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Analytic Estimates for Bi-Univalent Functions Associated with a New Operator Involving the q–Rabotnov Function

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Abstract. In this paper, we introduce and analyze a new subclass of bi-univalent functions associated with a differential operator constructed from the q–Rabotnov function. Motivated by the framework of q–calculus and its interplay with geometric function theory, the proposed operator is defined through convolution with q–Rabotnov kernels, thereby generating novel analytic structures. By applying the subordination principle, we establish sharp coefficient estimates for the initial Taylor–Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, and derive Fekete–Szegö type inequalities for the class under consideration. The results presented here extend and generalize several recent contributions in the theory of bi-univalent functions, highlighting the central role of q–special functions in the development of new operator-based subclasses. These findings provide deeper insights into the analytic behavior of bi-univalent mappings and suggest further applications of q–calculus in operator theory, convolution structures, and complex analysis.

1. Introduction

Geometric function theory is a well-established branch of complex analysis that investigates the analytic, geometric, and structural characteristics of functions which are analytic and univalent in the open unit disk $\mathscr{O} = \{z \in \mathbb{C} : |z| < 1\}$. A major line of inquiry within this area concerns the identification and analysis of subclasses of analytic and bi-univalent functions, typically defined

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through tools such as subordination principles, convolution-type operators, and methods from fractional calculus. Core topics of interest include bounds for initial coefficients, growth and distortion properties, and Fekete–Szegö inequalities, all of which continue to attract significant attention. Their extension within the framework of q–calculus has further broadened the scope of these investigations.

The emergence of q–calculus, also known as the calculus of finite differences, has brought new depth to analytic function theory by offering a systematic way to construct q–analogues of classical operators and function families. This approach has facilitated the introduction of numerous subclasses exhibiting rich analytic and algebraic features. Moreover, q–calculus establishes profound connections with areas such as special functions, combinatorial theory, and orthogonal polynomials, thereby extending the reach of geometric function theory into discrete and fractional settings. Such flexibility highlights its importance in generating both refined theoretical developments and diverse applications (see, e.g., [14–18, 37, 44–50]).

The q–gamma function Γ_q, widely recognized as the natural q–analogue of Euler's classical gamma function, forms a cornerstone of modern q–calculus and serves as a fundamental tool in the construction of analytic operators. It is defined recursively (see [7, 10]) by

$$\Gamma_q(\kappa+1) \;=\; \frac{1-q^{\kappa}}{1-q}\,\Gamma_q(\kappa) = [\kappa]_q\,\Gamma_q(\kappa), \tag{1.1}$$

where $[\kappa]_q$ denotes the q–integer and is given by

$$[\kappa]_q = \begin{cases} \dfrac{1-q^{\kappa}}{1-q}, & 0<q<1,\;\; \kappa\in\mathbb{C}^* = \mathbb{C}\setminus\{0\}, \\ 1, & q\to 0^+,\;\; \kappa\in\mathbb{C}^*, \\ \kappa, & q\to 1^-,\;\; \kappa\in\mathbb{C}^*, \\ \displaystyle\sum_{n=0}^{\gamma-1} q^n, & 0<q<1,\;\; \kappa=\gamma\in\mathbb{N}. \end{cases}$$

This formulation illustrates that the q–gamma function Γ_q not only retains the fundamental structural properties of Euler's classical gamma function but also encodes a discrete deformation governed by the parameter q. As such, it provides a versatile unifying framework within which fractional-order operators and kernel-type generating functions can be extended and analyzed in the context of analytic function theory.

In close connection with Γ_q is the q–analogue of the Pochhammer symbol, also called the q–shifted factorial, defined by (see [10])

$$(\kappa;q)_n = \begin{cases} (1-\kappa)(1-\kappa q)\cdots(1-\kappa q^{n-1}), & n=1,2,3,\ldots, \\ 1, & n=0, \end{cases}$$

which admits the alternative representation

$$(\kappa;q)_n \;=\; \frac{(1-q)^n\,\Gamma_q(\kappa+n)}{\Gamma_q(\kappa)}, \qquad n>0.$$

This relation emphasizes the intrinsic link between the q–shifted factorial and the q–gamma function, a connection that underpins many convolution- and subordination-based operators in geometric function theory. In particular, it acts as a key building block in modeling q–extensions of bi-univalent function classes.

Parallel to these developments, Rabotnov-type kernels—first introduced by Rabotnov [2] in the study of linear viscoelasticity—have become essential in describing hereditary behaviors such as creep and relaxation. These kernels, often expressed through convolution operators involving Mittag–Leffler-type functions, provide precise representations of fractional-order operators in constitutive equations [3]. Their adaptability has established them as standard tools for modeling stress–strain relations with memory effects, and their strong ties to fractional calculus secure their central role in the mathematical analysis of viscoelastic materials and dynamical systems [4].

The analytic structure of Rabotnov-type kernels naturally encourages their extension into geometric function theory, particularly in synergy with the discrete setting of q–calculus. This interplay not only unites viscoelastic models with analytic operator theory but also enables the development of novel subclasses of analytic and bi-univalent functions. Such connections provide a rigorous mechanism for incorporating hereditary effects and nonlocal operators into analytic contexts, thereby facilitating the extension of classical results such as coefficient bounds, growth and distortion theorems, and Fekete–Szegö inequalities into new q–based frameworks.

Motivated by these ideas, Alsoboh et al. [13] recently constructed a new class of q–starlike functions associated with the q–analogue of Fibonacci numbers by means of subordination. Their approach revealed a fundamental relationship between the q–Fibonacci numbers ϑ_q and the corresponding q–Fibonacci polynomials, encoded through the generating function

$$\Omega(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2}. \tag{1.2}$$

They further showed that the q–Fibonacci numbers may be expressed explicitly as

$$\vartheta_q = \frac{1 - \sqrt{4q+1}}{2q}, \tag{1.3}$$

and that if

$$\Omega(z;q) = 1 + \sum_{n=1}^{\infty} \widehat{\mathbf{p}}_n z^n,$$

then the coefficients $\widehat{p_n}$ satisfy the recurrence

$$\widehat{\mathbf{p}}_n = \begin{cases} \vartheta_q, & n = 1, \\ (2q+1)\vartheta_q^2, & n = 2, \\ (3q+1)\vartheta_q^3, & n = 3, \\ \Big(\delta_{n+1}(q) + q\,\delta_{n-1}(q)\Big)\vartheta_q^n, & n \geq 4. \end{cases} \tag{1.4}$$

In this work, we introduce and study a new subclass of bi-univalent functions generated jointly by the q–Rabotnov function and the q–analogue of Fibonacci numbers. Employing subordination principles, we derive sharp bounds for the initial Taylor–Maclaurin coefficients and establish corresponding Fekete–Szegö type inequalities. The results obtained not only unify and generalize several recent frameworks in geometric function theory but also forge new links between fractional viscoelastic modeling, q–calculus, and the analytic theory of bi-univalent functions.

2. Preliminaries

Let $\mathcal{A}$ denote the family of analytic functions defined on the open unit disk

$$\mathscr{O} = \{ z \in \mathbb{C} : |z| < 1 \},$$

where $z = a + ib$ with $a, b \in \mathbb{R}$. Geometrically, $\mathscr{O}$ represents all points in the complex plane that lie strictly inside the unit circle centered at the origin.

Functions $f \in \mathcal{A}$ are normalized by the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1,$$

which guarantee uniqueness and facilitate coefficient analysis within $\mathscr{O}$. Every $f \in \mathcal{A}$ admits a Taylor–Maclaurin expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \qquad z \in \mathscr{O}. \tag{2.1}$$

A function g is called a *Schwarz function* if it is analytic in $\mathscr{O}$, satisfies $g(0) = 0$, and $|g(z)| < 1$ for all $z \in \mathscr{O}$. For $f_1, f_2 \in \mathcal{A}$, the function f_1 is said to be *subordinate* to f_2, denoted $f_1 \prec f_2$, if there exists a Schwarz function g such that $f_1(z) = f_2(g(z))$ for all $z \in \mathscr{O}$.

Denote by S the subclass of $\mathcal{A}$ consisting of univalent (injective) functions in $\mathscr{O}$. Let P denote the family of functions in $\mathcal{A}$ with positive real part, i.e.,

$$\mathbf{p}(z) = 1 + \sum_{n=1}^{\infty} \mathbf{p}_n z^n = 1 + \mathbf{p}_1 z + \mathbf{p}_2 z^2 + \mathbf{p}_3 z^3 + \cdots, \tag{2.2}$$

where the coefficients satisfy

$$|\mathbf{p}_n| \le 2, \qquad n \ge 1, \tag{2.3}$$

according to Carathéodory's lemma (see [11]). Equivalently, $\mathbf{p} \in \mathsf{P}$ if and only if $\mathbf{p}(z) \prec \frac{1+z}{1-z}$ in $\mathscr{O}$.

The class P provides a foundation for constructing many important subclasses of analytic functions. For each $f \in \mathsf{S}$, there exists an inverse function f^{-1} defined in a neighborhood of the origin, satisfying

$$z = f^{-1}(f(z)), \quad \xi = f(f^{-1}(\xi)), \qquad \left(|\xi| < r_0(f) \le \tfrac{1}{4}\right). \tag{2.4}$$

The expansion of f^{-1} is given by

$$f^{-1}(\xi) = \xi - \alpha_2 \xi^2 + (2\alpha_2^2 - \alpha_3)\xi^3 - (5\alpha_2^3 + \alpha_4 - 5\alpha_2\alpha_3)\xi^4 + \cdots. \tag{2.5}$$

A function $f \in \mathsf{S}$ is called *bi-univalent* if both f and its inverse f^{-1} are univalent in $\mathscr{O}$. The family of such functions, denoted by Σ, is a natural and significant subclass of S. For example:

$$f_1(z) = \frac{z}{1+z}, \quad f_1^{-1}(z) = \frac{z}{1-z},$$

$$f_2(z) = -\log(1-z), \quad f_2^{-1}(z) = \frac{e^{2z}-1}{e^{2z}+1},$$

$$f_3(z) = \tfrac{1}{2}\log\Big(\tfrac{1+z}{1-z}\Big), \quad f_3^{-1}(z) = \frac{e^{z}-1}{e^{z}}.$$

These examples illustrate the structural duality between a bi-univalent function and its inverse, underscoring the analytic richness of the class Σ within geometric function theory.

Definition 2.1 ([1])**.** *Let $\beta, \delta, \lambda \in \mathbb{C}$ with $\Re(\beta) > 0$, $\Re(\delta) > 0$, $\Re(\lambda) > 0$, and $|q| < 1$. The generalized q–Mittag–Leffler function $\mathrm{E}_{\beta,\lambda}^{\delta}$ is defined by*

$$\mathrm{E}_{\beta,\lambda}^{\delta}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\delta};q)_n}{(q;q)_n} \frac{z^n}{\Gamma_q(\beta n + \lambda)}, \tag{2.6}$$

where Γ_q denotes the q–gamma function given in (1.1)*.*

In the limit $q \to 1^-$, $\mathrm{E}_{\beta,\lambda}^{\delta}(z;q)$ reduces to the classical generalized Mittag–Leffler function, providing a bridge between the discrete q–framework and its continuous analogue. Motivated by this connection, we now define the q–Rabotnov function.

Definition 2.2. *Let $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, $\lambda > 0$, and $|q| < 1$. The q–Rabotnov function $\Phi_{\beta,\lambda}^{\delta}(z;q)$ is defined by*

$$\Phi_{\beta,\lambda}^{\delta}(z;q) = z^{\beta} \sum_{n=0}^{\infty} \frac{(q^{\delta};q)_n}{(q;q)_n} \frac{[\lambda]_q^n}{\Gamma_q\big((n+1)(1+\beta)\big)} z^{n(1+\beta)}. \tag{2.7}$$

For $q \to 1^-$, the function $\Phi_{\beta,\lambda}^{\delta}(z;q)$ reduces to the classical Rabotnov function $\Phi_{\beta,\lambda}(z)$ (see [2]). Since $\Phi_{\beta,\lambda}^{\delta}(z;q)$ is not normalized, we introduce the normalized form

$$\begin{aligned} \mathbb{R}_{\beta,\lambda}^{\delta}(z;q) &= z^{\frac{1}{1+\beta}+1}\, \Gamma_q(1+\beta)\, \Phi_{\beta,\lambda}^{\delta}\big(z^{1/(1+\beta)};q\big) \\ &= z + \sum_{n=2}^{\infty} \frac{(q^{\delta};q)_{n-1}}{(q;q)_{n-1}} \frac{[\lambda]_q^{n-1}\, \Gamma_q(1+\beta)}{\Gamma_q(n(1+\beta))} z^n, \qquad z \in \mathscr{O}. \end{aligned} \tag{2.8}$$

Remark 2.1. *The function $\Phi_{\beta,\lambda}^{\delta}(z;q)$ serves as a q–analogue of kernel-type generating functions that frequently appear in the theory of analytic and bi-univalent functions. It combines the generalized q–Mittag–Leffler structure with the deformation parameter λ, and for $q \to 1^-$ it coincides with its classical analogue involving Euler's gamma function. Such kernels are fundamental in constructing subclasses defined via subordination, convolution, and operators related to fractional q–calculus.*

We now introduce a Hadamard–convolution operator associated with the q–Rabotnov kernel.

Definition 2.3. *For $\beta, \delta \in \mathbb{C}$ with $\Re(\beta) > 0$ and $\lambda > 0$, the operator $\mathcal{F}^{\delta}_{\beta,\lambda} : \mathcal{A} \to \mathcal{A}$ is defined by*

$$\begin{aligned}\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q) &= \mathbb{R}^{\delta}_{\beta,\lambda}(z;q) * f(z)\\ &= z + \sum_{n=2}^{\infty} \frac{(q^{\delta};q)_{n-1}}{(q;q)_{n-1}} \frac{[\lambda]_q^{n-1}\,\Gamma_q(1+\beta)}{\Gamma_q(n(1+\beta))}\,\alpha_n z^n,\end{aligned} \tag{2.9}$$

where f has the form (2.1)*, and $*$ denotes the Hadamard (coefficient-wise) product of power series. In particular,*

$$\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q) = z + \frac{[\delta]_q[\lambda]_q\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))}\alpha_2 z^2 + \frac{[\delta]_q[\delta+1]_q[\lambda]_q^2\Gamma_q(1+\beta)}{[2]_q\Gamma_q(3(1+\beta))}\alpha_3 z^3 + O(z^4).$$

Remark 2.2. *The operator $\mathcal{F}^{\delta}_{\beta,\lambda}$ extends classical convolution operators by incorporating q–Rabotnov kernels. Such operators play a pivotal role in defining and analyzing subclasses of analytic and bi-univalent functions, especially in deriving coefficient estimates and Fekete–Szegö type inequalities.*

The development of q–calculus has significantly enriched analytic function theory by enabling the discovery of new subclasses endowed with rich geometric and algebraic features. This framework not only broadens the reach of classical results but also offers a bridge between discrete and continuous approaches. Consequently, q–calculus provides a robust foundation for further progress in complex analysis, operator theory, and their applications [14–24, 44–50].

3. Definition and examples

Motivated by q-Fibonacci numbers and the q–Rabotnov operator, this section will now look at a novel subclasses of bi-univalent functions related to shell-like curves.

Definition 3.1. *Let $\mu \geq 0$ and $\varrho \geq 0$. A function $f \in \Sigma$, defined by* (2.1)*, is said to belong to the class $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$ if the following subordinations are satisfied:*

$$\mu\,\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} + \varrho\, z\,\partial_q^2\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) \prec \Omega(z;q), \quad (z \in \mathscr{O}), \tag{3.1}$$

and

$$\mu\,\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} + \varrho\,\xi\,\partial_q^2\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) \prec \Omega(\xi;q), \qquad (\xi \in \mathscr{O}), \tag{3.2}$$

where $\Omega(z;q)$ is specified in (1.2), $\eta = f^{-1}$ denotes the inverse of f, ∂_q represents the q–derivative, and ϑ_q is specified in (1.3)*.*

By prescribing suitable specializations of the parameters μ, ϱ and q, one can recover a variety of familiar subclasses of the bi-univalent function class Σ. For clarity, we present below several representative examples, illustrating how the general class $\Re_{\Sigma_q^{\mu}}(\beta,\delta,\lambda)$ reduces to well-known families under particular parameter choices.

Example 3.1. *If we take $\varrho = 0$ in Definition 3.1, then a function $f \in \Sigma$ is said to belong to the class $\Re_{\Sigma_q^{\mu,0}}(\beta,\delta,\lambda)$ if the following subordinations are satisfied:*

$$\mu\, \partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} \prec \Omega(z;q) := \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q z - q\vartheta_q^2 z^2}, \qquad (z \in \mathscr{O}), \tag{3.3}$$

and

$$\mu\, \partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} \prec \Omega(\xi;q) := \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathscr{O}), \tag{3.4}$$

where $\eta = f^{-1}$ denotes the inverse of f, ∂_q represents the q–derivative, and ϑ_q is specified in (1.3).

Example 3.2. *If we take $\mu = 1$ and $\varrho = 0$ in Definition 3.1, then a function $f \in \Sigma$ is said to belong to the class $\Re_{\Sigma_q^{1}}(\beta,\delta,\lambda)$ whenever the following subordinations hold:*

$$\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big) \prec \Omega(z;q) := \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q z - q\vartheta_q^2 z^2}, \qquad (z \in \mathscr{O}), \tag{3.5}$$

and

$$\partial_q\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big) \prec \Omega(\xi;q) := \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathscr{O}), \tag{3.6}$$

where $\eta = f^{-1}$ denotes the inverse of f, ∂_q is the q–derivative, and ϑ_q is given by (1.3).

Example 3.3. *If we take $\mu = 0$ and $\varrho = 0$ in Definition 3.1, then a function $f \in \Sigma$ is said to belong to the class $\Re_{\Sigma_q^{0}}(\beta,\delta,\lambda)$ whenever the following subordinations hold:*

$$\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} \prec \Omega(z;q) := \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q z - q\vartheta_q^2 z^2}, \qquad (z \in \mathscr{O}), \tag{3.7}$$

and

$$\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} \prec \Omega(\xi;q) := \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathscr{O}), \tag{3.8}$$

where $\eta = f^{-1}$ denotes the inverse of f, and ϑ_q is given by (1.3).

Example 3.4. *If we let $q \to 1^-$ in Definition 3.1, then the class $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$ reduces to its classical analogue $\Re_{\Sigma^{\mu,\varrho}}(\beta,\delta,\lambda)$. In this case, a function $f \in \Sigma$ belongs to the class if the subordinations*

$$\mu\,\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big)' + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)}{z} + \varrho z\,\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(f(z);q)\Big)'' \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z - \vartheta^2 z^2}, \ (z \in \mathscr{O}), \tag{3.9}$$

and

$$\mu\,\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big)' + (1-\mu)\,\frac{\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)}{\xi} + \varrho \xi\,\Big(\mathcal{F}^{\delta}_{\beta,\lambda}(\eta(\xi);q)\Big)'' \prec \frac{1+\vartheta^2 \xi^2}{1-\vartheta \xi - \vartheta^2 \xi^2}, \ (\xi \in \mathscr{O}), \tag{3.10}$$

hold, where $\eta = f^{-1}$ is the inverse of f, and $\vartheta = \frac{1-\sqrt{5}}{2} = \lim_{q\to 1^-} \vartheta_q$. Here the operator ∂_q is replaced by the classical derivative.

4. Main Results

In this section, we obtain the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_3|$ for the bi-univalent starlike and convex subclass $\Re_{\Sigma_q^\mu}(\beta,\delta,\lambda)$.

Firstly, let $\mathbf{p}(z) = 1 + \mathbf{p}_1 z + \mathbf{p}_2 z^2 + \mathbf{p}_3 z^3 + \dots$, and $\mathbf{p}(z) \prec \Omega(z;q)$. Then there exist $\delta \in \mathsf{P}$ such that $|\varepsilon(z)| < 1$ in $\mathscr{O}$ and $\mathbf{p}(z) = \Omega(\varepsilon(z);q)$, we have

$$\hbar(z) = (1+\varepsilon(z))(1-\varepsilon(z))^{-1} = 1 + \ell_1 z + \ell_2 z^2 + \dots \in \mathsf{P} \qquad (z \in \mathscr{O}). \tag{4.1}$$

It follows that

$$\varepsilon(z) = \frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots, \tag{4.2}$$

and

$$\begin{aligned}\Omega(\varepsilon(z);q) &= 1 + \widehat{\mathbf{p}_1}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right] \\ &\quad + \widehat{\mathbf{p}_2}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^2 \\ &\quad + \widehat{\mathbf{p}_3}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^3 + \cdots \\ &= 1 + \frac{\widehat{\mathbf{p}_1}\ell_1}{2} z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathbf{p}_1} + \frac{\ell_1^2}{2}\widehat{\mathbf{p}_2}\right] z^2 \\ &\quad + \frac{1}{2}\left[\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right)\widehat{\mathbf{p}_1} + \ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathbf{p}_2} + \frac{\ell_1^3}{4}\widehat{\mathbf{p}_3}\right] z^3 + \cdots.\end{aligned} \tag{4.3}$$

And similarly, there exists an analytic function ν such that $|\nu(\xi)| < 1$ in $\mathscr{O}$ and $\mathbf{p}(\xi) = \Omega(\nu(\xi);q)$. Therefore, the function

$$\kappa(\xi) = (1+\nu(\xi))(1-\nu(\xi))^{-1} = 1 + \tau_1\xi + \tau_2\xi^2 + \dots \in \mathsf{P}. \tag{4.4}$$

It follows that

$$\nu(\xi) = \frac{\tau_1\xi}{2} + \left(\tau_2 - \frac{\tau_1^2}{2}\right)\frac{\xi^2}{2} + \left(\tau_3 - \tau_1\tau_2 - \frac{\tau_1^3}{4}\right)\frac{\xi^3}{2} + \cdots, \tag{4.5}$$

and

$$\begin{aligned}\Omega(\nu(\xi);q) &= 1 + \frac{\widehat{\mathbf{p}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathbf{p}_1} + \frac{\tau_1^2}{2}\widehat{\mathbf{p}_2}\right]\xi^2 \\ &\quad + \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\widehat{\mathbf{p}_1} + \tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathbf{p}_2} + \frac{\tau_1^3}{4}\widehat{\mathbf{p}_3}\right]\xi^3 + \cdots.\end{aligned} \tag{4.6}$$

In the following theorem we determine the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_2|$ for the class $\Re_{\Sigma_q^\mu}(\beta,\delta,\lambda)$. Later we will reduce these bounds to other classes for special cases.

Theorem 4.1. *Let f given by* (2.1) *be in the class $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$. Then*

$$|\alpha_2| \leq \min\left\{ \frac{|\vartheta_q|}{\sqrt{\left|\mathscr{K}\,\vartheta_q\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)-\mathscr{T}^2\left((2q+1)\vartheta_q-1\right)\left(1+\mu q+\varrho[2]_q\right)^2\right|}}, \frac{|\vartheta_q|}{\left|\mathscr{T}\left(1+\mu q+\varrho[2]_q\right)\right|} \right\},$$

and

$$\left|\alpha_3\right| \leq \frac{\vartheta_q^2}{\mathscr{T}^2\left(1+\mu q+\varrho[2]_q\right)^2} + \frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)\right|}.$$

where

$$\mathscr{T} = \frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))}, \qquad \mathscr{K} = \frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q(3(1+\beta))}.$$

Proof. Let $f \in \Re_{\Sigma_q^{\mu}}(\beta,\delta,\lambda)$ and $\eta = f^{-1}$. Considering (3.3) and (3.10) we have

$$\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big) + (1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)}{z} + \varrho\, z\,\partial_q^2\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big) = \Omega(\varepsilon(z);q), \qquad (z\in\mathscr{O}), \quad (4.7)$$

and

$$\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)\Big) + (1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)}{\xi} + \varrho\,\xi\,\partial_q^2\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)\Big) = \Omega(\nu(\xi);q), \qquad (\xi\in\mathscr{O}). \quad (4.8)$$

Using (2.8), we have

$$\begin{aligned}
&\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big) + (1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)}{z} + \varrho\, z\,\partial_q^2\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(f(z);q)\Big)\\
&= 1 + \frac{\Big(1+\mu q+\varrho[2]_q\Big)[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\Big(2(1+\beta)\Big)}\,\alpha_2\, z \qquad (4.9)\\
&\quad + \frac{\Big(1+\mu\, q\,[2]_q+\varrho[3]_q[2]_q\Big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\Big(3(1+\beta)\Big)}\,\alpha_3\, z^2 + O(z^3).
\end{aligned}$$

and

$$\begin{aligned}
&\mu\,\partial_q\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)\Big) + (1-\mu)\,\frac{\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)}{\xi} + \varrho\,\xi\,\partial_q^2\Big(\mathcal{F}_{\beta,\lambda}^{\delta}(\eta(\xi);q)\Big)\\
&= 1 - \frac{\Big(1+\mu\, q+\varrho[2]_q\Big)[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\Big(2(1+\beta)\Big)}\,\alpha_2\,\xi \qquad (4.10)\\
&\quad + \frac{\Big(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\Big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\Big(3(1+\beta)\Big)}\Big(2\,\alpha_2^2-\alpha_3\Big)\,\xi^2 + O(\xi^3).
\end{aligned}$$

By comparing (4.7) and (4.9), along (4.3), yields

$$\frac{\Big(1+\mu q+\varrho[2]_q\Big)[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\Big(2(1+\beta)\Big)}\,\alpha_2\,z+\frac{\Big(1+\mu\,q\,[2]_q+\varrho[3]_q[2]_q\Big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\Big(3(1+\beta)\Big)}\,\alpha_3\,z^2+\cdots$$
$$=\frac{\widehat{\mathbf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathbf{p}_1}+\frac{\ell_1^2}{2}\widehat{\mathbf{p}_2}\right]z^2+\cdots. \tag{4.11}$$

Besied that, by comparing (4.3) and (4.10), along (4.6), yields

$$-\frac{\Big(1+\mu q+\varrho[2]_q\Big)[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q\Big(2(1+\beta)\Big)}\,\alpha_2\,\xi$$
$$+\frac{\Big(1+\mu\,q\,[2]_q+\varrho[3]_q[2]_q\Big)[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q\Big(3(1+\beta)\Big)}\Big(2\alpha_2^2-\alpha_3\Big)\xi^2+\cdots \tag{4.12}$$
$$=\frac{\widehat{\mathbf{p}_1}\,\tau_1}{2}\,\xi+\frac{1}{2}\left[\left(\tau_2-\tfrac{\tau_1^2}{2}\right)\widehat{\mathbf{p}_1}+\tfrac{\tau_1^2}{2}\,\widehat{\mathbf{p}_2}\right]\xi^2+\cdots.$$

Equating the corresponding coefficients in (4.11) and (4.12), we arrive at the following system of relations. For convenience, we introduce the constants

$$\mathscr{T}=\frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))},\qquad \mathscr{K}=\frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q(3(1+\beta))}.$$

With these notations, the relations can be expressed in the compact form:

$$\mathscr{T}\Big(1+\mu q+\varrho[2]_q\Big)\alpha_2=\tfrac{1}{2}\,\widehat{\mathbf{p}_1}\,\ell_1, \tag{4.13}$$

$$-\mathscr{T}\Big(1+\mu q+\varrho[2]_q\Big)\alpha_2=\tfrac{1}{2}\,\widehat{\mathbf{p}_1}\,\tau_1, \tag{4.14}$$

$$\mathscr{K}\Big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\Big)\alpha_3=\tfrac{1}{2}\left[\left(\ell_2-\tfrac{\ell_1^2}{2}\right)\widehat{\mathbf{p}_1}+\tfrac{\ell_1^2}{2}\,\widehat{\mathbf{p}_2}\right], \tag{4.15}$$

$$\mathscr{K}\Big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\Big)\Big(2\alpha_2^2-\alpha_3\Big)=\tfrac{1}{2}\left[\left(\tau_2-\tfrac{\tau_1^2}{2}\right)\widehat{\mathbf{p}_1}+\tfrac{\tau_1^2}{2}\,\widehat{\mathbf{p}_2}\right]. \tag{4.16}$$

From (4.13) and (4.14), we have

$$\ell_1=-\tau_1 \quad\Longleftrightarrow\quad \ell_1^2=\tau_1^2, \tag{4.17}$$

and using (1.4), we have

$$\alpha_2^2=\frac{\vartheta_q^2}{8\mathscr{T}^2\Big(1+\mu q+\varrho[2]_q\Big)^2}\Big(\ell_1^2+\tau_1^2\Big), \tag{4.18}$$

or equivalent to

$$\Big(\ell_1^2+\tau_1^2\Big)=\frac{8\mathscr{T}^2\Big(1+\mu q+\varrho[2]_q\Big)^2}{\vartheta_q^2}\alpha_2^2, \tag{4.19}$$

Now, by summing (4.15) and (4.16), we obtain

$$2\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)\alpha_2^2=\frac{(\ell_2+\tau_2)\vartheta_q}{2}+\left[\frac{(2q+1)\vartheta_q^2}{4}-\frac{\vartheta_q}{4}\right]\left(\ell_1^2+\tau_1^2\right). \tag{4.20}$$

By putting (4.19) in (4.20), with doing some calculations, yields to

$$\alpha_2^2=\frac{\vartheta_q^2\,(\ell_2+\tau_2)}{4\left[\mathscr{K}\,\vartheta_q\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)-\mathscr{T}^2\left((2q+1)\vartheta_q-1\right)\left(1+\mu q+\varrho[2]_q\right)^2\right]}. \tag{4.21}$$

Using (2.3) for (4.21), we have

$$|\alpha_2|\leq\frac{\left|\vartheta_q\right|}{\sqrt{\left|\mathscr{K}\,\vartheta_q\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)-\mathscr{T}^2\left((2q+1)\vartheta_q-1\right)\left(1+\mu q+\varrho[2]_q\right)^2\right|}}. \tag{4.22}$$

Besided that, from (4.18)

$$|\alpha_2|\leq\frac{\left|\vartheta_q\right|}{\left|\mathscr{T}\left(1+\mu q+\varrho\,[2]_q\right)\right|}.$$

Now, so as to find the bound on $|\alpha_3|$, let's subtract from (4.15) and (4.16) along (4.18), we obtain

$$\alpha_3=\alpha_2^2+\frac{(\ell_2-\tau_2)\,\vartheta_q}{4\,\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)}. \tag{4.23}$$

Hence, we get

$$\left|\alpha_3\right|=\left|\alpha_2\right|^2+\frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)\right|}. \tag{4.24}$$

Then, in view of (4.18), we obtain

$$\left|\alpha_3\right|\leq\frac{\vartheta_q^2}{\mathscr{T}^2\left(1+\mu q+\varrho[2]_q\right)^2}+\frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)\right|}. \tag{4.25}$$

□

In the following theorem, we find the Fekete-Szegö functional for $f\in\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$.

Theorem 4.2. *Let f given by* (2.1) *be in the class $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$ and $\rho\in\mathbb{R}$. Then we have*

$$\left|\alpha_3-\rho\,\alpha_2^2\right|\leq\begin{cases}\dfrac{|\vartheta_q|}{\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)}, & 0\leq\left|\mathscr{D}(\rho)\right|\leq\dfrac{1}{\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)}\\ \left|\vartheta_q\right|\left|\mathscr{D}(\rho)\right|, & \left|\mathscr{D}(\rho)\right|\geq\dfrac{1}{\mathscr{K}\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)}\end{cases}.$$

where

$$\mathscr{D}(\rho)=\frac{(1-\rho)\vartheta_q}{\mathscr{K}\,\vartheta_q\left(1+\mu\, q[2]_q+\varrho[3]_q[2]_q\right)-\mathscr{T}^2\left((2q+1)\vartheta_q-1\right)\left(1+\mu q+\varrho[2]_q\right)^2} \tag{4.26}$$

Proof. Let $f \in \Re_{\Sigma_q^{\mu}}(\beta,\delta,\lambda)$, from (4.21) and (4.23) we have

$$\begin{aligned}\alpha_3-\rho\,\alpha_2^2 &= \frac{(1-\rho)\vartheta_q^2\,(\ell_2+\tau_2)}{4\Big[\mathscr{K}\,\vartheta_q\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)-\mathscr{T}^2\big((2q+1)\vartheta_q-1\big)\big(1+\mu q+\varrho[2]_q\big)^2\Big]}\\ &\quad+\frac{(\ell_2-\tau_2)\,\vartheta_q}{4\,\mathscr{K}\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)}\\ &=\frac{\vartheta_q}{4}\Bigg[\Bigg(\mathscr{D}(\rho)+\frac{1}{\mathscr{K}\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)}\Bigg)\ell_2\\ &\quad+\Bigg(\mathscr{D}(\rho)-\frac{1}{\mathscr{K}\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)}\Bigg)\tau_2\Bigg]\end{aligned} \tag{4.27}$$

where $\mathscr{D}(\rho)$ is given by (4.26).

Then, by taking modulus of (4.27), we conclude that

$$\left|\alpha_3-\rho\,\alpha_2^2\right| \le \begin{cases} \dfrac{|\vartheta_q|}{\mathscr{K}\big(1+\mu\,q[2]_q+\varrho\,[3]_q[2]_q\big)}, & 0\le\left|\mathscr{D}(\rho)\right|\le\dfrac{1}{\mathscr{K}\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)}\\ \left|\vartheta_q\right|\left|\mathscr{D}(\rho)\right|, & \left|\mathscr{D}(\rho)\right|\ge\dfrac{1}{\mathscr{K}\big(1+\mu\,q[2]_q+\varrho[3]_q[2]_q\big)} \end{cases}.$$

□

5. Corollaries

The general coefficient estimates established in Theorems 4.1 and 4.2 give rise to several noteworthy special cases under suitable choices of the parameters μ,ϱ and q. In particular, when one considers the purely q–differential subclass $(\mu=1)$, the ratio-type subclass $(\mu=0)$, the ratio-type subclass $(\varrho=0)$ and the classical limiting case $(q\to 1^-)$, the results simplify to the following corollaries.

Corollary 5.1 ($\varrho = 1$). *Let f given by* (2.1) *be in the class $\Re_{\Sigma_q^{\mu,0}}(\beta,\delta,\lambda)$ and let $\rho\in\mathbb{R}$. Then*

$$|\alpha_2| \;\le\; \min\left\{\begin{array}{l}\dfrac{\left|\vartheta_q\right|}{\sqrt{\left|\mathscr{K}\,\vartheta_q\big(1+\mu\,q[2]_q\big)-\mathscr{T}^2\big((2q+1)\vartheta_q-1\big)\big(1+\mu q\big)^2\right|}},\\ \dfrac{\left|\vartheta_q\right|}{\left|\mathscr{T}\big(1+\mu q\big)\right|}\end{array}\right\},$$

and

$$\left|\alpha_3\right| \;\le\; \frac{\vartheta_q^2}{\mathscr{T}^2\big(1+\mu q\big)^2}\;+\;\frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\big(1+\mu\,q[2]_q\big)\right|},$$

where

$$\mathscr{T} = \frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))}, \qquad \mathscr{K} = \frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q(3(1+\beta))}.$$

Moreover,

$$\left|\alpha_3 - \rho\,\alpha_2^2\right| \le \begin{cases} \dfrac{\left|\vartheta_q\right|}{\mathscr{K}\left(1+\mu\,q[2]_q\right)}, & 0 \le \left|\mathscr{D}_0(\rho)\right| \le \dfrac{1}{\mathscr{K}\left(1+\mu\,q[2]_q\right)}, \\ \left|\vartheta_q\right|\,\left|\mathscr{D}_0(\rho)\right|, & \left|\mathscr{D}_0(\rho)\right| \ge \dfrac{1}{\mathscr{K}\left(1+\mu\,q[2]_q\right)}, \end{cases}$$

where

$$\mathscr{D}_0(\rho) = \frac{(1-\rho)\,\vartheta_q}{\mathscr{K}\,\vartheta_q\left(1+\mu\,q[2]_q\right) - \mathscr{T}^2\left((2q+1)\vartheta_q - 1\right)\left(1+\mu q\right)^2}.$$

Corollary 5.2 ($\mu = 0$)**.** *Let f given by* (2.1) *be in the class $\Re_{\Sigma_q^{0,\varrho}}(\beta,\delta,\lambda)$ and let $\rho \in \mathbb{R}$. Then*

$$|\alpha_2| \le \min\left\{ \frac{\left|\vartheta_q\right|}{\sqrt{\left|\mathscr{K}\,\vartheta_q\left(1+\varrho[3]_q[2]_q\right) - \mathscr{T}^2\left((2q+1)\vartheta_q - 1\right)\left(1+\varrho[2]_q\right)^2\right|}}, \frac{\left|\vartheta_q\right|}{\left|\mathscr{T}\left(1+\varrho[2]_q\right)\right|} \right\},$$

and

$$\left|\alpha_3\right| \le \frac{\vartheta_q^2}{\mathscr{T}^2\left(1+\varrho[2]_q\right)^2} + \frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\left(1+\varrho[3]_q[2]_q\right)\right|},$$

where

$$\mathscr{T} = \frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))}, \qquad \mathscr{K} = \frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q(3(1+\beta))}.$$

Moreover,

$$\left|\alpha_3 - \rho\,\alpha_2^2\right| \le \begin{cases} \dfrac{\left|\vartheta_q\right|}{\mathscr{K}\left(1+\varrho[3]_q[2]_q\right)}, & 0 \le \left|\mathscr{D}_{\mu=0}(\rho)\right| \le \dfrac{1}{\mathscr{K}\left(1+\varrho[3]_q[2]_q\right)}, \\ \left|\vartheta_q\right|\left|\mathscr{D}_{\mu=0}(\rho)\right|, & \left|\mathscr{D}_{\mu=0}(\rho)\right| \ge \dfrac{1}{\mathscr{K}\left(1+\varrho[3]_q[2]_q\right)}, \end{cases}$$

where

$$\mathscr{D}_{\mu=0}(\rho) = \frac{(1-\rho)\,\vartheta_q}{\mathscr{K}\,\vartheta_q\left(1+\varrho[3]_q[2]_q\right) - \mathscr{T}^2\left((2q+1)\vartheta_q - 1\right)\left(1+\varrho[2]_q\right)^2}.$$

Corollary 5.3 ($\mu = 1$)**.** *Let f given by* (2.1) *be in the class $\Re_{\Sigma_q^{1,\varrho}}(\beta,\delta,\lambda)$ and $\rho \in \mathbb{R}$. Then*

$$|\alpha_2| \le \min\left\{ \frac{\left|\vartheta_q\right|}{\sqrt{\left|\mathscr{K}\,\vartheta_q\left(1+q[2]_q+\varrho[3]_q[2]_q\right) - \mathscr{T}^2\left((2q+1)\vartheta_q - 1\right)\left(1+q+\varrho[2]_q\right)^2\right|}}, \frac{\left|\vartheta_q\right|}{\left|\mathscr{T}\left(1+q+\varrho[2]_q\right)\right|} \right\},$$

and

$$\left|\alpha_3\right| \le \frac{\vartheta_q^2}{\mathscr{T}^2\left(1+q+\varrho[2]_q\right)^2} + \frac{\left|\vartheta_q\right|}{\left|\mathscr{K}\left(1+q[2]_q+\varrho[3]_q[2]_q\right)\right|}.$$

Moreover,

$$\left|\alpha_3-\rho\,\alpha_2^2\right| \le \begin{cases} \dfrac{\left|\vartheta_q\right|}{\mathscr{K}\left(1+q[2]_q+\varrho[3]_q[2]_q\right)}, & 0\le\left|\mathscr{D}_{\mu=1}(\rho)\right|\le\dfrac{1}{\mathscr{K}\left(1+q[2]_q+\varrho[3]_q[2]_q\right)},\\ \left|\vartheta_q\right|\left|\mathscr{D}_{\mu=1}(\rho)\right|, & \left|\mathscr{D}_{\mu=1}(\rho)\right|\ge\dfrac{1}{\mathscr{K}\left(1+q[2]_q+\varrho[3]_q[2]_q\right)}, \end{cases}$$

where

$$\mathscr{D}_{\mu=1}(\rho)=\frac{(1-\rho)\,\vartheta_q}{\mathscr{K}\,\vartheta_q\left(1+q[2]_q+\varrho[3]_q[2]_q\right)-\mathscr{T}^2\left((2q+1)\vartheta_q-1\right)\left(1+q+\varrho[2]_q\right)^2},$$

and

$$\mathscr{T}=\frac{[\delta]_q\,[\lambda]_q\,\Gamma_q(1+\beta)}{\Gamma_q(2(1+\beta))},\qquad \mathscr{K}=\frac{[\delta]_q\,[\delta+1]_q\,[\lambda]_q^2\,\Gamma_q(1+\beta)}{[2]_q\,\Gamma_q(3(1+\beta))}.$$

Corollary 5.4 (Classical limit $q\to1^-$). *Assume f given by* (2.1) *belongs to the limit class $\Re_{\Sigma^{\mu,\varrho}}(\beta,\delta,\lambda)$ obtained from $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$ as $q\to1^-$, and set*

$$\mathscr{T}_1:=\lim_{q\to1^-}\mathscr{T}=\frac{\delta\,\lambda\,\Gamma(1+\beta)}{\Gamma\big(2(1+\beta)\big)},\qquad \mathscr{K}_1:=\lim_{q\to1^-}\mathscr{K}=\frac{\delta(\delta+1)\lambda^2\,\Gamma(1+\beta)}{2\,\Gamma\big(3(1+\beta)\big)}.$$

Let $\vartheta:=\lim_{q\to1^-}\vartheta_q$ (whenever this limit exists). Then

$$|\alpha_2|\le\min\left\{\frac{|\vartheta|}{\sqrt{\left|\mathscr{K}_1\,\vartheta\left(1+2\mu+6\varrho\right)-\mathscr{T}_1^2\left(3\vartheta-1\right)\left(1+\mu+2\varrho\right)^2\right|}},\ \frac{|\vartheta|}{\left|\mathscr{T}_1\left(1+\mu+2\varrho\right)\right|}\right\},$$

and

$$|\alpha_3|\le\frac{\vartheta^2}{\mathscr{T}_1^2\left(1+\mu+2\varrho\right)^2}+\frac{|\vartheta|}{\left|\mathscr{K}_1\left(1+2\mu+6\varrho\right)\right|}.$$

Moreover,

$$\left|\alpha_3-\rho\,\alpha_2^2\right| \le \begin{cases} \dfrac{|\vartheta|}{\mathscr{K}_1\left(1+2\mu+6\varrho\right)}, & 0\le\left|\mathscr{D}_1(\rho)\right|\le\dfrac{1}{\mathscr{K}_1\left(1+2\mu+6\varrho\right)},\\ |\vartheta|\left|\mathscr{D}_1(\rho)\right|, & \left|\mathscr{D}_1(\rho)\right|\ge\dfrac{1}{\mathscr{K}_1\left(1+2\mu+6\varrho\right)}, \end{cases}$$

where

$$\mathscr{D}_1(\rho)=\frac{(1-\rho)\,\vartheta}{\mathscr{K}_1\,\vartheta\left(1+2\mu+6\varrho\right)-\mathscr{T}_1^2\left(3\vartheta-1\right)\left(1+\mu+2\varrho\right)^2}.$$

6. Conclusion

In this work, we have introduced and studied the class $\Re_{\Sigma_q^{\mu,\varrho}}(\beta,\delta,\lambda)$, constructed through convolution operators involving the q–Rabotnov function and subordinated to the q–Fibonacci structure. A key feature of our investigation is the definition of a new q–derivative operator based on q–Rabotnov kernels, which provides a flexible framework for analyzing subclasses of bi-univalent functions. Within this setting, we have derived sharp coefficient estimates for the initial Taylor–Maclaurin coefficients and established corresponding Fekete–Szegö type inequalities.

The general results obtained in Theorems 4.1 and 4.2 unify and extend several recent contributions to the theory of bi-univalent functions, while naturally reducing to important special cases under suitable parameter choices. In particular, the framework recovers the purely q–differential subclass ($\mu = 1$), the ratio-type subclass ($\mu = 0$), the ratio-type subclass ($\varrho = 0$), and the classical limit ($q \to 1^-$), thereby illustrating both the flexibility and the unifying character of the class $\mathfrak{R}_{\Sigma_q^\mu}(\beta, \delta, \lambda)$ in geometric function theory.

For future research, it would be of significant interest to develop analogous subclasses generated by other q–special functions or higher-order convolution operators, and to examine possible applications of the proposed q–derivative operator in operator theory, multivariable geometric mappings, and related analytic inequalities.

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