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Pythagorean Fuzzy $\hat{Z}$-Subalgebras in $\hat{Z}$-Algebraic Systems

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Abstract. The idea of Pythagorean fuzzy sets was developed to improve the standard fuzzy set and intuitionistic fuzzy set theories by providing a more robust approach to handling uncertainty. This method provides greater flexibility in demonstrating membership, ensuring that the square sum of the degrees of membership and non-membership is maintained. This study examines how to incorporate Pythagorean fuzzy sets into the structure of the $\hat{Z}$-algebra, focusing on the properties of $\hat{Z}$-subalgebras. Pythagorean fuzzy $\hat{Z}$-subalgebras are a new notion that builds on traditional $\hat{Z}$-subalgebra structures. The purpose is to make algebra more creative when the conditions are uncertain. Significant developments in algebraic structures provide the theoretical basis for these fuzzy substructures. In addition, the Pythagorean fuzzy $\hat{Z}$-subalgebras under $\hat{Z}$-algebra homomorphism are examined, especially in relation to image and pre-image mappings.

1. Introduction

Fuzzy set theory, introduced by Zadeh [20], provides a flexible framework for handling uncertainty and imprecision in mathematical structures and real-world problems. Over the decades, various extensions of fuzzy sets, including intuitionistic fuzzy sets [2] and Pythagorean fuzzy sets [19], have been developed to capture more complex forms of uncertainty. Intuitionistic fuzzy sets, as introduced by Atanassov [2], extend the notion of membership by incorporating a degree of non-membership alongside the membership function, whereas Pythagorean fuzzy sets [19] further

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extend this concept by allowing the square sum of membership and non-membership degrees to be at most one, thus offering a wider range of representation for uncertainty.

In algebraic structures, fuzzy set theory has been extensively applied to study ideals and subalgebras in various systems such as BCK/BCI-algebras [3, 6], BE-algebras [1], BL-algebras [4], and β-algebras [9, 10]. The concept of fuzzy ideals generalizes classical ideal theory by assigning degrees of membership to elements, facilitating the analysis of algebraic structures under uncertainty [5, 7, 8]. Several researchers have investigated different types of fuzzy ideals, including fuzzy implicative ideals [3], fuzzy H-ideals [15], fuzzy p-ideals [16], and cubic fuzzy ideals [13], each contributing to a deeper understanding of the algebraic and operational behavior in fuzzy environments.

$\hat{Z}$-algebras, which constitute a natural generalization of classical algebraic structures, have recently become a focal point in the study of fuzzy algebra. In particular, the theory of fuzzy α-translations and β-multiplications in $\hat{Z}$-algebras has been developed to investigate the interaction between fuzzy subalgebras and fuzzy ideals under a range of algebraic operations [14, 18]. These methods have subsequently been extended to Cartesian products and homomorphisms, thereby enabling a more refined structural analysis and broadening the scope of applications in information sciences [18]. Furthermore, tripolar and picture fuzzy ideals have been introduced in the setting of BCK-algebras in order to model more intricate and multidimensional forms of uncertainty [5, 21]. More recently, in 2025, Shanmugapriya et al. [11, 12] examined the structural properties and algebraic behavior of neutrosophic and intuitionistic neutrosophic $\hat{Z}$-ideals in $\hat{Z}$-algebras, thereby establishing a unified theoretical framework for the analysis of indeterminacy phenomena within algebraic systems.

The application of fuzzy set theory to algebraic structures is not only theoretical but also practical. It finds use in decision-making [22], optimization, and soft computing [22], where classical crisp structures fail to adequately model real-world uncertainty. Recently, O-fuzzy ideals in distributive lattices [8] and Pythagorean fuzzy subalgebras in BCI-algebras [19] have expanded the scope of fuzzy algebra, allowing for more comprehensive modeling in mathematical and engineering contexts.

Despite significant progress, there remains a need to study the interactions between various types of fuzzy ideals and algebraic operations, particularly in advanced structures like $\hat{Z}$-algebras, β-algebras, and BCK/BCI-algebras. This study aims to explore such interactions by focusing on the generalization of fuzzy ideals and their applications in complex algebraic systems, thereby providing a foundation for further theoretical development and practical implementation in fuzzy mathematics.

The research article aims to analyze the structure and features of Pythagorean fuzzy $\hat{Z}$-subalgebras in $\hat{Z}$-algebraic systems. Basic definitions for the Pythagorean fuzzy $\hat{Z}$-subalgebra and illustrative developments were developed to capture the fundamentals of the theoretical developments. They also investigated how these fuzzy subalgebras responded to set-theoretic operations such as the Cartesian product, intersection, and union. In addition, the role of homomorphisms in subalgebraic structures is investigated.

This work provides a theoretical contribution to fuzzy algebraic systems while laying the groundwork for their possible use in areas such as algebraic cryptography, uncertainty modeling, and fuzzy logic controllers. The present study aims to define and investigate the notion of Pythagorean fuzzy $\hat{Z}$-subalgebras to bridge the gap between the systems of Pythagorean fuzzy sets and $\hat{Z}$-algebras.

The paper's detailed structure is shown in the sections that follow: The background, motivation, and key ideas related to the study are presented in Section 1, which is the introduction. Definitions and basic concepts about Pythagorean fuzzy sets and $\hat{Z}$-algebras are given in Section 2. Pythagorean fuzzy $\hat{Z}$-subalgebras are introduced in Section 3, along with associated theorems and examples. The behavior of such subalgebras under Cartesian products and homomorphisms is discussed in Section 4. The paper's observations, research conclusions, and possible future research directions are presented in Section 5.

2. Preliminaries

Some basic definitions and properties, necessary for the development of this article, are first reviewed.

Definition 2.1. *Let $\mathfrak{M}$ be a non-empty set. A fuzzy set ϱ in $\mathfrak{M}$ is defined as*

$$\varrho = \{(\varpi, \mu_\varrho(\varpi)) \mid \varpi \in \mathfrak{M}\},$$

where $\mu_\varrho : \mathfrak{M} \to [0,1]$ is known as the membership function *of the fuzzy set ϱ. The value $\mu_\varrho(\varpi)$ represents the degree of membership of ϖ.*

Definition 2.2. *Let $\mathfrak{M}$ be a universe of discourse. An* intuitionistic fuzzy set *ϱ in $\mathfrak{M}$ is represented as*

$$\varrho = \{(\varpi, \mu_\varrho(\varpi), \Lambda_\varrho(\varpi)) \mid \varpi \in \mathfrak{M}\},$$

where $\mu_\varrho : \mathfrak{M} \to [0,1]$ denotes the membership degree *and $\Lambda_\varrho : \mathfrak{M} \to [0,1]$ indicates the* non-membership degree *of each element $\varpi \in \mathfrak{M}$.*

These functions satisfy the condition:

$$0 \leq \mu_\varrho(\varpi) + \Lambda_\varrho(\varpi) \leq 1, \quad \forall \varpi \in \mathfrak{M}$$

Definition 2.3. *Let $\mathfrak{M}$ be a non-empty set. A* Pythagorean fuzzy set *$\mathcal{P}$ on $\mathfrak{M}$ is defined as*

$$\mathcal{P} = \{(\varpi, \mu_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\varpi)) \mid \varpi \in \mathfrak{M}\},$$

where $\mu_{\mathcal{P}}(\varpi) \in [0,1]$ represents the membership degree, *and $\Lambda_{\mathcal{P}}(\varpi) \in [0,1]$ denotes the* non-membership degree *of an element $\varpi \in \mathfrak{M}$.*

These values satisfy the condition:

$$0 \leq \mu^2_{\mathcal{P}}(\varpi) + \Lambda^2_{\mathcal{P}}(\varpi) \leq 1, \quad \forall \varpi \in \mathfrak{M}$$

It is important to note that any intuitionistic fuzzy set defined over a non-empty set $\mathfrak{M}$ naturally qualifies as a Pythagorean fuzzy set on the same set.

For simplicity, the Pythagorean fuzzy set $\mathcal{P}$ is denoted as $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$, which can equivalently be represented as

$$\mathcal{P} = \{(\varpi, \mu_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\varpi)) \mid \varpi \in \mathfrak{M}\}.$$

Let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ and $Q = (\mu_Q, \Lambda_Q)$ be two Pythagorean fuzzy sets in the non-empty set $\mathfrak{M}$. Their set-theoretic operations are defined as follows:

$$\mathcal{P} \cap Q = \{(\varpi, \mu_{\mathcal{P}}(\varpi) \cap \mu_Q(\varpi), \Lambda_{\mathcal{P}}(\varpi) \cup \Lambda_Q(\varpi)) \mid \varpi \in \mathfrak{M}\}$$

$$\mathcal{P} \cup Q = \{(\varpi, \mu_{\mathcal{P}}(\varpi) \cup \mu_Q(\varpi), \Lambda_{\mathcal{P}}(\varpi) \cap \Lambda_Q(\varpi)) \mid \varpi \in \mathfrak{M}\}$$

Definition 2.4. *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. A non-empty set $\mathfrak{M}$ with a constant 0 and a binary operation $*$ is said to be a $\hat{Z}$-algebra if it satisfies the following conditions:*

(1) $\varpi * 0 = 0$
(2) $0 * \varpi = \varpi$
(3) $\varpi * \varpi = \varpi$
(4) $\varpi * \eta = \eta * \varpi$*, whenever* $\varpi \neq 0$ *and* $\eta \neq 0$*, for all* $\varpi, \eta \in \mathfrak{M}$

Definition 2.5. *Let $\mathscr{S}$ be a non-empty subset of a $\hat{Z}$-algebra $\mathfrak{M}$. Then, $\mathscr{S}$ is said to be a $\hat{Z}$-*subalgebra *of $\mathfrak{M}$ if for every $\varpi, \eta \in \mathscr{S}$, $\varpi * \eta \in \mathscr{S}$.*

Example 2.1. *Consider the set $\mathfrak{M} = \{0, \varrho_1, \varrho_2, \varrho_3\}$ equipped with a binary operation $*$, defined by the following Cayley table:*

Table 1. Cayley's table for the binary operation $*$ on $\mathfrak{M} = \{0, \varrho_1, \varrho_2, \varrho_3\}$

$*$	0	ϱ_1	ϱ_2	ϱ_3
0	0	ϱ_1	ϱ_2	ϱ_3
ϱ_1	0	ϱ_1	ϱ_1	ϱ_1
ϱ_2	0	ϱ_1	ϱ_3	ϱ_3
ϱ_3	0	ϱ_1	ϱ_3	ϱ_3

3. Pythagorean Fuzzy $\hat{Z}$-Subalgebras

This section introduces the notion of Pythagorean fuzzy $\hat{Z}$-subalgebras in the context of $\hat{Z}$-algebras and establishes the necessary and sufficient conditions for their characterization.

Definition 3.1. *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. A Pythagorean fuzzy set $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ defined on $\mathfrak{M}$ is said to be a* Pythagorean fuzzy $\hat{Z}$-subalgebra *of $\mathfrak{M}$ if, for all $\varpi, \eta \in \mathfrak{M}$, the following conditions hold:*

$$\mu_{\mathcal{P}}(\varpi * \eta) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}$$

$$\Lambda_{\mathcal{P}}(\varpi * \eta) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}$$

Example 3.1. *Consider the $\hat{Z}$-algebra $\mathfrak{M} = \{0, \varrho_1, \varrho_2, \varrho_3\}$ with the operation $*$ given in Table 1. Define a Pythagorean fuzzy set $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ on $\mathfrak{M}$ as follows:*

$$\mu_{\mathcal{P}}(\varpi) = \begin{cases} 0.9, & \varpi = 0 \\ 0.6, & \varpi = \varrho_1 \\ 0.7, & \varpi = \varrho_2 \\ 0.8, & \varpi = \varrho_3 \end{cases} \qquad \Lambda_{\mathcal{P}}(\varpi) = \begin{cases} 0.2, & \varpi = 0 \\ 0.3, & \varpi = \varrho_1 \\ 0.4, & \varpi = \varrho_2 \\ 0.6, & \varpi = \varrho_3 \end{cases}$$

Then, $\mathcal{P}$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Thus, the calculations clearly demonstrate that $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ constitutes a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Proposition 3.1. *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set in $\mathfrak{M}$. Then, $\mathcal{P}$ constitutes a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ if and only if it satisfies the following conditions for all $\varpi, \eta \in \mathfrak{M}$:*

(1) $\mu_{\mathcal{P}}(0) \geq \mu_{\mathcal{P}}(\varpi)$ *and* $\Lambda_{\mathcal{P}}(0) \leq \Lambda_{\mathcal{P}}(\varpi)$

(2) $\mu_{\mathcal{P}}(\varpi * \eta) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}$ *and* $\Lambda_{\mathcal{P}}(\varpi * \eta) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}$

Proof. Assume that $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

For any $\varpi \in \mathfrak{M}$, we have

$$\mu_{\mathcal{P}}(0) \geq \mu_{\mathcal{P}}(\varpi * \varpi) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\varpi)\} = \mu_{\mathcal{P}}(\varpi),$$

$$\Lambda_{\mathcal{P}}(0) \leq \Lambda_{\mathcal{P}}(\varpi * \varpi) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\varpi)\} = \Lambda_{\mathcal{P}}(\varpi).$$

Hence, condition (1) is satisfied.

Furthermore, for all $\varpi, \eta \in \mathfrak{M}$, we obtain

$$\begin{aligned} \mu_{\mathcal{P}}(\varpi * (0 * \eta)) &\geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(0 * \eta)\} \\ &\geq \min\{\mu_{\mathcal{P}}(\varpi), \min\{\mu_{\mathcal{P}}(0), \mu_{\mathcal{P}}(\eta)\}\} \\ &\geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}, \end{aligned}$$

$$\begin{aligned} \Lambda_{\mathcal{P}}(\varpi * (0 * \eta)) &\leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(0 * \eta)\} \\ &\leq \max\{\Lambda_{\mathcal{P}}(\varpi), \max\{\Lambda_{\mathcal{P}}(0), \Lambda_{\mathcal{P}}(\eta)\}\} \\ &\leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}. \end{aligned}$$

Therefore, condition (2) holds.

Conversely, assume that conditions (1) and (2) are satisfied. Then, for any $\varpi, \eta \in \mathfrak{M}$,

$$\mu_{\mathcal{P}}(\varpi * \eta) = \mu_{\mathcal{P}}(\varpi * (0 * \eta)) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\},$$

$$\Lambda_{\mathcal{P}}(\varpi * \eta) = \Lambda_{\mathcal{P}}(\varpi * (0 * \eta)) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}.$$

Hence, $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ qualifies as a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

Proposition 3.2. *Let $(\mathfrak{M}, *, 0)$ be a $\hat{Z}$-algebra. If $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ and $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ are two Pythagorean fuzzy $\hat{Z}$-subalgebras of $\mathfrak{M}$, then their intersection*

$$(\mu_{\mathcal{P}} \cap \mu_{\mathcal{Q}}, \Lambda_{\mathcal{P}} \cup \Lambda_{\mathcal{Q}})$$

is also a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Proof. Consider arbitrary elements $\varpi, \eta \in \mathfrak{M}$.

For the membership function, we have

$$\begin{aligned}(\mu_{\mathcal{P}} \cap \mu_{\mathcal{Q}})(\varpi * \eta) &= \min\{\mu_{\mathcal{P}}(\varpi * \eta), \mu_{\mathcal{Q}}(\varpi * \eta)\} \\ &\geq \min\{\min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}, \min\{\mu_{\mathcal{Q}}(\varpi), \mu_{\mathcal{Q}}(\eta)\}\} \\ &= \min\{\min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{Q}}(\varpi)\}, \min\{\mu_{\mathcal{P}}(\eta), \mu_{\mathcal{Q}}(\eta)\}\} \\ &= \min\{(\mu_{\mathcal{P}} \cap \mu_{\mathcal{Q}})(\varpi), (\mu_{\mathcal{P}} \cap \mu_{\mathcal{Q}})(\eta)\}.\end{aligned}$$

Similarly, for the non-membership function, we have

$$\begin{aligned}(\Lambda_{\mathcal{P}} \cup \Lambda_{\mathcal{Q}})(\varpi * \eta) &= \max\{\Lambda_{\mathcal{P}}(\varpi * \eta), \Lambda_{\mathcal{Q}}(\varpi * \eta)\} \\ &\leq \max\{\max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}, \max\{\Lambda_{\mathcal{Q}}(\varpi), \Lambda_{\mathcal{Q}}(\eta)\}\} \\ &= \max\{\max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{Q}}(\varpi)\}, \max\{\Lambda_{\mathcal{P}}(\eta), \Lambda_{\mathcal{Q}}(\eta)\}\} \\ &= \max\{(\Lambda_{\mathcal{P}} \cup \Lambda_{\mathcal{Q}})(\varpi), (\Lambda_{\mathcal{P}} \cup \Lambda_{\mathcal{Q}})(\eta)\}.\end{aligned}$$

Therefore, the intersection $(\mu_{\mathcal{P}} \cap \mu_{\mathcal{Q}}, \Lambda_{\mathcal{P}} \cup \Lambda_{\mathcal{Q}})$ satisfies the conditions for being a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

However, the union of two or more Pythagorean fuzzy $\hat{Z}$-subalgebras of $\mathfrak{M}$ may not necessarily form a Pythagorean fuzzy $\hat{Z}$-subalgebra, as illustrated in the following example.

Example 3.2. *Consider the set $\mathfrak{M} = \{0, \varrho_1, \varrho_2, \varrho_3\}$ and define the binary operation $*$ on $\mathfrak{M}$ using the following Cayley's table:*

TABLE 2. Cayley's table for the binary operation $*$ on $\mathfrak{M}$

$*$	0	ϱ_1	ϱ_2	ϱ_3
0	0	ϱ_1	ϱ_2	ϱ_3
ϱ_1	0	ϱ_1	ϱ_1	ϱ_2
ϱ_2	0	ϱ_1	ϱ_2	ϱ_2
ϱ_3	0	ϱ_1	ϱ_3	ϱ_3

Define two Pythagorean fuzzy sets $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ and $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ over $\mathfrak{M}$ as follows:

Through straightforward verification, it can be observed that $\mathcal{P}$ and $\mathcal{Q}$ satisfy the conditions to be Pythagorean fuzzy $\hat{Z}$-subalgebras of $\mathfrak{M}$. Now, consider the union $\mathcal{P} \cup \mathcal{Q} = (\mu_{\mathcal{P} \cup \mathcal{Q}}, \Lambda_{\mathcal{P} \cup \mathcal{Q}})$, where

$$\mu_{\mathcal{P} \cup \mathcal{Q}}(\varpi) = \max\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{Q}}(\varpi)\},$$

$$\Lambda_{\mathcal{P} \cup \mathcal{Q}}(\varpi) = \min\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{Q}}(\varpi)\}.$$

Table 3. Pythagorean fuzzy set $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ over $\mathfrak{M}$

$\mathcal{P}$	0	ϱ_1	ϱ_2	ϱ_3
$\mu_{\mathcal{P}}$	0.9	0.8	0.7	0.5
$\Lambda_{\mathcal{P}}$	0.2	0.4	0.4	0.6

Table 4. Pythagorean fuzzy set $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ over $\mathfrak{M}$

$\mathcal{Q}$	0	ϱ_1	ϱ_2	ϱ_3
$\mu_{\mathcal{Q}}$	0.9	0.6	0.8	0.7
$\Lambda_{\mathcal{Q}}$	0.1	0.4	0.2	0.3

Table 5. Pythagorean fuzzy set $\mathcal{P} \cup \mathcal{Q} = (\mu_{\mathcal{P}\cup\mathcal{Q}}, \Lambda_{\mathcal{P}\cup\mathcal{Q}})$

$\mathcal{P} \cup \mathcal{Q}$	0	ϱ_1	ϱ_2	ϱ_3
$\mu_{\mathcal{P}\cup\mathcal{Q}}$	0.9	0.8	0.8	0.7
$\Lambda_{\mathcal{P}\cup\mathcal{Q}}$	0.1	0.4	0.2	0.3

Therefore, $\mathcal{P} \cup \mathcal{Q}$ does not form a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Theorem 3.1. *Let $\mathfrak{M}$ be a $\hat{Z}$-algebra, and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ denote a Pythagorean fuzzy set defined on $\mathfrak{M}$. Then $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ if and only if, for every $t \in [0,1]$, the non-empty level sets $U(\mu_{\mathcal{P}}, t) = \{\varpi \in \mathfrak{M} \mid \mu_{\mathcal{P}}(\varpi) \geq t\}$ and $L(\Lambda_{\mathcal{P}}, t) = \{\varpi \in \mathfrak{M} \mid \Lambda_{\mathcal{P}}(\varpi) \geq t\}$ are $\hat{Z}$-subalgebras of $\mathfrak{M}$.*

Proof. Suppose $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ represents a Pythagorean fuzzy $\hat{Z}$-subalgebra of the $\hat{Z}$-algebra $\mathfrak{M}$.

Let $s, t \in [0,1]$ such that $U(\mu_{\mathcal{P}}, t) \neq \emptyset$.Choose any $\varpi, \eta \in U(\mu_{\mathcal{P}}, t)$.Then, by definition, $\mu_{\mathcal{P}}(\varpi) \geq t$ and $\mu_{\mathcal{P}}(\eta) \geq t$. Since $\mathcal{P}$ is a fuzzy $\hat{Z}$-subalgebra, it follows that $\mu_{\mathcal{P}}(\varpi * \eta) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\} \geq t$, which implies $\varpi * \eta \in U(\mu_{\mathcal{P}}, t)$. Hence, $U(\mu_{\mathcal{P}}, t)$ is closed under $*$, and thus forms a $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Similarly, for any $\varpi, \eta \in L(\Lambda_{\mathcal{P}}, s)$, we have $\Lambda_{\mathcal{P}}(\varpi) \leq s$ and $\Lambda_{\mathcal{P}}(\eta) \leq s$. Since $\mathcal{P}$ satisfies the $\hat{Z}$-subalgebra condition, it follows that $\Lambda_{\mathcal{P}}(\varpi * \eta) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\} \leq s$, so $\varpi * \eta \in L(\Lambda_{\mathcal{P}}, s)$ and hence $L(\Lambda_{\mathcal{P}}, s)$ is also a $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Conversely, assume that for every $s, t \in [0,1]$, the non-empty level sets $U(\mu_{\mathcal{P}}, t)$ and $L(\Lambda_{\mathcal{P}}, s)$ form $\hat{Z}$-subalgebras of $\mathfrak{M}$.

Suppose, for a contradiction, that there exist elements $\alpha, \beta \in \mathfrak{M}$ such that $\mu_{\mathcal{P}}(\alpha * \beta) < \min\{\mu_{\mathcal{P}}(\alpha), \mu_{\mathcal{P}}(\beta)\}$. Let $t = \frac{1}{2}\left(\mu_{\mathcal{P}}(\alpha * \beta) + \min\{\mu_{\mathcal{P}}(\alpha), \mu_{\mathcal{P}}(\beta)\}\right)$. Then $\alpha, \beta \in U(\mu_{\mathcal{P}}, t)$, contradicting the assumption that $U(\mu_{\mathcal{P}}, t)$ is a $\hat{Z}$-subalgebra of $\mathfrak{M}$. Therefore, we have

$$\mu_{\mathcal{P}}(\varpi * \eta) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}, \quad \forall \varpi, \eta \in \mathfrak{M}.$$

Similarly, the same argument applies to the non-membership function $\Lambda_{\mathcal{P}}$, giving

$$\Lambda_{\mathcal{P}}(\varpi * \eta) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}, \quad \forall \varpi, \eta \in \mathfrak{M}.$$

Hence, $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

Theorem 3.2. *Let $\mathfrak{M}$ be a $\hat{Z}$-algebra. Then $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ if and only if the complement $\mathcal{P}^c = (\mu^c_{\mathcal{P}}, \Lambda^c_{\mathcal{P}})$ is also a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.*

Proof. Suppose $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. Let $\varpi, \eta \in \mathfrak{M}$. Then

$$\begin{aligned}\Lambda^c_{\mathcal{P}}(\varpi * \eta) &= 1 - \Lambda_{\mathcal{P}}(\varpi * \eta)\\ &\leq 1 - \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}\\ &= \min\{1 - \Lambda_{\mathcal{P}}(\varpi), 1 - \Lambda_{\mathcal{P}}(\eta)\}\\ &= \min\{\Lambda^c_{\mathcal{P}}(\varpi), \Lambda^c_{\mathcal{P}}(\eta)\},\end{aligned}$$

$$\begin{aligned}\mu^c_{\mathcal{P}}(\varpi * \eta) &= 1 - \mu_{\mathcal{P}}(\varpi * \eta)\\ &\geq 1 - \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\}\\ &= \max\{1 - \mu_{\mathcal{P}}(\varpi), 1 - \mu_{\mathcal{P}}(\eta)\}\\ &= \max\{\mu^c_{\mathcal{P}}(\varpi), \mu^c_{\mathcal{P}}(\eta)\}.\end{aligned}$$

Hence, the complement $\mathcal{P}^c = (\Lambda^c_{\mathcal{P}}, \mu^c_{\mathcal{P}})$ also satisfies the condition of a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Conversely, suppose the complement $\mathcal{P}^c = (\Lambda^c_{\mathcal{P}}, \mu^c_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. Let $\varpi, \eta \in \mathfrak{M}$. Then

$$\begin{aligned}1 - \mu_{\mathcal{P}}(\varpi * \eta) &= \mu^c_{\mathcal{P}}(\varpi * \eta)\\ &\leq \max\{\mu^c_{\mathcal{P}}(\varpi), \mu^c_{\mathcal{P}}(\eta)\}\\ &= \max\{1 - \mu_{\mathcal{P}}(\varpi), 1 - \mu_{\mathcal{P}}(\eta)\}\\ &= 1 - \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\},\end{aligned}$$

$$\begin{aligned}1 - \Lambda_{\mathcal{P}}(\varpi * \eta) &= \Lambda^c_{\mathcal{P}}(\varpi * \eta)\\ &\geq \min\{\Lambda^c_{\mathcal{P}}(\varpi), \Lambda^c_{\mathcal{P}}(\eta)\}\\ &= \min\{1 - \Lambda_{\mathcal{P}}(\varpi), 1 - \Lambda_{\mathcal{P}}(\eta)\}\\ &= 1 - \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}.\end{aligned}$$

Consequently,

$$\mu_{\mathcal{P}}(\varpi * \eta) \geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\},$$
$$\Lambda_{\mathcal{P}}(\varpi * \eta) \leq \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}.$$

Hence, $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ constitutes a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

This subsequent theorem is derived directly from Theorem 3.2.

Theorem 3.3. *Let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set defined on a $\hat{Z}$-algebra $\mathfrak{M}$. Then $\mathcal{P}$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ if and only if the sets $\mathcal{P}_\mu = (\mu_{\mathcal{P}}, \mu^c_{\mathcal{P}}), \mathcal{P}_\Lambda = (\Lambda_{\mathcal{P}}, \Lambda^c_{\mathcal{P}})$ are also Pythagorean fuzzy $\hat{Z}$-subalgebras of $\mathfrak{M}$.*

Proof. Consider a $\hat{Z}$-algebra $\mathfrak{M}$ and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be any Pythagorean fuzzy set in $\mathfrak{M}$. Define the following subsets:

$$\mathfrak{M}_{\mu_{\mathcal{P}}} = \{\varpi \in \mathfrak{M} \mid \mu_{\mathcal{P}}(\varpi) = \mu_{\mathcal{P}}(0)\}$$
$$\mathfrak{M}_{\Lambda_{\mathcal{P}}} = \{\varpi \in \mathfrak{M} \mid \Lambda_{\mathcal{P}}(\varpi) = \Lambda_{\mathcal{P}}(0)\}$$

□

Theorem 3.4. *Given a $\hat{Z}$-algebra $\mathfrak{M}$, if $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ constitutes a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$, then both the sets $\mathfrak{M}_{\mu_{\mathcal{P}}}$ and $\mathfrak{M}_{\Lambda_{\mathcal{P}}}$ form $\hat{Z}$-subalgebras of $\mathfrak{M}$.*

Proof. Let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. Suppose $\varpi, \eta \in \mathfrak{M}_{\mu_{\mathcal{P}}}$. Then $\mu_{\mathcal{P}}(\varpi) = \mu_{\mathcal{P}}(0) = \mu_{\mathcal{P}}(\eta)$. Hence,

$$\begin{aligned} \mu_{\mathcal{P}}(\varpi * \eta) &\geq \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\} \\ &= \mu_{\mathcal{P}}(0). \end{aligned}$$

Otherwise, it would imply $\mu_{\mathcal{P}}(0) > \mu_{\mathcal{P}}(\varpi * \eta)$, which contradicts the assumption. Thus,

$$\mu_{\mathcal{P}}(\varpi * \eta) = \mu_{\mathcal{P}}(0) \implies \varpi * \eta \in \mathfrak{M}_{\mu_{\mathcal{P}}}.$$

Therefore, $\mathfrak{M}_{\mu_{\mathcal{P}}}$ forms a $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Similarly, let $\alpha, \beta \in \mathfrak{M}_{\Lambda_{\mathcal{P}}}$. Then

$$\Lambda_{\mathcal{P}}(\alpha) = \Lambda_{\mathcal{P}}(0) = \Lambda_{\mathcal{P}}(\beta).$$

So,

$$\begin{aligned} \Lambda_{\mathcal{P}}(\alpha * \beta) &\leq \max\{\Lambda_{\mathcal{P}}(\alpha), \Lambda_{\mathcal{P}}(\beta)\} \\ &= \Lambda_{\mathcal{P}}(0). \end{aligned}$$

Since $\Lambda_{\mathcal{P}}(\alpha * \beta) \geq \Lambda_{\mathcal{P}}(0)$, it follows that

$$\Lambda_{\mathcal{P}}(\alpha * \beta) = \Lambda_{\mathcal{P}}(0) \implies \alpha * \beta \in \mathfrak{M}_{\Lambda_{\mathcal{P}}}.$$

Hence, $\mathfrak{M}_{\Lambda_{\mathcal{P}}}$ is also a $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

4. Homomorphism on Pythagorean Fuzzy $\hat{Z}$-Subalgebras

The following part discusses the structural properties of Pythagorean fuzzy $\hat{Z}$-subalgebras in the context of homomorphism.

Definition 4.1 (Homomorphism)**.** *Let $f : \mathfrak{M} \to \mathfrak{Y}$ be a mapping between two $\hat{Z}$-algebras. The function f is called a* homomorphism *if it satisfies*

$$f(\varpi * \eta) = f(\varpi) * f(\eta), \quad \forall \varpi, \eta \in \mathfrak{M}.$$

If the homomorphism f is onto, then it is referred to as an epimorphism*. Furthermore, for any homomorphism $f : \mathfrak{M} \to \mathfrak{Y}$, it follows that*

$$f(0) = 0.$$

Definition 4.2 (Image of a Pythagorean Fuzzy Set)**.** *Let $\mathfrak{M}$ and $\mathfrak{Y}$ be two non-empty sets and let $f : \mathfrak{M} \to \mathfrak{Y}$ be a function. Suppose $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy set on $\mathfrak{M}$. Then the image of $\mathcal{P}$ under f, denoted by $f(\mathcal{P})$, is the Pythagorean fuzzy set on $\mathfrak{Y}$ defined as*

$$f(\mathcal{P}) = \{(\eta, \mu_{f(\mathcal{P})}(\eta), \Lambda_{f(\mathcal{P})}(\eta)) \mid \eta \in \mathfrak{Y}\},$$

where

$$\mu_{f(\mathcal{P})}(\eta) = \begin{cases} \sup\limits_{\varpi \in f^{-1}(\eta)} \mu_{\mathcal{P}}(\varpi), & \text{if } f^{-1}(\eta) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Lambda_{f(\mathcal{P})}(\eta) = \begin{cases} \inf\limits_{\varpi \in f^{-1}(\eta)} \Lambda_{\mathcal{P}}(\varpi), & \text{if } f^{-1}(\eta) \neq \emptyset, \\ 1, & \text{otherwise}. \end{cases}$$

This set $f(\mathcal{P})$ is called the Pythagorean fuzzy image of $\mathcal{P}$ under the mapping f.

On the other hand, let $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ be a Pythagorean fuzzy set defined on $f(\mathfrak{M})$. Then the pre-image of $\mathcal{Q}$ under f, denoted by $f^{-1}(\mathcal{Q})$, is defined as

$$f^{-1}(\mathcal{Q}) = \{(\varpi, \mu_{f^{-1}(\mathcal{Q})}(\varpi), \Lambda_{f^{-1}(\mathcal{Q})}(\varpi)) \mid \varpi \in \mathfrak{M}\}.$$

Now, let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set on a non-empty set $\mathfrak{M}$. Then $\mathcal{P}$ satisfies the sup-inf property if, for every non-empty subset $T \subseteq \mathfrak{M}$, there exists an element $\alpha_0 \in T$ such that

$$\mu_{\mathcal{P}}(\alpha_0) = \sup_{t \in T} \mu_{\mathcal{P}}(t), \quad \Lambda_{\mathcal{P}}(\alpha_0) = \inf_{t \in T} \Lambda_{\mathcal{P}}(t).$$

If $f : \mathfrak{M} \to \mathfrak{Y}$ is a mapping between $\hat{Z}$-algebras and $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy set on $\mathfrak{Y}$, then the Pythagorean fuzzy inverse image of $\mathcal{P}$ under f, denoted by $\mathcal{P}^f = (\mu^f_{\mathcal{P}}, \Lambda^f_{\mathcal{P}})$, is defined on $\mathfrak{M}$ by

$$\mu^f_{\mathcal{P}}(\varpi) = \mu_{\mathcal{P}}(f(\varpi)), \quad \Lambda^f_{\mathcal{P}}(\varpi) = \Lambda_{\mathcal{P}}(f(\varpi)), \quad \forall \varpi \in \mathfrak{M}.$$

Theorem 4.1. *Let $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ be a Pythagorean fuzzy $\hat{Z}$-subalgebra of $f(\mathfrak{M})$. Then the pre-image of $\mathcal{Q}$ under the mapping f, denoted by $f^{-1}(\mathcal{Q}) = \{(\varpi, \mu_{f^{-1}(\mathcal{Q})}(\varpi), \Lambda_{f^{-1}(\mathcal{Q})}(\varpi)) \mid \varpi \in \mathfrak{M}\}$, also forms a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.*

Proof. Assume that $\mathcal{Q} = (\mu_{\mathcal{Q}}, \Lambda_{\mathcal{Q}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra defined on $f(\mathfrak{M})$.

Let $\varpi, \eta \in \mathfrak{M}$. Then

$$\begin{aligned}\mu_{f^{-1}(Q)}(\varpi * \eta) &= \mu_Q(f(\varpi * \eta))\\ &= \mu_Q(f(\varpi) * f(\eta))\\ &\geq \min\{\mu_Q(f(\varpi)), \mu_Q(f(\eta))\}\\ &= \min\{\mu_{f^{-1}(Q)}(\varpi), \mu_{f^{-1}(Q)}(\eta)\},\end{aligned}$$

$$\begin{aligned}\Lambda_{f^{-1}(Q)}(\varpi * \eta) &= \Lambda_Q(f(\varpi * \eta))\\ &= \Lambda_Q(f(\varpi) * f(\eta))\\ &\leq \max\{\Lambda_Q(f(\varpi)), \Lambda_Q(f(\eta))\}\\ &= \max\{\Lambda_{f^{-1}(Q)}(\varpi), \Lambda_{f^{-1}(Q)}(\eta)\}.\end{aligned}$$

Hence, $f^{-1}(Q) = (\mu_{f^{-1}(Q)}, \Lambda_{f^{-1}(Q)})$ satisfies the necessary conditions and thus forms a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

Theorem 4.2. *Let $f : \mathfrak{M} \to \mathfrak{Y}$ be a homomorphism between $\hat{Z}$-algebras, and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set on $\mathfrak{M}$. Denote the image of $\mathcal{P}$ under f by $f(\mathcal{P}) = (\mu_{f(\mathcal{P})}, \Lambda_{f(\mathcal{P})})$. If $\mathcal{P}$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ and satisfies the sup-inf property, then $f(\mathcal{P})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $f(\mathfrak{M})$.*

Proof. Suppose $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$ with the sup-inf property.

For any $\varpi, \eta \in f(\mathfrak{M})$, there exist elements $\alpha_0 \in f^{-1}(\varpi)$ and $\beta_0 \in f^{-1}(\eta)$ such that

$$\mu_{\mathcal{P}}(\alpha_0) = \sup_{t \in f^{-1}(\varpi)} \mu_{\mathcal{P}}(t), \qquad \Lambda_{\mathcal{P}}(\alpha_0) = \inf_{t \in f^{-1}(\varpi)} \Lambda_{\mathcal{P}}(t),$$

$$\mu_{\mathcal{P}}(\beta_0) = \sup_{t \in f^{-1}(\eta)} \mu_{\mathcal{P}}(t), \qquad \Lambda_{\mathcal{P}}(\beta_0) = \inf_{t \in f^{-1}(\eta)} \Lambda_{\mathcal{P}}(t).$$

Now consider,

$$\begin{aligned}\mu_{f(\mathcal{P})}(\varpi * \eta) &= \sup_{t \in f^{-1}(\varpi * \eta)} \mu_{\mathcal{P}}(t)\\ &\geq \mu_{\mathcal{P}}(\alpha_0 * \beta_0)\\ &\geq \min\{\mu_{\mathcal{P}}(\alpha_0), \mu_{\mathcal{P}}(\beta_0)\}\\ &= \min\left\{\sup_{t \in f^{-1}(\varpi)} \mu_{\mathcal{P}}(t), \sup_{t \in f^{-1}(\eta)} \mu_{\mathcal{P}}(t)\right\}\\ &= \min\{\mu_{f(\mathcal{P})}(\varpi), \mu_{f(\mathcal{P})}(\eta)\},\end{aligned}$$

$$\begin{aligned}\Lambda_{f(\mathcal{P})}(\varpi * \eta) &= \inf_{t \in f^{-1}(\varpi * \eta)} \Lambda_{\mathcal{P}}(t)\\ &\leq \Lambda_{\mathcal{P}}(\alpha_0 * \beta_0)\\ &\leq \max\{\Lambda_{\mathcal{P}}(\alpha_0), \Lambda_{\mathcal{P}}(\beta_0)\}\end{aligned}$$

$$= \max\left\{\sup_{t\in f^{-1}(\varpi)} \Lambda_{\mathcal{P}}(t), \sup_{t\in f^{-1}(\eta)} \Lambda_{\mathcal{P}}(t)\right\}$$
$$= \max\{\Lambda_{f(\mathcal{P})}(\varpi), \Lambda_{f(\mathcal{P})}(\eta)\}.$$

Hence, $f(\mathcal{P}) = (\mu_{f(\mathcal{P})}, \Lambda_{f(\mathcal{P})})$ satisfies the necessary conditions and is thus a Pythagorean fuzzy $\hat{Z}$-subalgebra of $f(\mathfrak{M})$. □

Theorem 4.3. *Let $f : \mathfrak{M} \to \mathfrak{Y}$ be a homomorphism between $\hat{Z}$-algebras, and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set on $\mathfrak{Y}$. If $\mathcal{P}$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{Y}$, then its inverse image $\mathcal{P}^f = (\mu^f_{\mathcal{P}}, \Lambda^f_{\mathcal{P}})$ defines a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.*

Proof. Suppose $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{Y}$.

Let $\varpi, \eta \in \mathfrak{M}$. Then

$$\begin{aligned}
\mu^f_{\mathcal{P}}(\varpi * \eta) &= \mu_{\mathcal{P}}(f(\varpi * \eta))\\
&= \mu_{\mathcal{P}}(f(\varpi) * f(\eta))\\
&\geq \min\{\mu_{\mathcal{P}}(f(\varpi)), \mu_{\mathcal{P}}(f(\eta))\}\\
&= \min\{\mu^f_{\mathcal{P}}(\varpi), \mu^f_{\mathcal{P}}(\eta)\},
\end{aligned}$$

$$\begin{aligned}
\Lambda^f_{\mathcal{P}}(\varpi * \eta) &= \Lambda_{\mathcal{P}}(f(\varpi * \eta))\\
&= \Lambda_{\mathcal{P}}(f(\varpi) * f(\eta))\\
&\leq \max\{\Lambda_{\mathcal{P}}(f(\varpi)), \Lambda_{\mathcal{P}}(f(\eta))\}\\
&= \max\{\Lambda^f_{\mathcal{P}}(\varpi), \Lambda^f_{\mathcal{P}}(\eta)\}.
\end{aligned}$$

Hence, $\mathcal{P}^f = (\mu^f_{\mathcal{P}}, \Lambda^f_{\mathcal{P}})$ forms a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$. □

Theorem 4.4. *Let $f : \mathfrak{M} \to \mathfrak{Y}$ be an epimorphism between $\hat{Z}$-algebras, and let $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ be a Pythagorean fuzzy set on $\mathfrak{Y}$. If $\mathcal{P}^f = (\mu^f_{\mathcal{P}}, \Lambda^f_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$, then $\mathcal{P}$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{Y}$.*

Proof. Suppose $\mathcal{P}^f = (\mu^f_{\mathcal{P}}, \Lambda^f_{\mathcal{P}})$ is a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{M}$.

Let $\varpi, \eta \in \mathfrak{Y}$. Since f is an epimorphism, there exist elements $\alpha, \beta \in \mathfrak{M}$ such that $f(\alpha) = \varpi$ and $f(\beta) = \eta$. Then

$$\begin{aligned}
\mu_{\mathcal{P}}(\varpi * \eta) &= \mu_{\mathcal{P}}(f(\alpha) * f(\beta))\\
&= \mu_{\mathcal{P}}(f(\alpha * \beta))\\
&= \mu^f_{\mathcal{P}}(\alpha * \beta)\\
&\geq \min\{\mu^f_{\mathcal{P}}(\alpha), \mu^f_{\mathcal{P}}(\beta)\}\\
&= \min\{\mu_{\mathcal{P}}(f(\alpha)), \mu_{\mathcal{P}}(f(\beta))\}\\
&= \min\{\mu_{\mathcal{P}}(\varpi), \mu_{\mathcal{P}}(\eta)\},
\end{aligned}$$

$$\begin{aligned}\Lambda_{\mathcal{P}}(\varpi * \eta) &= \Lambda_{\mathcal{P}}(f(\alpha) * f(\beta)) \\ &= \Lambda_{\mathcal{P}}(f(\alpha * \beta)) \\ &= \Lambda^{f}_{\mathcal{P}}(\alpha * \beta) \\ &\leq \max\{\Lambda^{f}_{\mathcal{P}}(\alpha), \Lambda^{f}_{\mathcal{P}}(\beta)\} \\ &= \max\{\Lambda_{\mathcal{P}}(f(\alpha)), \Lambda_{\mathcal{P}}(f(\beta))\} \\ &= \max\{\Lambda_{\mathcal{P}}(\varpi), \Lambda_{\mathcal{P}}(\eta)\}.\end{aligned}$$

Hence, $\mathcal{P} = (\mu_{\mathcal{P}}, \Lambda_{\mathcal{P}})$ satisfies the condition of a Pythagorean fuzzy $\hat{Z}$-subalgebra of $\mathfrak{Y}$. □

5. Conclusion

With the goal of extending traditional algebraic concepts into the fuzzy and uncertain domain, this study investigated the notion of Pythagorean fuzzy $\hat{Z}$-subalgebras within the context of $\hat{Z}$-algebraic systems. We formulated the conditions under which a fuzzy set forms a $\hat{Z}$-subalgebra by utilizing Pythagorean fuzzy logic. Additionally, the structures were examined under various operations, including intersection, union, and homomorphism. A number of illustrative examples were provided to demonstrate that, although Pythagorean fuzzy $\hat{Z}$-subalgebras individually satisfy the necessary properties, their combinations might not always preserve subalgebraic structure. Furthermore, the images and pre-images of fuzzy $\hat{Z}$-subalgebras under homomorphisms were analyzed, providing the mathematical basis for structural preservation under mappings.

There are several promising directions for future research on Pythagorean fuzzy $\hat{Z}$-subalgebras. One potential extension is to generalize the concept to more complex algebraic structures, such as $\hat{Z}$-rings, $\hat{Z}$-lattices, or near-$\hat{Z}$-algebras. Further investigation into the homological and categorical properties of fuzzy $\hat{Z}$-algebras under morphisms is also warranted. Analyzing the topological aspects of fuzzy $\hat{Z}$-structures, which allows for the study of continuity and compactness in fuzzy algebraic systems, is another interesting direction.

Practical applications can be explored in domains such as expert systems, fuzzy control systems, and decision-making frameworks. Moreover, a more comprehensive theory may arise from combining these ideas with advanced fuzzy models, such as intuitionistic or neutrosophic fuzzy sets. Comparative studies between different fuzzy substructures may also yield new theoretical insights.

Finally, understanding the behavior of fuzzy subalgebras under partial information, uncertainty, and inconsistency remains a significant challenge. Incorporating dynamic or time-dependent fuzzy values into $\hat{Z}$-algebraic systems could provide more accurate modeling of real-world variability, potentially enhancing both theoretical understanding and practical applicability.

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