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Iterative Methods for General Bivariational Inequalities

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Abstract. Some new classes of general bivariational inequalities, which can be viewed as a novel important special case of variational equalities, are investigated. Projection method, auxiliary principle and dynamical systems coupled with finite difference approach are used to suggest and analyzed a number of new and known numerical techniques for solving bivariational inequalities. Convergence analysis of these methods is investigated under suitable conditions. One can obtain a number of new classes of bivariational inequalities by interchanging the role of operators. Sensitivi anaysis of the bivariational inequalities is also discussed. Some important special cases are highlighted. Several open problems are suggested for future research.

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory was introduced by Lions and Stampacchia [24] in early sixties, can be viewed as a novel generalization of the variational principles. By variational principles, we mean maximum and minimum problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. Many basic equations of mathematical physics result from variational problems. It is known that the gauge fields theories are a continuation of Einstein's concept of describing physical effects mathematically in terms of differential geometry. These theories play a fundamental role in the modern theory of elementary particles and are right tool of building up a unified theory of elementary particles, which includes

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all kind of known interactions. Variatitional principles have played a leading role in the developments of computational methods for solving complicated and complex problems arising in game theory, mechanics, geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas. It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities, which occur in various branches of pure and applied sciences.

It is a well known fact that the variational inequalities are equivalent to the fixed point problem. This equivalent formulation has played an important role to study the existence of the solution and to develop efficient numerical methods for solving variational inequalities and related optimization problems. Noor [34, 37] has proposed and suggested three (multi step) forward-backward iterative methods for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These Noor(three-step) schemes are a natural generalization of the splitting methods for solving partial differential equations. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special cases. It has been established [8, 16, 34, 36, 56] that Noor(three-step) iterations perform better than two-step(Ishikawa iteration) and one step method Mann iteration. Ashish et al. [3,4], Cho et al. [9] and Kwuni et al. [23] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations. We would like to point out Noor(three-step) iterations have influenced the research in the fixed point theory, optimization and will continue to inspire further research in fixed point theory, compressive sensing, image in painting, fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. For recent developments and applications of the variational inequalities and their variant forms, see [1, 2, 9, 63, 65] and the references therein.
It is known that the projection methods and their variant forms can not be used to suggest iterative schemes for some classes of variational inequalities and equilibrium problems. To overcome these deficiencies, one usually uses the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [24] and Glowinski et al [15], as developed by Noor [33, 37], Noor et al. [44, 50–52] and Patricksson [55].To prove the existence of a solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. Another novel feature of this approach is that this technique enables us to suggest some iterative methods for solving the variational inequalities. Noor [33] has modified the auxiliary principle technique involving an arbitrary operator. For more details and applications of the modified auxiliary principle technique, see Patrickson [55] and the references therein.

Variational inequality theory has been generalized and extended in several directions using novel and innovative ideas to tackle complex and complicated problems. Noor [30,31] considered two new classes of variational inequalities involving two arbitrary operators in 1988, which are known as general variational inequalities and have applications in oceanography, non-positive and non-symmetric differential equations theory. Noor [39, 40] introduced and studied the concepts of general convex set and general convex functions, which are known as g-convexity. Noor [39, 40] also proved the optimal conditions for the differentiable general convex functions can be characterized by the general variational inequality. For the applications and shape properties of Noor's g-convexity, see Cristescu et al. [12] and the references therein. An important special case of these general variational inequalities known as inverse variational inequalities has been considered in [5, 13, 19, 20, 46, 47, 58, 59, 64], which have applications in traffic assignment and network equilibrium control problems.

It is known that a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences are being studied in the unified framework of variational inequalities and equilibrium problems. The behaviour of such problems as a result of changes in the problem data is always of concern. This problem is called sensitivity analysis. The sensitivity analysis provides useful information for designing or planning various equilibrium systems. From a mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied and can stimulate new ideas and techniques for problem solving. Due to these and other reasons, one investigates the sensitivity analysis of bifunction variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed. Dafermos [13] and Noor [39] studied the sensitivity anaysis of the variational inequalities uisng quite diffferent techniques.

Motivated and inspired by ongoing recent research in variational inequalities, we consider some new classes of bivariational inequalities, which include general variational inequalities involving two arbitrary operators, see Noor [30,31]. New concepts of convex-like sets and convex-like convex functions are introduced. The optimality conditions of the differentiable convex-like functions are characterized by bifunction variational inequalities. Several special cases are discussed as applications of the bifunction variational inequalities in Section 2. In Section 3, we establish the equivalence with the fixed point problem, which is used to discuss the unique existence of the solution as well as to suggest several inertial iterative methods along with the convergence analysis. We also apply the auxiliary principle technique involving an arbitrary operator to consider some iterative schemes for solving the bivariational inequalities in Section 4. In Section 5, a dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods for solving the bivariational inequalities exploring the finite difference idea. The sensitivity of the bivariational inequalities is investigated in Section 6. Our results can be viewed as significant refinements and improvements of the new results for bivariational inequalities and their variant forms.

We have given only a brief introduction to this fast growing field. The interested reader is advised to explore this field further and discover novel, fascinating applications of bivariational inequalities in other areas of science such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic processes, financial analysis, and related other optimization problems.

2. Formulations and Basic Facts

Let Ω be a nonempty closed convex set in a real Hilbert space $\mathcal{H}$.. Let $N(.,.) : \mathcal{H} \times \mathcal{H} \Longrightarrow \mathcal{H}$ be an arbitrary bifunction and $g : \mathcal{H} \Longrightarrow \mathcal{H}$ be an operator. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [11, 12, 27, 48], which are needed in the derivation of the main results.

Definition 2.1. *[11, 27] A set Ω in $\mathcal{H}$ is said to be a convex set, if*

$$\mu + \lambda(\nu - \mu) \in \Omega, \qquad \forall \mu, \nu \in \Omega, \lambda \in [0, 1].$$

Definition 2.2. *[11, 27] A function Φ is said to be a convex function if*

$$\Phi((1-\lambda)\mu + \lambda\nu) \leq (1-\lambda)\Phi(\mu) + \lambda\Phi(\nu), \quad \forall \mu, \nu \in \Omega, \quad \lambda \in [0, 1].$$

Convex functions are closely related to the integral inequalities and variational inequalities. If the convex function Φ is differentiable, then $\mu \in \Omega$ is the minimum of the function Φ, if and only if, $\mu \in \Omega$ satisfies the inequality

$$\langle \Phi'(\mu), \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.1}$$

The inequalities of the type (2.1) are called the variational inequalities. Lions and Stampacchia [24] considered a more general variational inequality of which (2.1) is a special case.

To be more precise, for a given nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega. \tag{2.2}$$

which is called the variational inequality.
In many cases, the function may not be a convex function. To overcome these drawbacks, several new convex sets and convex functions have been considered.

We now introduce some new concepts for the convex-like sets and functions.

Definition 2.3. *The set Ω_b is said to be a convex-like set in $\mathcal{H}$, if*

$$N(\mu, \mu) + t(\nu - g(\mu)) \in \Omega_b \quad \nu, \mu \in \Omega_b, \quad t \in [0, 1],$$

where $N(.,.) : \mathcal{H} \times \mathcal{H} \Longrightarrow \mathcal{H}$ is an arbitrary bifunction and $g : \mathcal{H} \Longrightarrow \mathcal{G}$ is an operator. Clearly, for $t = 0$, we have $N(\mu, \mu) = 0$ and for $t = 1, N(\nu, \mu) = g(\mu)$, we obtain

$$g(\mu) + t(\mu - g(\mu)) \in \Omega_g,$$

which is known as a general convex set, which was introduced by Noor [38, 39] with respect to an arbitrary function g. These are called Noor's g-convex sets and g-convex functions.

Definition 2.4. *A function Φ is said to be a convex-like function if*

$$\Phi\Big(N(\mu, \mu) + t(\nu - g(\mu))\Big) \leq \Phi(N(\mu, \mu)) + t(\Phi(\nu) - \Phi(g(\mu)), \quad \forall \nu, \mu \in \Omega_{bg}.$$

For $N(\mu,\mu) = g(\mu)$, the convex-like convex functions become:

Definition 2.5. *A function Φ is said to be a general convex function if*

$$\Phi\Big(g(\mu) + t(\mu - g(\mu))\Big) \leq \Phi(\mu) + t(\Phi(\nu) - \Phi(g(\mu)), \quad \forall \nu, \mu \in \Omega_g,$$

which appears to be a new concept. It is an open problem to explore the applications of the convex-like sets and convex-like functions in data sciences, information, machine learning, signal processing, and medical images.

Using the technique of convex analysis, one can discuss the optimality conditions of the differentiable convex-like functions.

Theorem 2.1. *Let Φ be a differentiable general convex function on the general biconvex set Ω_{bg}. Then the minimum $\mu \in \Omega_{bg}$ of the function Φ, if and only if, $\mu \in \Omega_{bg}$ satisfies the inequality*

$$\langle \Phi'(N(\mu,\mu)), \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega_{bg}, \tag{2.3}$$

where $\Phi'(.)$ is the differential of Φ in the direction $\nu - g(\mu)$.

For the sake of simplicity and to convey the main ideas, we assume that Ω is a nonempty, closed, and convex set, unless otherwise specified.
We now consider a general bivariational inequality problem, from which one obtains the previously known and new classes of variational inequalities and mathematical programming problems as special cases.

Let $\Omega \subseteq \mathcal{H}$ be a closed convex set and $N(.,.) = \mathcal{H} \times \mathcal{H} \Longrightarrow \mathcal{H}$ be the bifunction. For given operators $\mathcal{B}, \mathcal{T}, g : \mathcal{H} \Longrightarrow \mathcal{H}$, consider the problem of finding $\mu \in \Omega$ such that

$$\langle N(\mathcal{B}\mu, \mathcal{T}\mu), \nu - g(\mu)\rangle \geq 0, \forall \nu \in \Omega, \tag{2.4}$$

which is called the general bivariational inequality.

Special Cases. We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2.4).

(1) This problem (2.4) can be viewed as finding the minimum of general convex-like function. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.
(2) For $\mathcal{T} = I, \mathcal{B} = I$, the bivariational inequality (2.4) reduces to finding $\mu \in \Omega$, such that

$$\langle N(\mu,\mu), \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.5}$$

which is called the generalized variational inequality.
(3) For $N(\mathcal{B}\mu, \mathcal{T}\mu) = \mathcal{T} + \mathcal{B}\mu, g = I$, the problem (2.4) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu + \mathcal{B}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.6}$$

which is called the mildly nonlinear variational inequality involving the sum (difference) of two operators, introduced and studied by Noor [28]. For different and suitable choice of the operators mildly nonlinear variational inequalities include Absolute value, hemivariational inequalities and representing theorems such as Lax-Milgram lemma and Riesz-Fréchet theorem as special cases. It has been shown that the minimum of the difference of two convex functions can be studied in the unified frame work of mildly variational inequalities. Also, see [7, 17, 18] for the applications of the DC-problems and optimization problems.

(4) If $N(\mu,\mu) = \mathcal{T}\mu$ then the problem (2.6) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.7}$$

which is called the general variational inequality, introduced and studied by Noor [30, 31] in 1988.

(5) For $g = I$, the general variational inequality (2.7) reduces to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.8}$$

which is called the well known variational inequality, introduced and investigated by Lions and Stampacchia [16]. For the applications, formulations,generalizations and numerical methods, see [1, 5, 6, 8, 9, 13–16, 19, 52, 54–63, 65] and the references therein.

(6) If $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega\}$ is a polar (dual) cone of a convex cone Ω in $\mathcal{H}$, then the problem (2.4) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega, \quad N(\mathcal{B}\mu, \mathcal{T}\mu) \in \Omega^* \quad \text{and} \quad \langle N(\mathcal{B}\mu, \mathcal{T}\mu), g(\mu)\rangle = 0, \tag{2.9}$$

which is known as general bi-complementarity problems and appears to be a new one.

(7) For $N(\mathcal{B}\mu, \mathcal{T}\mu) = \mathcal{T}\mu$, the problem (2.9) reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{2.10}$$

which is called the general nonlinear complementarity problem, see Noor [30, 37].

Obviously quasi complementarity problems include the nonlinear complementarity problems and linear complementarity problems. The complementarity problems were introduced and studied by Cottle et al. [10], Noor [29, 30, 37] and Noor et al. [50, 51].

Remark 2.1. *It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, \mathcal{B}, g$, bifunction $N(.,.)$, convex set Ω and the spaces, one can obtain several classes of variational inequalities, complementarity problems, and optimization problems as special cases of the bivariational inequalities (2.4). This shows that the problem (2.4) is quite general and unifying one. It is an interesting problem to develop efficient and implementable numerical methods for solving the bivariational inequalities and their variants.*

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the bivariational inequalities and the fixed point problems.

Lemma 2.1. *[22] Let Ω be a closed and convex set in $\mathcal{H}$.. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.11}$$

if and only if,

$$\mu = \Pi_\Omega(z),$$

where Π_Ω is the projection of $\mathcal{H}$ onto the closed convex set Ω.

It is well known that the projection operator Π_Ω is nonexpansive, that is,

$$||\Pi_\Omega - \Pi_\Omega|| \leq ||\mu - \nu||, \quad \forall \mu, \nu \in \Omega. \tag{2.12}$$

Definition 2.6. *The bifunction $N(.,.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be:*

(1) *Strongly monotone, if there exist a constant $\alpha > 0$, such that*

$$\langle N(\mathcal{B}\mu, \mathcal{T}\mu) - N(\mathcal{B}\nu, \mathcal{T}\nu), \mu - \nu \rangle \geq \alpha ||\mu - \nu||^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(2) *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*

$$||N(\mathcal{B}\mu, \mathcal{T}\mu) - N(\mathcal{B}\nu, \mathcal{T}\nu)|| \leq \beta ||\mu - \nu||, \quad \forall \mu, \nu \in \mathcal{H}.$$

(3) *partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle N(\mathcal{B}\mu, T\mu) - N(\mathcal{B}\nu, T\nu), z - \nu \rangle \geq -\alpha ||\mu - z||^2$$

(4) *partially relaxed monotone, if*

$$\langle N\mathcal{B} - N(\mathcal{B}\nu, T\nu), z - \nu \rangle \geq 0.$$

(5) **co-coercive***, if there exists a constant $\mu > 0$ such that*

$$\langle N(\mathcal{B}\mu, T\mu) - N(\mathcal{B}\nu, T\nu), \mu - \nu \rangle \geq \mu ||\mu - \nu||^2.$$

We remark that if $z = u$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator $N(.,.)$. For $N(\mathcal{B}u, \mathcal{T}u) \equiv Tu$, Definition 2.6 reduces to the standard definition of g-partially relaxed strongly monotonicity, and g-co-coercivity of the operator. It can be shown that g-co-coercivity implies g-partially relaxed strongly monotonicity. This shows that partially relaxed strongly monotonicity is a weaker condition than co-coercivity.

3. Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the bivariational inequalities.
Using Lemma 2.1, one can show that the bivariational inequalities are equivalent to the fixed point problems.

Lemma 3.1. *The function $\mu \in \Omega$ is a solution of the bivariational inequality (2.4), if and only if, $\mu \in$ satisfies the relation*

$$g(\mu) = \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)], \tag{3.1}$$

where Π_Ω is the projection operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \Omega$ be a solution of the problem (2.4), then

$$\rho\langle N(\mathcal{B}\mu.\mathcal{T}) + g(\mu) - g(\mu), \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega,$$

from which, using Lemma 2.1, we obtain

$$g(\mu) = \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)],$$

the required result (3.1). □

Lemma 3.1 implies that the bivariational inequality (2.4) is equivalent to the fixed point problem (3.1). From the equation (3.1), we have

$$u = u - g(\mu) + \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)].$$

We define the function Φ associated with (3.1) as

$$\Phi(\mu) = \mu - g(\mu) + \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)], \tag{3.2}$$

To prove the unique existence of the solution of the problem (2.4), it is enough to show that the map Φ defined by (3.2) has a fixed point.

Theorem 3.1. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. If the bifunction $N(,..)$ is Lipschitz continuous with constant β and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta} \quad k < 1, \tag{3.3}$$

where

$$\theta = \rho\beta + k \tag{3.4}$$
$$k = \sqrt{1 - 2\sigma + \zeta^2} + \zeta. \tag{3.5}$$

then there exists a unique solution of the problem (2.4).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.4) are equivalent. Thus it is enough to show that the map $\Phi(\mu)$, defined by (3.2) has a fixed point.
For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned}\|\Phi(\mu) - \Phi(\nu)\| &= \|\mu - \nu - (g(\mu) - g(\nu))\| \\ &+\Pi_\Omega\|[\mu - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)] - \Pi_\Omega[\nu - \rho N(\mathcal{B}\nu, \mathcal{T}\nu)]\| \\ &\leq \|\mu - \nu - (g(\mu) - g(\nu))\|\end{aligned}$$

$$+\|g(\nu) - g(\mu) - \rho(N(\mathcal{B}\nu, \mathcal{T}\nu) - N(\mathcal{B}\mu, \mathcal{T}\mu))\|$$
$$\leq \|\mu - \nu - (g(\mu) - g(\nu))\|$$
$$+\|g(\nu) - g(\mu)\| + \rho\|(N(\mathcal{B}\nu, \mathcal{T}\nu) - N(\mathcal{B}\mu, \mathcal{T}\mu))\|$$
$$\leq \|\mu - \nu - (g(\mu) - g(\nu)\| + (\rho\beta + \zeta)\|\nu - \mu\|. \quad (3.6)$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned} \|\mu - \nu - (g(\mu) - g(\nu))\|^2 &\leq \|\mu - \nu\|^2 - 2\langle g(\mu) - g(\nu), \mu - \nu\rangle + \zeta^2\|g(\mu) - g(\nu)\|^2 \\ &\leq (1 - 2\sigma + \zeta^2)\|\mu - \nu\|^2. \end{aligned} \quad (3.7)$$

From (3.6) and (3.7), we have

$$\|\Phi(\mu) - \Phi(\nu)\| \leq \left\{\sqrt{(1 - 2\sigma + \zeta^2)} + \rho\beta + \zeta\right\}\|\mu - \nu\| = \theta\|\mu - \nu\|,$$

where θ and k are defined by the relation (3.4) and 3.5), respectively.
From (3.3), it follows that $\theta < 1$, which implies that the map $\Phi(u)$ defined by (3.2) has a fixed point, which is the unique solution of (2.4). □

This alternative equivalent formulation (3.1) is used to suggest the following three-step iterative methods for solving the problem (2.4).

Algorithm 3.1. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = (1 - \gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \Pi_\Omega[g(\mu_n) - \rho N(\mathcal{B}\mu_n, \mathcal{T}\mu_n)]\} \quad (3.8)$$
$$w_n = (1 - \beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \Pi_\Omega[g(y_n - \rho N(\mathcal{B}y_n), \mathcal{T}y_n)]\} \quad (3.9)$$
$$\mu_{n+1} = (1 - \alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \Pi_\Omega[w_n - \rho N(\mathcal{B}w_n, \mathcal{T}w_n)]\}. \quad (3.10)$$

which are known as Noor iterations, which contain Ishikawa(two-set) iterations and Mann iteration(one-step) as special cases.
We now study the convergence analysis of Algorithm 3.1, which is the main motivation for our next result.

Theorem 3.2. *Let the operator g satisfy all the assumptions of Theorem 3.1. Then the approximate solution $\{u_n\}$ obtained from Algorithm 3.1 converges to the exact solution $\mu \in \Omega$ of the bivariational inequality (2.4) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \Omega$ of the bivariational inequalities (2.4). Let $\mu \in \Omega$ be the unique solution of (2.4). Then, using Lemma 3.1, we have

$$\mu = (1 - \alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)]\} \quad (3.11)$$
$$= (1 - \beta_n)\mu + \beta_n\{\mu - g(\mu) + \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)]\} \quad (3.12)$$
$$= (1 - \gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)]\}. \quad (3.13)$$

From (3.10) and (3.11), we have

$$\begin{aligned}
\|\mu_{n+1}-\mu\| &= \|(1-\alpha_n)(\mu_n-\mu)+\alpha_n(w_n-\mu-(g(w_n)-g(\mu)))\\
&+ \alpha_n(\Pi_{\Omega}[g(w_n)-\rho N(\mathcal{B}w_n,\mathcal{T}w_n]-\Pi_{\Omega)}[g(\mu)-\rho N(\mathcal{B}\mu,\mathcal{T}\mu])\}\|\\
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|w_n-\mu-(g(w_n)-g(\mu))\|\\
&+ \alpha_n\|\Pi_{\Omega(}[g(w_n)-\rho N(\mathcal{B}w_n,\mathcal{T}w_n)]-\Pi_{\Omega}[g(\mu_n)-\rho N(\mathcal{B}\mu_n,\mathcal{T}\mu_n\}\|\\
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|w_n-\mu-(g(w_n)-g(\mu))\|\\
&+ \alpha_n\|g(w_n)-g(\mu)-\rho(N(\mathcal{B}w_n,\mathcal{T}w_n)-N(\mathcal{B}\mu,\mathcal{T}\mu)\|\\
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\|w_n-\mu-(g(w_n)-g(\mu))\|\\
&+ \alpha_n(\|g(w_n)-g(\mu)\|+\rho\|(N(\mathcal{B}w_n,\mathcal{T}w_n)-N(\mathcal{B}\mu,\mathcal{T}\mu)\|\\
&\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n(k+\rho\beta)\|w_n-\mu\|\\
&= (1-\alpha_n)\|u_n-\mu\|+\alpha_n\theta\|w_n-\mu\|, \qquad (3.14)
\end{aligned}$$

where θ is defined by (3.4).
In a similar way, from (3.8) and (3.12), we have

$$\begin{aligned}
\|w_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+2\beta_n\theta\|y_n-\mu-(g(y_n)-g(\mu))\|\\
&+ \beta_n\|y_n-\mu-\rho(N(\mathcal{B}y_n,\mathcal{T}y_n-N(\mathcal{B}\mu,\mathcal{T}\mu))\|\\
&\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|y_n-\mu\|, \qquad (3.15)
\end{aligned}$$

where θ is defined by (3.3).
From (3.8) and (3.13), we obtain

$$\begin{aligned}
\|y_n-\mu\| &\leq (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\|\\
&\leq (1-(1-\theta)\gamma_n)\|\mu_n-\mu\|\leq\|\mu_n-\mu\|. \qquad (3.16)
\end{aligned}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}
\|w_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\|\\
&= (1-(1-\theta)\beta_n)\|\mu_n-\mu\|\leq\|\mu_n-\mu\|. \qquad (3.17)
\end{aligned}$$

From the above equations, we have

$$\begin{aligned}
\|\mu_{n+1}-\mu\| &\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\|\\
&= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\|\leq\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|.
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$,, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently, the sequence $\{u_n\}$ converges strongly to μ. From (3.16), and (3.17), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

One can rewrite (3.1) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\frac{\mu+\mu}{2}) - \rho N(\mathcal{B}(\frac{\mu+\mu}{2}), \mathcal{T}(\frac{\mu+\mu}{2}))]. \quad (3.18)$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.4).

Algorithm 3.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_\Omega[(\frac{\mu_n+\mu_{n+1}}{2}) - \rho N((\mathcal{B}(\frac{\mu_n+\mu_n}{2}), \mathcal{T}(\frac{\mu_n+\mu_{n+1}}{2}))].$$

To implement the implicit method, one uses the predictor-corrector technique to have the following new two-step method for solving the problem (2.4).

Algorithm 3.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mu_n - g(\mu_n) + \Pi_\Omega[\mu_n - \rho N(\mathcal{B}\mu_n, \mathcal{T}\mu_n)]\\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_\Omega\Big[g(\frac{\omega_n+\mu_n}{2}) - \rho N((\mathcal{B}(\frac{\omega_n+\mu_n}{2}), \mathcal{T}(\frac{\omega_n+\omega_n}{2}))\Big].\end{aligned}$$

For a parameter ξ, one can rewrite the (3.1) as

$$\mu = \mu - g(\mu) + \Pi_\Omega[g((1-\xi)\mu + \xi\mu)) - \rho N(\mathcal{B}(1-\xi)\mu + \xi\mu), \mathcal{T}((1-\xi)\mu + \xi\mu)].$$

This equivalent fixed point formulation enables us to suggest the following inertial method for solving the problem (2.4).

Algorithm 3.4. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\mu_{n+1} = \mu_n - g(\mu_n)\\ &+\Pi_\Omega[g((1-\xi)\mu_n + \xi\mu_{n-1})) - \rho N((\mathcal{B}(1-\xi)\mu_n + \xi\mu_{n-1}), \mathcal{T}(1-\xi)\mu_n + \xi\mu_{n-1}))].\end{aligned}$$

We now suggest some multi-step inertial methods for solving the bivariational inequalities (2.4).

Algorithm 3.5. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned}&\omega_n = \mu_n - \theta_n(\mu_n - \mu_{n-1}),\\ &y_n = (1-\gamma_n)\omega_n + \gamma_n\Big\{\omega_n - g(\omega_n)\\ &\quad+\Pi_\Omega\Big[g(\frac{\omega_n+\mu_n}{2}) - \rho N(\mathcal{B}(\frac{\omega_n+\mu_n}{2}), \mathcal{T}(\frac{\omega_n+\mu_n}{2}))\Big]\Big\},\\ &z_n = (1-\beta_n)y_n + \beta_n\Big\{y_n - g(y_n)\\ &\quad+\Pi_\Omega\Big[g(\frac{y_n+\omega_n+\mu_n}{3}) - \rho N(\mathcal{B}(\frac{y_n+\omega_n+\mu_n}{3}), \mathcal{T}\frac{y_n+\omega_n+\mu_n}{3}))\Big]\Big\},\\ &\mu_{n+1} = (1-\alpha_n)z_n + \alpha_n\Big\{z_n - g(z_n)\\ &\quad+\Pi_\Omega\Big[g(\frac{z_n+y_n+\omega_n+\mu_n}{4}) - \rho N(\mathcal{B}(\frac{y_n+z_n+\omega_n+\mu_n}{4}), \mathcal{T}(\frac{z_n+y_n+\omega_n+\mu_n}{4}))\Big]\Big\},\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1.$

Using the technique of Noor et al. [46,47], one can investigate the convergence analysis of these inertial projection methods. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving systems of bivariational inequalities, which is an interesting problem.
For $N(\mathcal{B}\mu, \mathcal{T}\mu) = \mathcal{T}\mu$, then Algorithm 3.5 reduces to

Algorithm 3.6. *For given* μ_0, μ_1, *compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned}
&\omega_n = \mu_n - \theta_n\left(\mu_n - \mu_{n-1}\right),\\
&y_n = (1-\gamma_n)\omega_n + \gamma_n\Big\{\omega_n - g(\omega_n) + \Pi_\Omega\Big[g(\frac{\omega_n + \mu_n}{2}) - \rho\mathcal{T}(\frac{\omega_n + \mu_n}{2})\Big]\Big\},\\
&z_n = (1-\beta_n)y_n + \beta_n\Big\{y_n - g(y_n) + \Pi_\Omega\Big[g(\frac{y_n + \omega_n + \mu_n}{3}) - \rho\mathcal{T}(\frac{y_n + \omega_n + \mu_n}{3})\Big]\Big\},\\
&\mu_{n+1} = (1-\alpha_n)z_n + \alpha_n\Big\{z_n - g(z_n)\\
&+\Pi_\Omega\Big[g(\frac{z_n + y_n + \omega_n + \mu_n}{4}) - \rho\mathcal{T}(\frac{z_n + y_n + \omega_n + \mu_n}{4})\Big]\Big\},
\end{aligned}$$

for solving the general variational inequality.
We note that the Algorithm 3.6 contains Noor iteration, Ishikawa iterations, Mann iteration, and Picard method as special cases. For the applications and various modifications of Noor iterations in various disciplines of pure and applied sciences such as Chaos, fractal geometry, logistic maps, green energy, imaging, signal processing, data sciences, machine learning, and information, see [3–6, 11, 23, 26, 57, 61] and the references therein.
From (3.1), we have

$$g(\mu) = \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)],$$

or equivalently

$$g(\mu) = \Pi_\Omega\Big[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu) + \eta(\mu - \mu)\Big], \tag{3.19}$$

where η is a constant, which can be used to suggest and consider the inertial type methods for solving the general bivariational inequalities (2.4).

Algorithm 3.7. *For given* μ_0, μ_1, *compute* μ_{n+1} *by the recurrence relation*

$$\begin{aligned}
y_n &= \mu_n - \theta_n\left(\mu_n - \mu_{n-1}\right),\\
\mu_{n+1} &= \mu_n + g(\mu_n) + \Pi_\Omega\Big[g(y_n) - \rho N(\mathcal{B}y_n, \mathcal{T}y_n)\Big]
\end{aligned}$$

which is a two-step inertial method.
In this section, we have tried to convey the idea that the alternative equivalent fixed point formulation of the bivariational inequalities can be written several ways to suggest and investigate several extragradient type computational methods. It is expected to play an important role in developing

numerically efficient methods for variational inequalities, equilibrium problems and nonconvex optimization problems.

4. Auxiliary Principle Technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [15] as developed in [33,40,56,57,60]. Noor [33] has modified the auxiliary principle technique involving an arbitrary function M.. For the readers' convenience, we recall some basic properties of the modified distance functions, which were introduced by Noor [33] in 1992 and discussed in Patrikson [55]. For a strongly monotone operator M, we define the distance function as

$$\begin{aligned} \mathcal{M}(v,u) &= M(v) - M(u), v-u\rangle, \forall u,v \in K. \\ &\geq \zeta\|v-u\|^2, \quad u,v \in K, \end{aligned} \tag{4.1}$$

where ζ is the strongly monotonicity constant. It is important to emphasize that various types of function M give different modified distance functions. We give the following important examples of some practically important types of operator M and their corresponding distance functions, see [33]

Examples

(1) If $\Phi(v) = \|v\|^2$, then $\mathcal{M}(v,u) = \|v-u\|'$ which is the squared Euclidean distance (SE).

(2) If $\Phi(v) = \sum_{i=1}^{n} a_i \log(v_i)$, which is known as Shannon entropy, then its corresponding $\mathcal{M}$ distance function is given as

$$\mathcal{M}(v,u) = \sum_{i=1}^{n}\left(v_i \log(\frac{v_i}{u_i}) + u_i - v_i\right),$$

This distance is called Kullback-Leibler distance (KL) and as become a very important tool in several areas of applied mathematics such as machine learning.

(3) If $\Phi(v) = -\sum_{i=1}^{n} \log(v_i)$, which is called Burg entropy, then its corresponding Bregman distance function is given as

$$\mathcal{M}(v,u) = \sum_{i=1}^{n}\left(\log(\frac{v_i}{u_i}) + \frac{v_i}{u_i} - 1\right).$$

This is called Itakura-Saito distance (IS),which is very important in information theory.

It is a challenging problem to explore the applications of modified $\mathcal{M}$ distance functions for other types of nonconvex functions as biconvex, k-convex functions and harmonic functions.

We apply the auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the problem (2.4).
For a given $\mu \in \Omega$ satisfying (2.4), find $w \in \Omega$ such that

$$\begin{aligned}&\langle \rho N(\mathcal{B}(w+\eta(\mu-w)),\mathcal{T}(w+\eta(\mu-w))+g(w)-g(u)),\nu-g(w)\rangle\\&+\langle M(w)-M(\mu),\nu-w\rangle \geq 0, \quad \forall \nu \in \Omega, \qquad (4.2)\end{aligned}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (4.2) is called the auxiliary bifunction variational inequality.
If $w = \mu$, then w is a solution of (2.4). This simple observation enables us to suggest the following iterative method for solving (2.4).

Algorithm 4.1. *For a given $\mu_0 \in \Omega$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(\mathcal{B}(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})),\mathcal{T}\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))\\&\quad +g(\mu_{n+1})-g(\mu),\nu-g(\mu_{n+1})\rangle\\&\quad +\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega. \qquad (4.3)\end{aligned}$$

Algorithm 4.1 is called the hybrid proximal point algorithm for solving the bivariational inequalities (2.4).
Special Cases: We now discuss some special cases.
(I). For $\eta = 0$, Algorithm 4.1 reduces to

Algorithm 4.2. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(\mathcal{B}(\mu_{n+1}),\mathcal{T}\mu_{n+1})+g(\mu_{n+1})-g(\mu_n),\nu-g(\mu_{n+1})\rangle\\&+\langle M(\mu_{n+1})-M(\mu),\nu-\mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega, \qquad (4.4)\end{aligned}$$

is called the implicit iterative methods for solving the problem (2.4).
(II). For $\eta = \frac{1}{2}$, Algorithm 4.1 becomes:

Algorithm 4.3. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(\mathcal{B}(\frac{\mu_{n+1}+\mu_n}{2}),\mathcal{T}(\frac{\mu_{n+1}+\mu_n}{2}))+g(\mu_{n+1})-g(\mu),\nu-g(\mu_{n+1})\rangle\\&+\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega,\end{aligned}$$

is known as the mid-point proximal method for solving the problem (2.4).
For the convergence analysis of Algorithm 4.2, we need the following concepts.

Definition 4.1. *An operator g is said to be pseudomonotone, if*

$$\langle \rho N(\mathcal{B}\mu,\mathcal{T}\mu)+g(\mu)-g(\nu),\nu-g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega,$$

implies that

$$-\langle \rho N(\mathcal{B}\nu,\mathcal{T}\nu)+g(\nu)-g(\mu),g(\mu)-\nu\rangle \geq 0, \quad \forall \nu \in \Omega.$$

Theorem 4.1. *Let the operator g be a pseudo-monotone. Let the approximate solution μ_{n+1} obtained in Algorithm 4.2 converge to the exact solution $\mu \in \Omega$ of the problem (2.4). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu - \mu_n\|. \tag{4.5}$$

Proof. Let $\mu \in \Omega$ be a solution of the problem (2.4). Then

$$-\langle \rho N(\nu, \mathcal{T}\nu) + g(\nu) - g(\mu), g(\mu) - \nu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{4.6}$$

since the operator g is pseudomonotone.
Taking $\nu = \mu_{n+1}$ in (4.6), we obtain

$$-\langle \rho N(\mu_{n+1}, \mathcal{T}\mu_{n+1}) + g(\mu_{n+1}) - g(\mu), g(\mu) - \mu_{n+1}\rangle \geq 0. \tag{4.7}$$

Setting $\nu = \mu$ in (4.4), we have

$$\begin{aligned} &\langle \rho N(\mu_{n+1}, \mathcal{T}\mu_{n+1}) + g(\mu_{n+1}) - g(\mu), \mu - g(\mu_{n+1})\rangle \\ &+\langle M(\mu_{n+1}) - M(\mu_{n+1}), \mu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega, \end{aligned} \tag{4.8}$$

Combining (4.8), (4.7) and (4.6), we have

$$\begin{aligned} \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle &\geq -\langle \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}\mu_{n+1}), \mu - u_{n+1}\rangle \\ &\geq 0. \end{aligned} \tag{4.9}$$

From the equation (4.9), we have

$$\begin{aligned} 0 &\leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle = \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strong monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (4.5). □

Theorem 4.2. *Let H be a finite dimensional space and all the assumptions of Theorem 4.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 4.2 converges to the exact solution $\mu \in \Omega$ of (2.4).*

Proof. Let $\mu \in \Omega$ be a solution of (2.4). From (4.5), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi \sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\| \leq \zeta \|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{4.10}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega$. Replacing w_n by μ_{n_j} in (4.4), taking the limit $n_j \longrightarrow \infty$ and using (4.10), we have

$$\langle \rho\hat{\mu} + g(\hat{u}) - \hat{\mu}, \nu - g(\hat{\mu})\rangle \geq 0, \qquad \forall \nu \in \Omega,$$

which implies that $\hat{u}$ solves the problem (2.4) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (\mu_n) = \hat{\mu}.$$

the required result. □

We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the bivariational inequalities (2.4).
For a given $\mu \in \Omega$ satisfying (2.4), find $w \in \Omega$ such that

$$\begin{aligned}&\langle \rho N(\mathcal{B}(w + \eta(\mu - w)), \mathcal{T}w + \eta(\mu - w)), \nu - g(w)\rangle \\ &\quad + \langle M(w) - M(\mu) + \alpha(\mu - \mu), \nu - w\rangle \geq 0, \quad \forall \nu \in \Omega,\end{aligned} \tag{4.11}$$

where $\rho > 0, \eta, \alpha \in [0, 1]$ are constants and M is a nonlinear operator.
Clearly $w = \mu$, implies that w is a solution of (2.4). This simple observation enables us to suggest the following iterative method for solving (2.4).

Algorithm 4.4. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho N(\mathcal{B}(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \mathcal{T}(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle \\ &\quad + \langle M(\mu_{n+1}) - M(\mu_n) + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega.\end{aligned}$$

Algorithm 4.4 is called the hybrid proximal point algorithm for solving the bivariational inequalities (2.4). For $\alpha = 0$, Algorithm 4.4 is exactly
the$Algorithm$4.1.
$IfM = 0$, then Algorithm 4.4 reduces to:

Algorithm 4.5. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho N(\mathcal{B}\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \mathcal{T}(\mu_{n+1} + \eta(\mu_n - \mu_{n+1})), \nu - g(\mu_{n+1})\rangle$$
$$+\alpha\langle(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall \nu \in \Omega,$$

which is called the inertial iterative method for solving the bivariational inequalities (2.4).

Remark 4.1. *For different and suitable choices of the parameters ρ, η, α, operators $\mathcal{B}, \mathcal{T}, g, M$ and convex-valued sets, one can recover new and known iterative methods for solving bifunction variational inequalities, bicomplementarity problems and related optimization problems. Using the technique and ideas of Theorem 4.1 and Theorem 4.2, one can analyze the convergence of Algorithm 4.4 and its special cases.*

.

5. Dynamical Systems Technique

This section explores the application of dynamical systems techniques to solve bivariational inequalities. The framework of projected dynamical systems, introduced in the context of variational inequalities by Dupuis and Nagurney [14], provides a foundational approach. These dynamical systems are characterized as first-order initial value problems. This connection establishes a powerful methodology, enabling the study of variational inequalities and a broad class of nonlinear problems from pure and applied sciences through the lens of differential equations. The utility of this paradigm is further demonstrated by its role in developing efficient numerical algorithms for solving variational inequalities and associated optimization problems. Building on this foundation, we now propose novel iterative methods for solving bivariational inequalities.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = g(\mu) - \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)]. \tag{5.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (**??**) if and only if $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{5.2}$$

We now consider a dynamical system associated with the bivariational inequalities. Using the equivalent formulation (3.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{5.3}$$

where λ is a parameter. The system of type (5.3) is called the projection dynamical system associated with the problem (2.4). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness, and continuous dependence of the solution of (2.4) can be studied.
The equilibrium point of the dynamical system (5.3) is defined as follows.

Definition 5.1. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.3), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the bivariational inequality (2.4), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

We use the dynamical system (5.3) to suggest some iterative for solving the bivariational inequalities (2.4).

For simplicity, we take $\lambda = 1$. Thus the dynamical system (5.3) becomes

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega}\Big[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)\Big], \quad \mu(t_0) = \alpha, \tag{5.4}$$

which is an initial value problem.

The forward difference scheme is used to construct the implicit iterative method. Discretizing (5.4), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega}[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))], \tag{5.5}$$

where h is the step size.

Now, we can suggest the following implicit iterative method for solving the bivariational inequality (2.4).

Algorithm 5.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega}\left[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1})) - \frac{\mu_{n+1} - \mu_n}{h}\right],$$

This is an implicit method, which is equivalent to the following two-step method.

Algorithm 5.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega}[\mu_n - \rho N(\mathcal{B}\mu_n, \mathcal{T}\mu_n)] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega}\Big[g(\mu_n) - \rho N(\mathcal{B}\omega_n, \mathcal{T}\omega_n) - \frac{\omega_n - \mu_n}{h}\Big], \end{aligned}$$

Discretizing (5.4), we have

$$\frac{\mu_{n+1} - \mu_n}{h} = \mu_n - g(\mu_n) + \Pi_{\Omega}[g(\mu_{n+1}) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))], \tag{5.6}$$

where h is the step size.

This helps us to suggest the following implicit iterative method for solving the problem (2.4).

Algorithm 5.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - g(\mu_n) + \Pi_{\Omega}[g(\mu_n) - \rho N(\mathcal{B}\mu_n, \mathcal{T}(\mu_n))] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega}\Big[g(\omega_n) - \rho N(\mathcal{B}\omega_n, \mathcal{T}(\omega_n))\Big]. \end{aligned}$$

We now introduce the second order dynamical system associated with the bivariational inequality (2.4). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} + g(\mu) = \lambda\{\Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)\},$$
$$\mu(a) = \alpha, \quad \mu(b) = \beta, \tag{5.7}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (5.7) is indeed a second order boundary value problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (5.7) is defined as follows.

Definition 5.2. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.7) if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the bivariational inequality (2.4), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

We can rewrite (5.7) as follows:

$$g(\mu) = \Pi_\Omega\Big[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu) + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\Big]. \tag{5.8}$$

For $\lambda = 1$, the problem (5.7) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} + g(\mu) = P_\Omega\Big[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)\Big], \quad \mu(a) = \alpha, \mu(b) = \beta. \tag{5.9}$$

The problem (5.9) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various areas is fruitful from numerical analysis in developing implementable numerical methods for finding the approximate solutions of the variational inequalities. Consequently, we can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the bivariational inequalities and related optimization problems.

We discretize the second-order dynamical systems (5.9) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_n)$$
$$= \Pi_\Omega[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))], \tag{5.10}$$

where h is the step size.

If $\gamma = 1, h = 1$, then, from equation(5.10) we have

Algorithm 5.4. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_\Omega[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))],$$

which is the extragradient method, which is equivalent to:

Algorithm 5.5. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_\Omega[g(\mu_n) - \rho N(\mathcal{B}y_n, \mathcal{T}(y_n))], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant. In a similar way, we have the following two-step method.

Algorithm 5.6. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_\Omega[g(y_n) - \rho N(\mathcal{B}y_n, \mathcal{T}(y_n))], \end{aligned}$$

which is also called the double projection method for solving the bivariational inequalities (2.4). We discretize the second-order dynamical systems (5.3 using central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_{n+1}) = \Pi_\Omega\Big[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))\Big],$$

where h is the step size.
Using this discrete form, we can suggest the following iterative method for solving the bivariational inequalities (2.4).

Algorithm 5.7. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\mu_{n+1} = \mu_n - g(\mu_{n+1}) \\ &+\Pi_\Omega\Big[g(\mu_{n+1}) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1})) - \gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h}\Big]. \end{aligned}$$

Algorithm 5.7 is called the hybrid inertial proximal method for solving the bivariational inequalities and related optimization problems. This is a new iterative method.
We now consider the third order dynamical systems associated with the bivariational inequalities of the type (2.4). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\begin{aligned} &\gamma \frac{d^3\mu}{dt^3} + \zeta \frac{d^2\mu}{dt^2} + \xi \frac{d\mu}{dt} + g(\mu) = \Pi_\Omega[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}(\mu))], \\ &u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = 0 \end{aligned} \tag{5.11}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem (5.11) is called a third-order dynamical system associated with bivariational inequalities (2.4).
The equilibrium point of the dynamical system (5.11) is defined as follows.

Definition 5.3. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (5.7), if,*

$$\gamma \frac{d^3\mu}{dt^3} + \zeta \frac{d^2\mu}{dt^2} + \xi \frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the bifunction variational inequality (2.4), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
Consequently, the problem (5.3) can be equivalently written as

$$g(\mu) = \Pi_\Omega\Big[\mu - \rho N(\mathcal{B}\mu, \mathcal{T}\mu) + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\Big]. \tag{5.12}$$

We discretize the third-order dynamical systems (5.11) using central finite difference and backward difference schemes to have

$$\begin{aligned}&\gamma\frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} + \zeta\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\\ &+\xi\frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h} + g(\mu_n) = \Pi_\Omega[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1}))],\end{aligned} \tag{5.13}$$

where h is the step size.
If $\gamma = 1, h = 1, \zeta = 1, \xi = 1$, then, from equation(5.13) after adjustment, we have

Algorithm 5.8. *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$u_{n+1} \quad = \quad \mu_n - g(\mu_n) + \Pi_\Omega\Big[g(\mu_n) - \rho N(\mathcal{B}\mu_{n+1}, \mathcal{T}(\mu_{n+1})) + \frac{\mu_{n-1} - 3u_n}{2}\Big],$$

which is an inertial type hybrid iterative method for solving the bivariational inequalities (2.4).

Remark 5.1. *For appropriate and suitable choice of the bifunction $N(.,.)$, operator g, convex set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving bivariational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [**?**, 47, 48], one can discuss the convergence analysis of the proposed methods.*

6. Sensitivity Analysis

In recent years, variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research, and engineering sciences. The behavior of such problems as a result of changes in the problem data is always of concern, which is known as the sensitivity analysis. Dafermos [13] investigated the sensitivity analysis of the variational inequalities using the fixed point theory. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors; sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from a mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied and can stimulate new ideas and techniques for solving the problems due to these and other reasons. In this section, we study the sensitivity analysis of the bifunction variational inequalities, that is, examining how solutions of such problems change when the data of the problems are changed.

We now consider the parametric versions of the problem (2.4). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $g(\mu,\lambda)$ be the given identity operator defined on $\mathcal{H}\times\mathcal{H}\times M$ and take values in $\mathcal{H}\times\mathcal{H}$.. From now onward, we denote $g_\lambda(.)\equiv g(.,\lambda)$ and $\mathcal{T}_\lambda(.)\equiv\mathcal{T}(.,\lambda)$, respectively, unless otherwise specified.

The parametric bivariational inequality problem is to find $(\mu,\lambda)\in\mathcal{H}\times M$ such that

$$\langle\rho N(\mathcal{B}_\lambda\mu,\mathcal{T}_\lambda(\mu))+g_\lambda(\mu)-g_\lambda(\mu),\nu-g_\lambda(\mu)\rangle\geq 0,\quad\forall w,\nu\in\Omega. \tag{6.1}$$

We also assume that, for some $\overline{\lambda}\in M$, problem (6.1) has a unique solution $\overline{\mu}$. From Lemma 3.1, we see that the parametric bifunction variational inequalities are equivalent to the fixed point problem:

$$g_\lambda(\mu)=\Pi_\Omega[g_\lambda(g(\mu)-\rho N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu)],$$

or equivalently

$$\mu=\mu-g_\lambda(\mu)+\Pi_\Omega[g_\lambda(\mu)-\rho N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu))].$$

We now define the mapping F_λ associated with the problem (6.1) as

$$\begin{aligned}F_\lambda(\mu)\ &=\ \mu-g_\lambda(\mu)\\ &\quad+\Pi_\Omega[g_\lambda(\mu)-\rho N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu))],\quad\forall(\mu,\lambda)\in X\times M.\end{aligned} \tag{6.2}$$

We leverage this equivalence to conduct a sensitivity analysis for the bivariational inequalities. It is assumed that for a given parameter $\overline{\lambda}\in M$, the problem (6.1) admits a solution $\overline{\mu}$, and that the set X constitutes the closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. Our objective is to derive conditions ensuring that for every λ in a neighborhood of $\overline{\lambda}$, the problem (6.1) possesses a unique solution $z(\lambda)$ in a neighborhood of $\overline{\mu}$, and that the resulting function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 6.1. *Let $\mathcal{T}_\lambda(.)$ be an operator on $X\times M$. Then, the bifunction $N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu)$ is said to:*
(a) Locally strongly monotone with constant $\sigma>0$, if

$$\langle N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu))-N(\nu,\mathcal{T}_\lambda(\nu)),\mu-\nu\rangle\geq\sigma\|\mu-\nu\|^2,\quad\forall\lambda\in M,w,\mu,\nu\in X.$$

(b) Locally Lipschitz continuous with constant $\zeta>0$, if

$$\|N(\mathcal{B}_\lambda(\mu),\mathcal{T}_\lambda(\mu))-N(\nu,\mathcal{T}_\lambda(\nu))\|\leq\zeta\|\mu-\nu\|,\quad\forall\lambda\in M,w,\mu,\nu\in X.$$

We consider the case when the solutions of the parametric bivariational inequality (6.1) lie in the interior of X. Following the ideas of Noor [39], we consider the map $F_\lambda(\mu)$ as defined by (6.2). We have to show that the map $F_\lambda(\mu)$ has a fixed point, which is a solution of the bivariational inequality (6.1). First of all, we prove that the map $F_\lambda(\mu)$, defined by (6.2), is a contraction map with respect to μ uniformly in $\lambda\in M$.

Lemma 6.1. *Let* $\}_\lambda(.)$ *be locally strongly monotone with constants* $\sigma > 0$ *and locally Lipschitz continuous with constants* $\zeta > 0$ *respectively. If bifunction* $N(,.,)$ *is locally Lipschitz continuous with constant* $\beta > 0$*,, we have*

$$\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

for

$$\rho < \frac{1-k}{\beta} \quad k < 1, \tag{6.3}$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \zeta^2} + \zeta + \rho\beta\right\} = \{k + \rho\beta\} \tag{6.4}$$

and

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \zeta. \tag{6.5}$$

Proof. In order to prove the existence of a solution of (6.1), it is enough to show that the mapping $F_\lambda(\mu)$, defined by (6.2), is a contraction mapping.
For $\mu_1 \neq \mu_2 \in \mathcal{H}$, we have

$$\begin{aligned}
&\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&+\|\Pi_\Omega[g_\lambda(\mu_1) - \rho(\mathcal{B}_\lambda N(\mu_1, \mathcal{T}_\lambda(\mu_1)] - \Pi_\Omega[g_\lambda(\mu_2) - \rho N(\mathcal{B}_\lambda\mu_2, \mathcal{T}_\lambda(\nu_2)]\| \\
\leq\ &\|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&+\|g_\lambda(\mu_1) - g_\lambda(\mu_2) - \rho\Big(N(\mathcal{B}_\lambda\mu_1, (\mathcal{T}_\lambda(\mu_1) - N(\mathcal{B}_\lambda\mu_2, \mathcal{T}_\lambda(\mu_2)\Big)\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + \|g_\lambda(\mu_1) - g_\lambda(\mu_2)\| \\
&+\rho\|N(\mathcal{B}_\lambda, \mathcal{T}_\lambda(\mu)) - N(\mathcal{B}_\lambda, \mathcal{T}_\lambda(\mu))\| \\
&\leq +\|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| + (\zeta + \rho\beta)\|\mu_1 - \mu_2)\|,
\end{aligned} \tag{6.6}$$

assuming the local Lipschitz continuity of the operator g_λ with constant $\zeta > 0$.
Since the operator g_λ is locally strongly monotone with constant $\sigma > 0$ and locally Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2)\|^2 \ &\leq\ \|u_1 - u_2\|^2 - 2\langle g_\lambda(\mu_1) - g_\lambda(\mu_2), \mu_1 - \mu_2\rangle \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\|^2 \\
&\leq\ (1 - 2\sigma + \zeta^2)\|\mu_1 - \mu_2\|^2.
\end{aligned} \tag{6.7}$$

From (6.5), (6.6) and (6.7), we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \ &\leq\ \left\{\zeta + \sqrt{(1 - 2\sigma + \zeta^2)} + \rho\beta\right\}\|\mu_1 - \mu_2\| \\
&=\ \theta\|\mu_1 - \mu_2\|,
\end{aligned}$$

where

$$\theta = k + \rho\beta.$$

From (6.3), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(\mu)$, defined by (6.2), is a contraction mapping and consequently it has a fixed point, which belongs to $\Omega(\mu)$ satisfying the bivariational inequality (6.1), the required result. □

Remark 6.1. *From Lemma 3.1, we see that the map $F_\lambda(\mu)$ defined by (6.2) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = F_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric bifunction variational inequality (6.1). Again using Lemma 3.1, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(\mu)$ and it is also a fixed point of $F_{\overline{\lambda}}(\mu)$. Consequently, we conclude that*

$$\mu(\overline{\lambda}) = \overline{\mu} = F_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 3.1, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric bivariational inequality (6.1) using the technique of Dafermos [13] and Noor [39].

Lemma 6.2. *Assume that the bifunction $N(.,.)$ and the operator $g_\lambda(.)$ are locally Lipschitz continuous with respect to the parameter λ.. If the operator $g_\lambda(.)$ is locally Lipschitz continuous and the map $\lambda \rightarrow P_{K_\lambda}u$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (6.2) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

We now state and prove the main result of this paper and it is the motivation for our next result.

Theorem 6.1. *Let $\overline{\mu}$ be the solution of the parametric bivariational inequality (6.1) for $\lambda = \overline{\lambda}$. Let $N(.,.), g_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operator for all $\mu, \nu \in X$.. If the map $\lambda \rightarrow \Pi_{\Omega_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric bivariational inequality (6.2) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma 6.1, Lemma 6.2 and Remark 6.1. □

7. Generalizations and Future Research

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $\mathcal{A}(\cdot) : \mathcal{H} \times \mathcal{H} \Longrightarrow \mathcal{H}$ be a maximal monotone operator. For given bifunction $N(.,.) : \mathcal{H} \times \mathcal{H} \Longrightarrow \mathcal{H}$ and nonlinear operators $\mathcal{T}.\mathcal{B}, g : \mathcal{H} \Longrightarrow \mathcal{H}$ consider the problem of finding $\mu \in \mathcal{H}$ such that

$$\begin{aligned} 0 &\in N(\mathcal{B}\mu, \mathcal{T}(\mu)) + \mathcal{A}(g(\mu)) \\ &= N(\mathcal{B}\mu, \mathcal{T}(\mu)) + g(\mu) - g(\mu) + \mathcal{A}(g(\mu)), \end{aligned} \tag{7.1}$$

which is called the general bivariational inclusion. A number of problems arising in structural analysis, mechanics and economics can be studied in the framework of the general quasi bifunction variational inclusions in a unified manner.

We remark that, if f $\mathcal{A}(\cdot,\mu) = \partial\varphi(\cdot,\mu) : \mathcal{H} \Longrightarrow R \cup \{+\infty\}$, the subdifferential of a convex, proper and lower semi-continuous function $\varphi(\cdot,\mu)$, then problem (7.1) is equivalent to finding $\mu \in H$ such that

$$\langle N(\mathcal{B}\mu, \mathcal{T}(\mu)), g(\nu) - g(\mu)\rangle + \varphi(g(\nu)) - \varphi(g(\mu)) \geq 0, \quad \forall \quad \nu \in H, \tag{7.2}$$

which is called the general mixed bivariational inequality.
We also need the following concept.
Definition 7.1 If $\mathcal{A}$ is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with $\mathcal{A}$ is defined by

$$\mathcal{J}_{\mathcal{A}}(\mu) = (I + \rho\mathcal{A})^{-1}(\mu), \quad \forall \mu \in \mathcal{H},$$

where I is the identity operator.
We now establish the equivalence between the problem (7.1) and the fixed-point problem.

Lemma 7.1. *The function $\mu \in \mathcal{H}$ is a solution of the general bivariational inclusion (7.1), if and only if, $\mu \in \mathcal{H}$ satisfies the relation*

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)], \tag{7.3}$$

where $\mathcal{J}_{\mathcal{A}}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $\mu \in \mathcal{H}$ be a solution of (7.1), then, for a constant ρ,

$$\begin{aligned} \rho N(\mathcal{B}\mu, \mathcal{T}\mu) \quad &+ \quad \rho\mathcal{A}(g(\mu)) \ni 0, \\ &\Longleftrightarrow \\ -g(\mu) + \rho N(\mathcal{B}\mu, \mathcal{T}\mu) \quad &+ \quad g(\mu) + \rho\mathcal{A}(g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ g(\mu) = \quad &\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)], \end{aligned}$$

the required (7.3). □

Lemma 7.1 implies that the problem (7.1) is equivalent to the fixed point problem (7.3).
Rewriting (7.3), we have

$$\mu = \mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)]. \tag{7.4}$$

We now define the mapping F associated with general bivariational inclusion.

$$F(\mu) = \mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho N(\mathcal{B}\mu, \mathcal{T}\mu)], \tag{7.5}$$

which is used to study the existence of the unique solution of the general bivariational inclusion.

Using the technique of this paper, one can consider the existence of the solution of the problem (7.1), iterative methods, associated dynamical systems, sensitivity analysis, merit functions, error bounds and other aspects of the quasi bivariational inclusions. These are interesting and open problems, which require further research efforts.

Conclusion: In this paper, we have introduced and studied new classes of general bivariational inequalities. Several new concepts have been defined. We have established the equivalence between the general bivariational inequalities and fixed point problems. This alternative formulation has been used to study the existence of the unique solution and to suggest some new multi-step iterative methods for solving the bivariational inequalities and their special cases. These new methods include extragradient methods, modified double projection methods, and multi-step inertial methods, which are suggested using the techniques of projection methods, auxiliary techniques, and dynamical systems. Convergence analysis of the proposed method is discussed for suitable weaker conditions. It is an open problem to compare these proposed methods with other methods. Applying the technique and ideas discussed in [3, 4, 6, 23], can one explore the Julia set and Mandelbrot set in Noor orbit using the Noor (three-step) iterations in fixed point theory. It is an open interesting problem to discuss the applications of the bivariational inequalities and their variant forms in fuzzy set theory, stochastic control, quantum calculus, fractal, fractional, random traffic equilibrium, artificial intelligence, computer science, control engineering, management science, and operations research.

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