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Codimension-One Bifurcation Analysis and Chaos of a Discrete Competition Duopoly Game

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Abstract. This paper examines the temporal evolution of a competitive duopoly game model, emphasizing the identification of significant bifurcation points that influence the system's dynamics throughout time. The analysis centers on codimension-one bifurcations, specifically two primary categories: flip (period-doubling) bifurcations, potentially leading to quasiperiodic motion on invariant closed curves, and Neimark–Sacker bifurcations, indicative of the onset of periodic or chaotic dynamics. These bifurcations provide substantial insights into the stability, complexity, and strategic interactions between the two competing firms within a nonlinear duopoly framework.

1. Introduction

In a dynamic Cournot duopoly model, each firm adjusts its strategy based on the actions of its rival. The structure of such models is strongly influenced by how firms form expectations. Specifically, it depends on the decision-making rules they use to update future output. The original Cournot model assumed firms believed their competitors' output would stay constant. This assumption is simple but unrealistic. Later research showed that such simplicity can lead to a loss of important market information. It also fails to capture the complex nature of real-world competition [1–5].

Several researchers have highlighted constrained rationality as a more realistic behavioral assumption to address these constraints. This technique involves enterprises adjusting their output according to the marginal profit, defined as the partial derivative of the profit function relative to their production level. If the marginal profit is positive (negative), firms increase (decrease) their output in the following period. This adjustment rule is often referred to as the gradient adjustment

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or gradient compensation mechanism [6]. Bischi et al. [6] proposed a general framework for such dynamically adaptive Cournot models with bounded rationality.

In addition to bounded rationality, adaptive expectations have also received significant attention. In this approach, as formalized by Agiza and Elsadany [7, 8] firms adjust their output by moving between their previous output and a perceived equilibrium output. The speed and direction of this adjustment are governed by an adaptive coefficient, which determines the degree of "oscillation" toward equilibrium. It's important to note that naïve expectations are a type of adaptive expectations where the adaptive coefficient has a certain number.

Firms may adopt different expectation schemes, either uniformly (homogeneous expectations) or individually (heterogeneous expectations). Several studies have examined models where firms share the same type of expectation, revealing a rich spectrum of dynamic behaviors, including cycles and chaos [9, 11]. Conversely, other researchers have explored duopoly games with heterogeneous expectations. Zhang et al. [12] examined a Cournot duopoly model with quadratic cost structures that included both bounded rationality and strategic flexibility. Similarly, Long et al. [13] studied a dynamic duopoly model that incorporates asymmetric spillover effects and diverse strategic behaviors. The coupled competition map mentioned and investigated numerically in [14] are as follows:

$$\begin{cases} x_{n+1} = x_n + \alpha_1 x_n(A_1 - 2x_n - By_n), \\ y_{n+1} = y_n + \alpha_2 y_n(A_2 - 2y_n + Bx_n), \end{cases} \tag{1.1}$$

This model exhibits flip, pitchfork, and Neimark-Sacker bifurcations, along with chaotic behavior, as demonstrated by numerical study. This research conducts a theoretical and analytical analysis of many types of codimension one bifurcation and their dynamical behaviors within this model. The manuscript's structure is delineated as follows: In Section 2, we demonstrated the existence and stability of fixed points in a Cournot model that includes relative profit maximization and homogeneous expectations. In Section 3, we investigate the flip bifurcation of this system, utilizing the center manifold theorem and normal form theory to establish the necessary and sufficient conditions for the parameter set during a flip bifurcation. This is accompanied by pertinent numerical simulations. The analytical findings presented in Section 4 are validated by numerical results. The results are summarized in Section 5.

2. Existence and stability of fixed points

This section examines the existence and stability of the fixed points of model (1.1). Clearly, model (1.1) mainly has the four fixed points: $E_0 = (0, 0)$, $E_1 = (\frac{A_1}{2}, 0)$, $E_2(0, \frac{A_2}{2})$, and $E^*(x^*, y^*)$ where $x^* = \frac{2A_1 - BA_2}{B^2+4}$ and $y^* = \frac{2A_2 + BA_1}{B^2+4}$. The last fixed point exists only if $2A_1 \geq BA_2$. The fixed points E_0, E_1, and E_2 are border fixed points, whereas the fixed point E^* is referred to as an interior fixed point or Nash equilibrium point. We will now examine the local stability of these fixed locations. The Jacobian matrix J of system (1.1), assessed at a fixed point $E(x,y)$, is defined as

$$J(E) = \begin{pmatrix} 1+\alpha_1(A_1-4x-By) & -\alpha_1 Bx \\ \alpha_2 By & 1+\alpha_2(A_2-4y+Bx) \end{pmatrix}.$$

The following lemma explains how stable equilibrium point $E(x,y)$ is:

Lemma 2.1. *Let $F(\lambda)=\lambda^2+P\lambda+Q$. Suppose that $F(1)>0$, $\lambda_i, i=1,2$ are the two roots of $F(\lambda)=0$. Then*

(1) *The fixed point is locally asymptotically stable i.e. $|\lambda_i|<1, i=1,2$ iff $F(-1)>0$, $Q<1$.*
(2) *The fixed point is saddle point i.e. $|\lambda_1|<1$ and $|\lambda_2|>1$ (or $|\lambda_1|>1$ and $|\lambda_2|<1$) iff $F(-1)<0$.*
(3) *The fixed point is a source i.e. $|\lambda_i|>1, i=1,2$ iff $F(-1)>0$ and $Q>1$.*
(4) *$\lambda_1=-1$ and $|\lambda_2|\neq 1$ iff $F(-1)=0$ and $P\neq 0,2$.*
(5) *λ_1 and λ_2 are complex conjugate numbers with $|\lambda_i|=1, i=1,2,$ iff $P^2-4Q<0$ and $Q=1$.*

In the last two cases, the fixed point is not hyperbolic. From the above lemma, we can come up with the following propositions.

Proposition 2.1. *The eigenvalues associated with the fixed point E_0 are $\lambda_1=1+\alpha_1A_1$ and $\lambda_2=1+\alpha_2A_2$. Thus, E_0 is a source point.*

Proposition 2.2. *The eigenvalues associated with the fixed point E_1 are $\lambda_1=1-\alpha_1A_1$ and $\lambda_2=1+\alpha_2(A_2+\frac{A_1B}{2})$. Then,*

(1) *E_1 is unstable saddle point if $0<\alpha_1A_1<2$.*
(2) *E_1 is unstable source point if $\alpha_1A_1>2$.*
(3) *E_1 is non-hyperbolic if $\alpha_1=\frac{2}{A_1}$.*

You can also get similar findings for the fixed point E_2. You may write the characteristic equation of the interior fixed point E^* as

$$\lambda^2-(2+G)\lambda+1+G+\alpha_1\alpha_2H=0,$$

where

$$\begin{aligned} G &= -2(\alpha_1x^*+\alpha_2y^*), \\ H &= (B^2+4)x^*y^*. \end{aligned}$$

Let

$$F(\lambda)=\lambda^2-(2+G)\lambda+1+G+\alpha_1\alpha_2H,$$

then

$$F(1)=\alpha_1\alpha_2H>0, \qquad F(-1)=4+2G+\alpha_1\alpha_2H.$$

At the fixed point E^*, we may handle the local dynamics by using Lemma (1).

Proposition 2.3. *A fixed point E^* in the interior is categorized:*

(1) *Sink if $4+2G+\alpha_1\alpha_2H>0$, and $G+\alpha_1\alpha_2H<0$.*

(2) *Saddle if* $4 + 2G + \alpha_1\alpha_2 H < 0$.

(3) *Source if* $4 + 2G + \alpha_1\alpha_2 H > 0$, *and* $G + \alpha_1\alpha_2 H > 0$.

(4) *Non-hyperbolic if any of the conditions that follow:*

- $4 + 2G + \alpha_1\alpha_2 H = 0$ *and* $G \neq -2, -4$.
- $G < 2\sqrt{\alpha_1\alpha_2 H}$ *and* $\alpha_1 = \frac{-G}{\alpha_2 H}$.

Let

$$F_B = \left\{(A_1, A_2, \alpha_1, \alpha_2, B) \in \mathbb{R}^5_+ |\ \alpha_1 = -\frac{4+2G}{\alpha_2 H} \text{ and } G \neq -2, -4\right\},$$

then, the interior fixed point E^* exhibits flip bifurcation behavior if the parameter α_1 is perturbed in a small neighborhood of F_B.

Let

$$NS_B = \left\{(A_1, A_2, \alpha_1, \alpha_2, B) \in \mathbb{R}^5_+ |\ \alpha_1 = -\frac{G}{\alpha_2 H}, \text{ and } G < 2\sqrt{\alpha_1\alpha_2 H}\right\}.$$

3. Codimension-one bifurcation analysis

3.1. **Bifurcation at** $E_1(\frac{A_1}{2}, 0)$**.** Our first topic is the flip bifurcation at $E_1(\frac{A_1}{2}, 0)$. Using α_1 as the bifurcation parameter. Flip bifurcation may occur at $E_1(\frac{A_1}{2}, 0)$ when $\alpha_1 = \frac{2}{A_1} = \alpha_1^*$. The eigenvalues of the Jacobian matrix at E_1 are $\lambda_1 = -1$ and $\lambda_2 = 1 + \alpha_2(A_2 + \frac{A_1 B}{2})$, with $|\lambda_2| \neq 1$. The fixed point E_1 is translated to the origin by setting $\tilde{x} = x - \frac{A_1}{2}$, $\tilde{y} = y$. Thus, system (1.1) becomes:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \to A(\alpha_1)\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + F(X), \tag{3.1}$$

where $A(\alpha_1) = J(E_1)$ and $F(X)$ is the nonlinear part of map (1.1). Its Taylor expansion close to the origin can be written like this:

$$F(X) = \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X), \quad X = (\tilde{x}, \tilde{y})^T.$$

where

$$F(X) = \begin{pmatrix} F_1(\tilde{x}, \tilde{y}, \alpha_1) \\ F_2(\tilde{x}, \tilde{y}, \alpha_1) \end{pmatrix},$$

$$\begin{aligned} F_1(\tilde{x}, \tilde{y}, \alpha_1) &= -2\alpha_1\tilde{x}^2 - B\alpha_1\tilde{x}\tilde{y} + O\|X\|^4, \\ F_2(\tilde{x}, \tilde{y}, \alpha_1) &= -2\alpha_2\tilde{y}^2 + B\alpha_2\tilde{x}\tilde{y} + O\|X\|^4. \end{aligned} \tag{3.2}$$

The multilinear functions $B(X, X)$ and $C(X, X, X)$ are defined as follows:

$$\begin{aligned} B_1(x, y) &= \sum_{j,k=1}^{2} \frac{\partial^2 F_1(\xi, \alpha_1)}{\partial\xi_j \partial\xi_k}\bigg|_{\xi=0} x_j y_k, \\ &= -\alpha_1(4x_1 y_1 + Bx_1 y_2 + Bx_2 y_1), \end{aligned} \tag{3.3}$$

$$B_2(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2(\xi, \alpha_1)}{\partial\xi_j \partial\xi_k}\bigg|_{\xi=0} x_j y_k,$$

$$= \alpha_2(Bx_1y_2 + Bx_2y_1 - 4x_2y_2),$$

$$C_1(x,y,u) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_1(\xi,\alpha_1)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0} x_jy_ku_l = 0,$$

$$C_2(x,y,u) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_2(\xi,\alpha_1)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0} x_jy_ku_l = 0.$$

Two eigenvectors $p, q \in \mathbb{R}^2$, each with the eigenvalue $\lambda_1 = -1$, are such that $A(\alpha_1^*)q = -q$ and $A^T(\alpha_1^*)p = -p$. After some calculations we obtain

$$\begin{aligned} q &\sim (1,0)^T, \\ p &\sim (2 + \alpha_2 A_2 + \frac{\alpha_2 A_1 B}{2}, B)^T. \end{aligned}$$

The normalization of these vectors is set to 1, where $\langle .,. \rangle$ is the standard scalar product in $\mathbb{R}^2$. So, we have

$$\begin{aligned} q &= (1,0)^T, \\ p &= (1, \frac{2}{4 + 2\alpha_2 A_2 + \alpha_2 A_1 B})^T, \end{aligned}$$

Based on the methods explained in [15], we change map (1.1) to the following normal form at α_1:

$$\xi \to -\xi + c(\alpha_1^*)\xi^3 + O(\xi^4),$$

where

$$c(\alpha_1^*) = \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\langle p, B(q,(A-I)^{-1}B(q,q))\rangle,$$

where I denotes the identity matrix of dimensions 2×2. Direct computations yield

$$c(\alpha_1) = \frac{-1}{3}(2+3B) - \frac{16(1+B)^2}{A_1^2} - \frac{8(1+B)(B-4)}{A_1(2A_2 + A_1 B)}. \tag{3.4}$$

Based on the above analysis and the theory in [15], we can say the following:

Theorem 3.1. *If $c(\alpha_1^*)$ is given by (3.4), then a flip bifurcation occurs at $E_1(\frac{A_1}{2}, 0)$ of map (1.1) at $\alpha_1 = \frac{2}{A_1}$. Further, if $c(\alpha_1^*) < 0$ (resp., $c(\alpha_1^*) > 0$), then the process of division is subcritical (or supercritical), which means that the two newborn periods are stable (or unstable).*

3.2. **Bifurcation at $E_3(x^*, y^*)$.** We initially examine the possibility of flip bifurcation at E^*, which may occur if the parameters are selected from the set F_B, where α_1 is regarded as the bifurcation parameter.. The eigenvalues of $J(E^*)$ are $\lambda_1 = -1$ and $\lambda_2 = 3 + G$. The condition $|\lambda_2| \neq 1$ results in $G \neq -2, -4$. Map (1.1) is rewritten as follows by transforming E^* to the origin using the translations $\hat{x} = x - x^*$ and $\hat{y} = y - y^*$:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \to \tilde{A}(\alpha_1)\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + F(X), \tag{3.5}$$

where $\tilde{A}(\alpha_1) = J(E^*)$, $X = (\hat{x}, \hat{y})^T$, and $F(X)$ denotes the nonlinear term of map (1.1), which is as similar as those given in (3.2). The multilinear functions $\tilde{B}(X,X)$ and $\tilde{C}(X,X,X)$ are identical to those provided in (3.3). Two eigenvectors $p, q \in \mathbb{R}^2$ are such that $\tilde{A}(\alpha_1)q = -q$ and $\tilde{A}^T(\alpha_1)p = -p$. Following the completion of calculations, we acquire

$$\begin{aligned} q &\sim (\alpha_1 B x^*, 2 - 2\alpha_1 x^*)^T, \\ p &\sim (2 - 2\alpha_2 y^*, \alpha_1 B x^*)^T. \end{aligned}$$

The corresponding normalized vectors p and q are

$$\begin{aligned} q &= (\alpha_1 B x^*, 2 - 2\alpha_1 x^*)^T, \\ p &= \left(\frac{2 - 2\alpha_2 y^*}{\alpha_1 B x^* [4 - 2(\alpha_1 x^* + \alpha_2 y^*)]}, \frac{1}{4 - 2(\alpha_1 x^* + \alpha_2 y^*)} \right)^T. \end{aligned}$$

We convert map (1.1) to the following normal form at α_1 in accordance with the algorithms introduced in [15]:

$$\xi \to -\xi + \tilde{c}(\alpha_1)\xi^3 + O(\xi^4),$$

where

$$\tilde{c}(\alpha_1) = \frac{1}{6}\langle p, \tilde{C}(q, q, q)\rangle - \frac{1}{2}\langle p, \tilde{B}(q, (A - I)^{-1}\tilde{B}(q, q))\rangle,$$

In this case, I is the unit matrix that is 2 by 2. The following theorem follows from the above analysis and the theorem in [15]:

Theorem 3.2. *If $G \neq -2, -4$ and $\tilde{c}(\alpha_1) \neq 0$, then a flip bifurcation occurs at E^* of map (1.1) at $\alpha_1 = -\frac{4+2G}{\alpha_2 H} = -\frac{4-4(\alpha_1 x^* + \alpha_2 y^*)}{\alpha_2 (B^2+4) x^* y^*}$. Further, if $c(\alpha_1) < 0$ (resp., $c(\alpha_1) > 0$), then the bifurcation is subcritical (or supercritical), which means that the period-2 orbits that split off from E^* are stable (or unstable).*

Next, we analyze Neimark-Sacker bifurcation at E^*. Consider map (1.1) with arbitrary $(\tilde{\alpha}_1, \alpha_2, A_1, A_2, B) \in NS_B$. The eigenvalues of $J(E^*)$ are a pair of complex conjugate integers λ and $\bar{\lambda}$ with modulus 1, as shown by Proposition 3.

$$\lambda, \bar{\lambda} = \frac{-P(\tilde{\alpha}_1) \pm i\sqrt{4Q(\tilde{\alpha}_1) - P^2(\tilde{\alpha}_1)}}{2}.$$

and

$$P(\tilde{\alpha}_1) = -\left(2 + G(\tilde{\alpha}_1)\right), \quad Q(\tilde{\alpha}_1) = 1 + G(\tilde{\alpha}_1) + \tilde{\alpha}_1 \alpha_2 H.$$

So, we have

$$|\lambda|_{\alpha_1 = \tilde{\alpha}_1} = \sqrt{Q(\tilde{\alpha}_1)} = 1, \quad \left.\frac{d|\lambda(\alpha_1)|}{d\alpha_1}\right|_{\alpha_1 = \tilde{\alpha}_1} = \frac{-2x^* + \alpha_2 H}{2} \neq 0.$$

In addition, it is required that $\lambda^k(\tilde{\alpha}_1), \bar{\lambda}^k(\tilde{\alpha}_1) \neq 1 (k = 1, 2, 3, 4)$, which leads to

$$G(\tilde{\alpha}_1) \neq -2, -3. \tag{3.6}$$

Let $q \in \mathbb{C}^2$ be an eigenvector of $A(\tilde{\alpha}_1)$ corresponding to the eigenvalue $\lambda(\tilde{\alpha}_1)$ such that $A(\tilde{\alpha}_1)q = e^{i\theta}q$, and $p \in \mathbb{C}^2$ be an eigenvector of the transposed matrix $A^T(\tilde{\alpha}_1)$ corresponding to the eigenvalue $\bar{\lambda}(\tilde{\alpha}_1)$ such that $A^T(\tilde{\alpha}_1)p = e^{-i\theta}p$. By direct calculation, we have

$$q = \begin{pmatrix} \tilde{\alpha}_1 B x^* \\ 1 - 2\tilde{\alpha}_1 x^* - \lambda \end{pmatrix},$$

$$p = \begin{pmatrix} \frac{\sqrt{4\tilde{\alpha}_1\alpha_2 H - G^2(\tilde{\alpha}_1)} - i(2\tilde{\alpha}_1 x^* - 2\alpha_2 y^*)}{2\tilde{\alpha}_1 B x^* \sqrt{4\tilde{\alpha}_1\alpha_2 H - G^2(\tilde{\alpha}_1)}} \\ -\frac{i}{\sqrt{4\tilde{\alpha}_1\alpha_2 H - G^2(\tilde{\alpha}_1)}} \end{pmatrix},$$

It is clear that $\langle p, q\rangle = 1$, where $\langle ., .\rangle$ means the standard scalar product in $\mathbb{C}^2$, $\langle p, q\rangle = \overline{p}_1 q_1 + \overline{p}_2 q_2$. Any vector $X \in \mathbb{R}^2$. can be represented for α_1 near $\tilde{\alpha}_1$ as $X = zq + \bar{z}\bar{q}$, for some complex z. Thus, system (3.3) can be transformed for sufficiently small $|\alpha_1|$ (near $\tilde{\alpha}_1$) into the following form:

$$z \to \lambda(\tilde{\alpha}_1)z + g(z, \bar{z}, \tilde{\alpha}_1),$$

where $\lambda(\tilde{\alpha}_1)$ can be written as $\lambda(\alpha_1) = (1 + \varphi(\alpha_1))e^{i\theta(\alpha_1)}$ (where $\varphi(\alpha_1)$ is a smooth function with $\varphi(\tilde{\alpha}_1) = 0$) and g is a complex-valued smooth function of z, $\bar{z}$, and α_1, whose Taylor expression with respect to $(z, \bar{z})$ contains quadratic and higher-order terms:

After some transformations similarly introduced in (), we can transform map (1.1) to the normal form on the center manifold at $\alpha_1 = \alpha_{1NS}$ as follows:

$$z \mapsto e^{i\theta(\alpha_{1NS})}z(1 + d_1|z|^2) + O(|z|^4),$$

where $e^{i\theta(\alpha_{1NS})} = \lambda(\alpha_{1NS}), z \in \mathbb{Z}^2$ and the real number $\tilde{d}(\alpha_{1NS}) = Re \;\; d_1$ is given as follows:

$$\begin{aligned} \tilde{d}(\alpha_{1NS}) \;\; &= \;\; \frac{1}{2} Re\{e^{-i\theta(\alpha_{1NS})}[\langle p, \tilde{C}(q, q, \overline{q})\rangle \\ &+ \;\; 2\langle p, \tilde{B}(q, (I - \tilde{A}(\alpha_{1NS}))^{-1}\tilde{B}(q, \overline{q}))\rangle] \\ &+ \;\; \langle p, \tilde{B}(\overline{q}, (e^{2i\theta(\alpha_{1NS})}I - \tilde{A}(\alpha_{1NS}))^{-1}\tilde{B}(q, q))\rangle]\}, \end{aligned}$$

Which decides whether the bifurcating closed invariant curve attracts or repels. The following theorem is obtained from [15].

Theorem 3.3. *If (3.6) and $\tilde{d}(\alpha_{1NS}) \neq 0$ hold, a Neimark-Sacker bifurcation occurs at E^* of map (1.1) when $\alpha_1 = \alpha_{1NS}$. The sign of $\tilde{d}(\alpha_{1NS})$ determines the stability of a closed invariant bifurcating curve. If $\tilde{d}(\alpha_{1NS}) < 0$ (resp., $\alpha_{1NS}) > 0$), the bifurcating closed invariant curve attracts (resp., repels) for $\alpha_1 > \alpha_{1NS}$ (resp., $\alpha_1 < \alpha_{1NS}$).*

4. Numerical results

4.1. **Numerical simulations.** In this section, we will illustrate the above analytical findings by means of phase portraits, bifurcation diagrams and Lyapunov exponent. This also will show the more complex dynamics of model (1.1). The codimension-one bifurcations will be investigated numerically by MATCONTM.

(1) Fix $A_1 = 5.9$, $A_2 = 5.8$, and let $B = 0.5$, $\alpha_1 = 0.27$, and $\alpha_2 = 0.41$. The fixed point $E_1 = (\frac{A_1}{2}, 0) = (2.85, 0.1)$, taking $(x_o, y_o) = (2.9, 0.1)$, we can see the E_1 is an unstable saddle as depicted in Fig.(1)(a). However, E_1 is an unstable source at $\alpha_1 = 0.37$ as seen in Fig.(1)(b). The fixed point $E^* = (x^*, y^*) = (2.0941, 3.4235)$, so if we start at $(x_o, y_o) = (2, 3.3)$, we can see that E^* is a spiral sink in Fig.(1)(c) if $\alpha_1 = 0.41$, and $\alpha_2 = 0.3$, while E^* is a source

when $\alpha_1 = 0.25$, and $\alpha_2 = 0.5$ as depicted in Fig.(1)(d). The chaotic attractor is shown in both Fig.(1)(e) and (f) for $\alpha_1 = 0.1$, $\alpha_2 = 0.42$ and $\alpha_1 = 0.49$, $\alpha_2 = 0.32$, respectively.

(2) Now let $B = 0.2$, $\alpha_2 = 0.25$ and vary α_1, according to Theorem 1, a flip bifurcation occurs at the fixed point E_1 when $\alpha_1 = \frac{2}{A_1} = 0.3390$ as shown in Fig. (2)(a) and the corresponding maximal Lyapunov exponent (MLE) is depicted in Fig. (2)(b). At $\alpha_2 = 0.42$, a flip bifurcation occurs at E^* when $\alpha_1 = 0.3649$ according to Theorem 2 which is clearly seen in Fig. (2)(c) and the corresponding MLE is shown in Fig. (2)(d). For $B = 0.6$ and $\alpha_2 = 0.28$, a Neimark-Saker bifurcation occurs at $\alpha_1 = 0.4551$ according to theorem 3. as seen in Fig. (2)(e) and its corresponding MLE is shown in Fig. (2)(f).

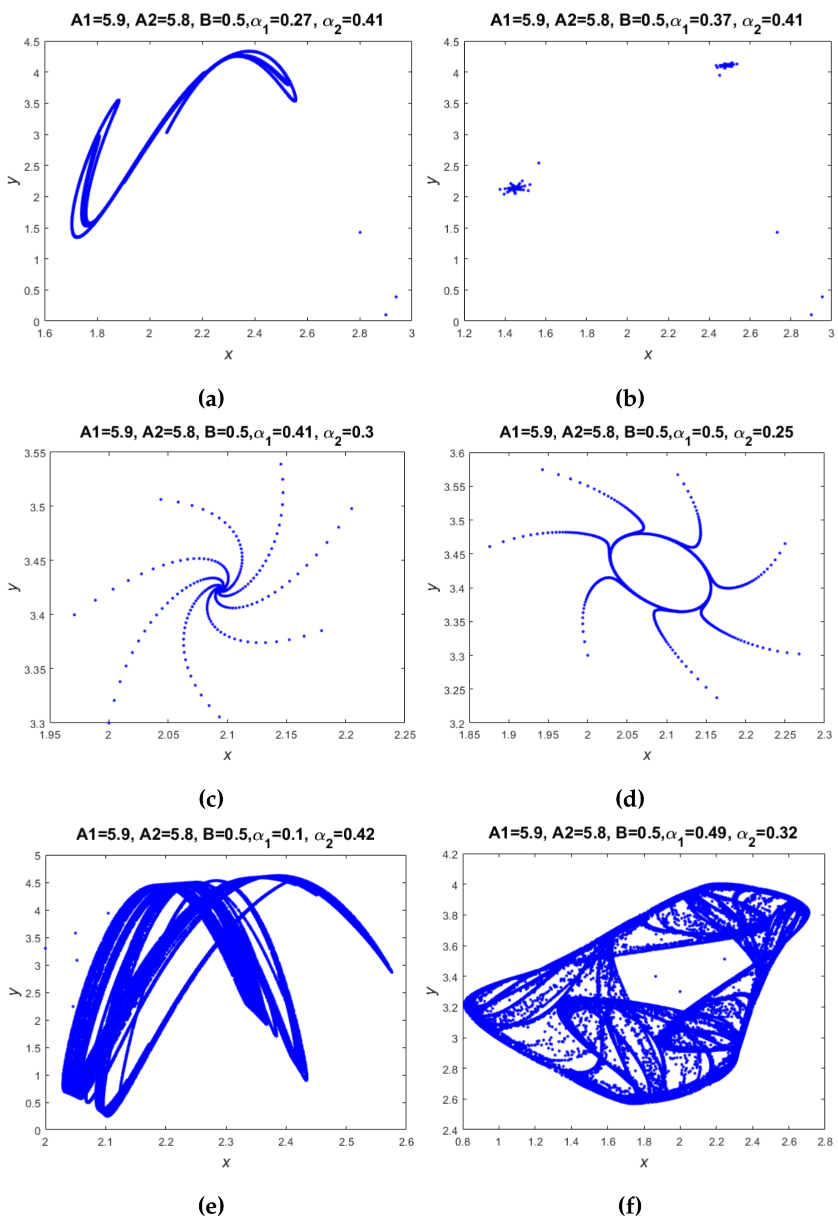


Figure 1. Phase portraits of model (1.1).

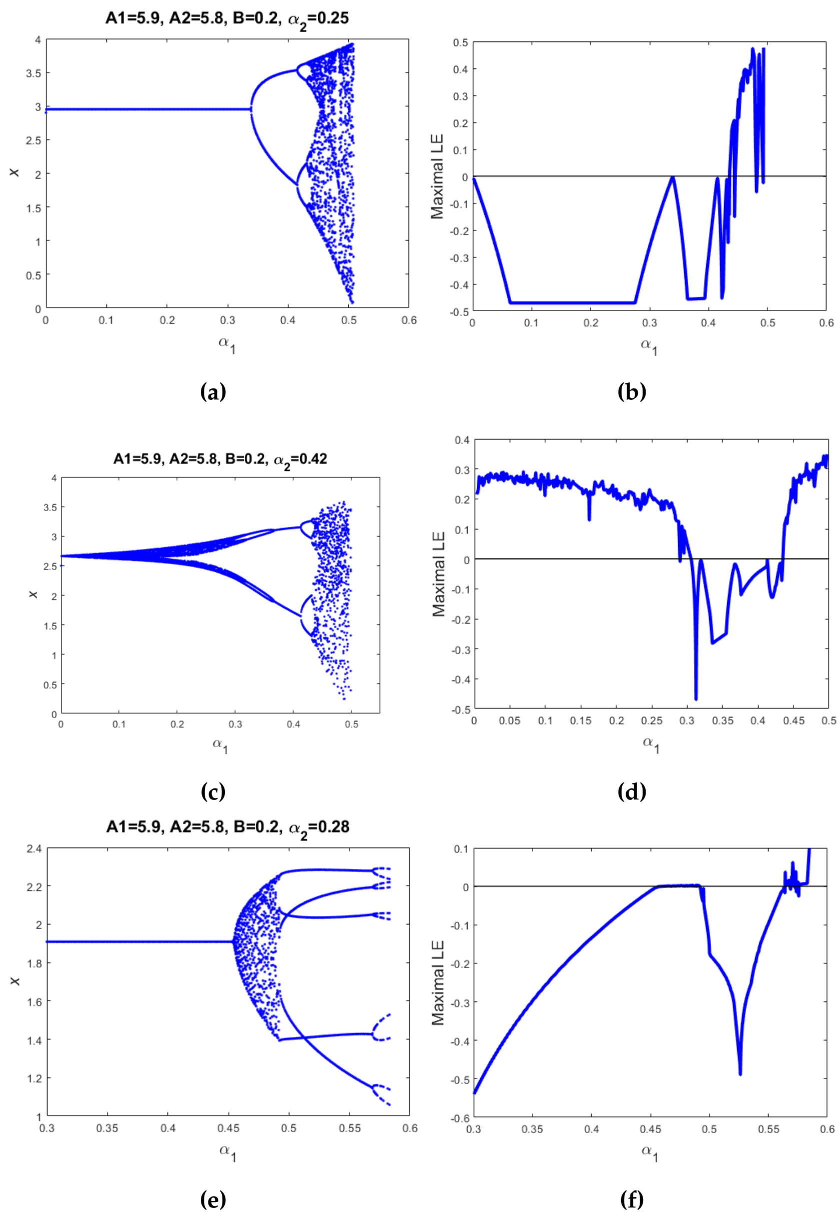


Figure 2. Bifurcation diagrams and corresponding maximal Lyapunov exponent (MLE) of model (1.1).

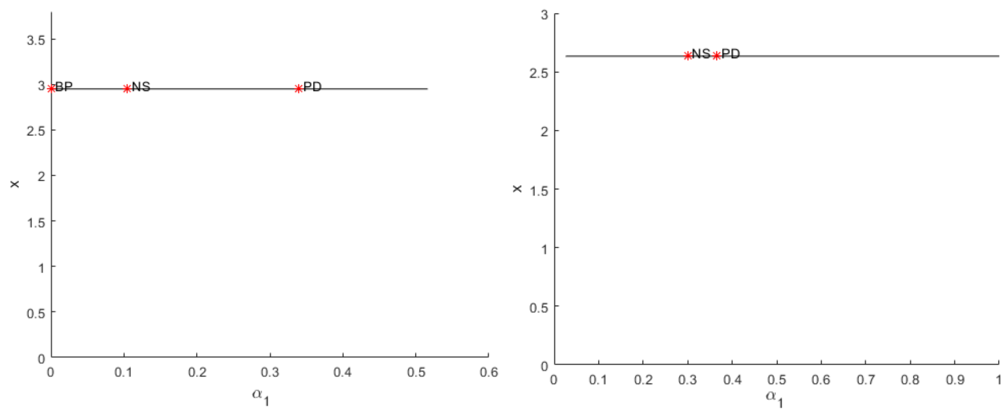
4.2. **Numerical continuation.** This part uses MATCONTM [15] to execute numerical continuation at fixed locations E_1 and E^*.. Example 1: Fix $A_1 = 5.9, A_2 = 5.8, B = 0.5, \alpha_2 = 0.25$, and vary α_1. The report of MATCONTM is as follows.

```
label PD, x= (2.950000 0.000000 0.338983),
Normal form coefficient for PD = 4.296380e-01 ,
label NS, x= (2.950000 0.000000 0.104240),
Neutral Saddle
label BP, x= (2.950000 0.000000 -0.000000).
```

According to Theorem 1, E_1 exhibits a flip bifurcation (Fig.(3)(a)) and a Neimark-sacker bifurcation (NS), which has not been demonstrated theoretically. Example 2: Fix $A_1 = 5.9$, $A_2 = 5.8, B = 0.2$, $\alpha_2 = 0.42$, and vary α_1. The report of MATCONTM is as follows.

```
label NS, x= (2.633663 3.163366 0.299604),
Neutral Saddle
label PD, x= (2.633663 3.163366 0.364944),
 Normal form coefficient for PD = 5.583321e-01.
```

The fixed point E^* has flip and Neimark-sacker bifurcations, which can be found in Theorems 2 and 3. Figure (3) shows this.



(a) **(b)**

Figure 3. Continuation of E_1 and E^* in (α_1, x)-plane. The branch point (BP), Neimark-Sacker point (NS) and period doubling point (PD) obtained at (a)Theorem 1, and (b)Theorem 2 and 3.

5. Conclusion

This article examines a nonlinear dynamical system that represents a duopoly competition game model. The model mimics the interaction strategies of two competitive firms under nonlinear adjustments. The system's equilibrium points were identified by rigorous mathematical analysis, and their local stability qualities were evaluated using linearization and eigenvalue analysis. Critical bifurcation possibilities in system dynamics were analyzed. Threshold parameters were established for pitchfork, flip (period-doubling), and Neimark–Sacker (quasi-periodic) bifurcations. Bifurcations indicate transitions from stable equilibria to cycles and quasi-periodic orbits, commonly observed in economic competition.

The results demonstrate how minute system attributes might affect the qualitative conduct of rival companies. This approach elucidates nonlinear dynamics in duopoly markets and offers a theoretical framework for forecasting and regulating complex competitive economic behaviors.

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References

[1] A. Cournot, Researches into the Mathematical Principles of the Theory of Wealth, in: Forerunners of Realizable Values Accounting in Financial Reporting, Routledge, 1982: pp. 3–13. https://doi.org/10.4324/9781003051091-2.

[2] D. Rand, Exotic Phenomena in Games and Duopoly Models, J. Math. Econ. 5 (1978), 173–184. https://doi.org/10.1016/0304-4068(78)90022-8.

[3] M. Kopel, Simple and Complex Adjustment Dynamics in Cournot Duopoly Models, Chaos Solitons Fractals 7 (1996), 2031–2048. https://doi.org/10.1016/s0960-0779(96)00070-7.

[4] G. Bischi, L. Stefanini, L. Gardini, Synchronization, Intermittency and Critical Curves in a Duopoly Game, Math. Comput. Simul. 44 (1998), 559–585. https://doi.org/10.1016/s0378-4754(97)00100-6.

[5] C. Hommes, Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems, Cambridge University Press, 2013. https://doi.org/10.1017/cbo9781139094276.

[6] G.I. Bischi, A. Naimzada, Global Analysis of a Dynamic Duopoly Game with Bounded Rationality, in: Advances in Dynamic Games and Applications, Birkhäuser, Boston, MA, 2000: pp. 361–385. https://doi.org/10.1007/978-1-4612-1336-9_20.

[7] H. Agiza, A. Elsadany, Chaotic Dynamics in Nonlinear Duopoly Game with Heterogeneous Players, Appl. Math. Comput. 149 (2004), 843–860. https://doi.org/10.1016/s0096-3003(03)00190-5.

[8] H. Agiza, A. Elsadany, Nonlinear Dynamics in the Cournot Duopoly Game with Heterogeneous Players, Physica: Stat. Mech. Appl. 320 (2003), 512–524. https://doi.org/10.1016/s0378-4371(02)01648-5.

[9] J. Grau-Climent, L. Garcia-Perez, R. Alonso-Sanz, J.C. Losada, Effect of Players' Expectations and Memory in a Quantum Cournot Game, Chaos, Solitons Fractals 175 (2023), 113950. https://doi.org/10.1016/j.chaos.2023.113950.

[10] H. Li, R. He, W. Zhou, Dynamic Behaviors in a Cournot Duopoly Model with Knowledge Spillover Effect Based on Constant Conjectural Variation, Math. Comput. Simul. 201 (2022), 305–323. https://doi.org/10.1016/j.matcom.2022.05.006.

[11] S.E.I. Bouzeraa, R. Bououden, M.S. Abdelouahab, Fractional Logistic Map with Fixed Memory Length, Int. J. Gen. Syst. 52 (2023), 653–663. https://doi.org/10.1080/03081079.2023.2201001.

[12] X. Zhang, D. Sun, W. Jiang, Dynamics of a Heterogeneous Quantum Cournot Duopoly with Adjusting Players and Quadratic Costs, Quantum Inf. Process. 19 (2020), 403. https://doi.org/10.1007/s11128-020-02911-2.

[13] J. Long, H. Huang, A Dynamic Stackelberg–Cournot Duopoly Model with Heterogeneous Strategies Through One-Way Spillovers, Discret. Dyn. Nat. Soc. 2020 (2020), 3251609. https://doi.org/10.1155/2020/3251609.

[14] A. Elsadany, Dynamics of a Cournot Duopoly Game with Bounded Rationality Based on Relative Profit Maximization, Appl. Math. Comput. 294 (2017), 253–263. https://doi.org/10.1016/j.amc.2016.09.018.

[15] Y.A. Kuznetsov, I. Karatzas, M. Yor, Elements of Applied Bifurcation Theory, Springer, New York, 1998. https://doi.org/10.1007/b98848.