup is an S-uni tri-bi-ideal, while the converse holds for the right (resp. left) zero semigroup. As a result, we showed that for an S-uni quasi-interior ideal to be an S-uni tri-bi-ideal, the semigroup must be left simple or right simple semigroup, and for the converse to be valid, the semigroup must be the zero. Furthermore, we have shown that the S-uni bi-quasi-interior ideal of a commutative semigroup is an S-uni tri-bi-ideal, but for the converse to hold the semigroup must be a regular idempotent. We have shown that an S-uni (left/right) tri-ideal of a band is equivalent with an S-uni tri-bi-ideal, and that an S-uni tri-bi-ideal of a right (resp. left) simple semigroup is an S-uni left (resp. right) tri-ideal. As a result, we showed that for an S-uni tri-bi-ideal of group is an S-uni tri-ideal. We also obtain a relation between tri-bi-ideal and its soft characteristic function, demonstrating how this idea connects classical semigroup theory and soft set theory. Furthermore, we present conceptual characterizations and analysis of the new concept in terms of soft set operations, soft image, and soft inverse image, supporting our assertions with illuminating examples. Further studies can be conducted on soft union ss-supplement submodules being inspired by the study of [85].

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where A→B denotes that A is B but B may not always be A.

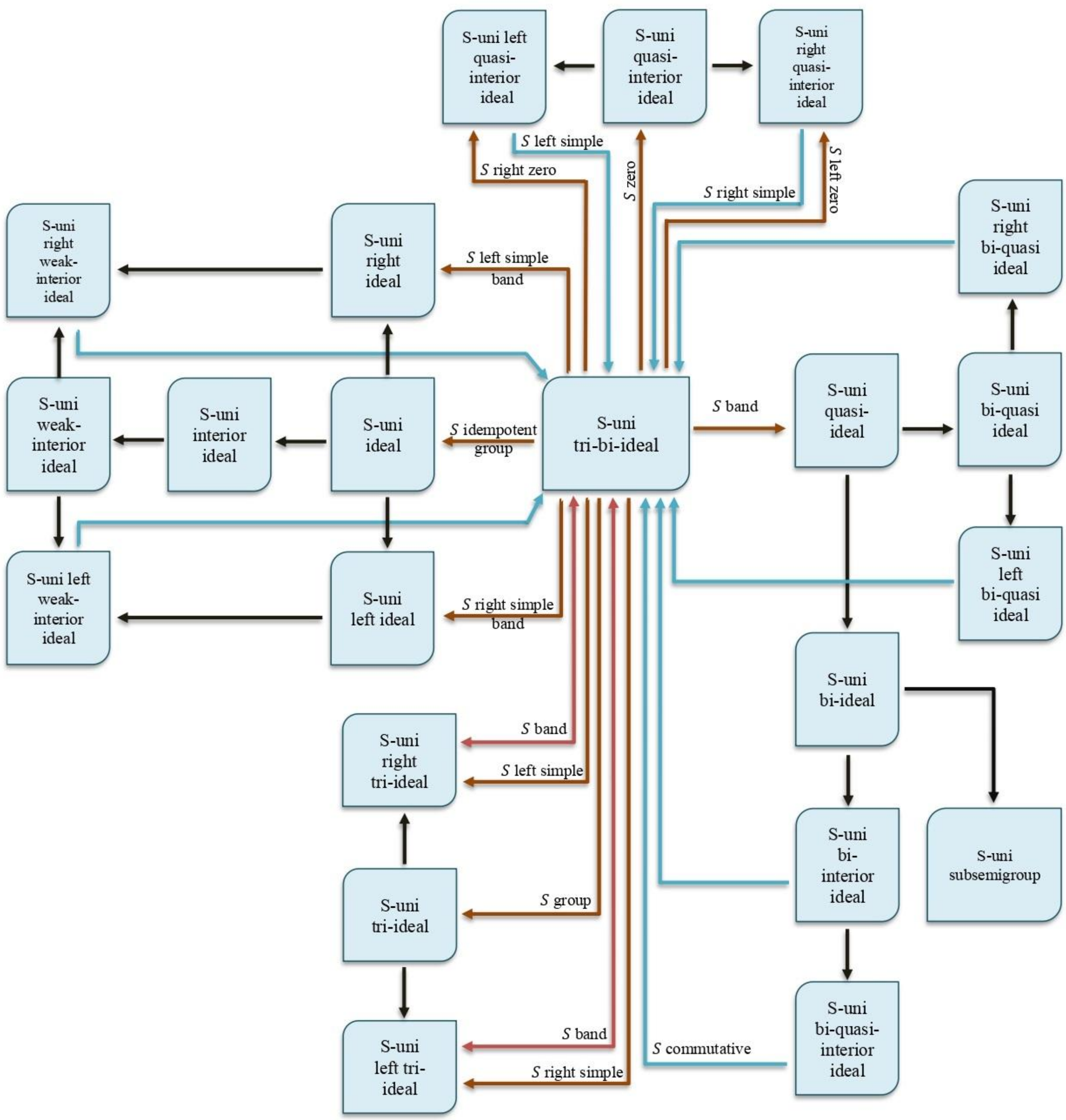

Figure 1. *Diagram showing the relationships between some S-uni ideals.*

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