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A NOTE ON FIXED POINT THEORY FOR CYCLIC WEAKER MEIR–KEELER FUNCTION IN COMPLETE METRIC SPACES

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Abstract. In this paper we consider, discuss, improve and complement recent fixed points results for so-called cyclical weaker Meir-Keeler functions, established by Chi-Ming Chen [Chi-Ming Chen, *Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces,* Fixed Point Theory Appl., 2012, 2012:17]. In fact, we prove that weaker Meir-Keeler notion is superfluous in results.

1. Introduction and preliminaries

The Banach contraction principle [1] has various applications in many branches of applied science. It ensures the existence and uniqueness of fixed point of a contraction on a complete metric space. After this interesting principle, several authors generalized it by introducing the various contractions on metric spaces (see, e.g., [2]-[14]). Rhoades [19], in his work compare several contractions defined on metric spaces.

Cyclic representations and cyclic contractions were introduced by Kirk et al. [9] and further used by several authors to obtain various interesting and significant fixed point results (see, e.g., [2], [3], [8], [11],[12], [13], [14], [16]-[18]). However, we have proved ([16]-[18]) the following result:

• *If some ordinary fixed point theorem in the setting of complete metric spaces has a true cyclic-type extension, then these both theorems are equivalent.*

In this paper we prove the similar things. Namely, we consider, discuss, improve and complement recent fixed points results for so-called cyclical weaker Meir-Keeler functions, established by Chi-Ming Chen in [4]. In fact, we prove that weaker Meir-Keeler notion introduced in [4], is superfluous in results.

It is well known that a function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. Chi-Ming Chen introduced weaker Meir-Keeler function:

Definition 1.1. [4] The function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a weaker Meir-Keeler function for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Also in [4], the author assume the following conditions for a weaker Meir-Keeler function $\psi : [0, +\infty) \to [0, +\infty)$:

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(ψ_1) $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$;
(ψ_2) for all $t \in [0,\infty)$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(ψ_3) for $t_n \in [0,\infty)$, we have that:
(**a**) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \psi(t_n) < \gamma$, and
(**b**) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \psi(t_n) = 0$.

Chi-Ming Chen in [4] suppose that $\varphi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing and continuous function satisfying:
(φ_1) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;
(φ_2) φ is subadditive, that is, for every $\mu_1, \mu_2 \in [0,+\infty), \varphi(\mu_1 + \mu_2) \leq \varphi(\mu_1) + \varphi(\mu_2)$;
$(\varphi_3)$for all $t \in (0,\infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \varphi(t_n) = 0$.

Author state the notion of cyclic weaker $(\psi \diamond \varphi)$ −contraction as follows:

Definition 1.2. [4] Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1,...,A_m$ be nonempty subsets of X and $X = \cup_{i=1}^m A_i$. An operator $f : X \to X$ is called a cyclic weaker $(\psi \diamond \varphi)$ −contraction if:
(i) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
(ii) for any $x \in A_i, y \in A_{i+1}, i \in \{1,2,...,m\}$,

$$\varphi(d(fx,fy)) \leq \psi(\varphi(d(x,y))), \tag{1.1}$$

where $A_{m+1} = A_1$.

In [4] author proved the following:

Theorem 1.3. *Let (X,d) be a complete metric space, $m \in \mathbb{N}, A_1,...,A_m$ be nonempty closed subsets of X and $X = \cup_{i=1}^m A_i$. Let $f : X \to X$ be a cyclic weaker $(\psi \diamond \varphi)$ −contraction. Then, f has a unique fixed point $z \in \cap_{i=1}^m A_i$.:*

The cyclic weaker (ψ, φ) −contraction is defined in [4]:

Definition 1.4. Let $\psi : [0,\infty) \to [0,\infty)$ be a weaker Meir-Keeler function satisfying conditions (ψ_1), (ψ_2) and (ψ_3). Also, let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing and continuous function satisfying (φ_1).

Definition 1.5. Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1,...,A_m$ be nonempty subsets of X and $X = \cup_{i=1}^m A_i$. An operator $f : X \to X$ is called a cyclic weaker (ψ, φ) −contraction if:
(i) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
(ii) for any $x \in A_i, y \in A_{i+1}, i \in \{1,2,...,m\}$,

$$d(fx,fy) \leq \psi(d(x,y)) - \varphi(d(x,y)), \tag{1.2}$$

where $A_{m+1} = A_1$.

In [4] author proved the following result for this type of operator:

Theorem 1.6. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1,...,A_m$ be nonempty closed subsets of X and $X = \cup_{i=1}^m A_i$. Let $f : X \to X$ be a cyclic weaker (ψ, φ) −contraction. Then, f has a unique fixed point $z \in \cap_{i=1}^m A_i$.*

Here we will use the following (new, useful and very significant) result for the proofs of cyclic-type results (see also [15]-[18]):

Lemma 1.7. *Let (X,d) be a metric space, $f : X \to X$ be a mapping and let $X = \cup_{i=1}^p A_i$ be a cyclic representation of X w.r.t. f. Assume that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{1.3}$$

where $x_{n+1} = fx_n, x_1 \in A_1$. If $\{x_n\}$ is not a Cauchy sequence then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the *following sequences tend to* ε^+ *when* $k \to \infty$:

$$d\left(x_{m(k)-j(k)}, x_{n(k)}\right),\ d\left(x_{m(k)-j(k)+1}, x_{n(k)}\right),\ d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right), \tag{1.4}$$

$d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right)$, *where* $j(k) \in \{1, 2, ..., p\}$ *is chosen so that* $n(k) - m(k) + j(k) \equiv 1 \pmod p$, *for each* $k \in \mathbb{N}$.

2. Main results

In this section, first of all, we announce the following remarks:

(**a**) *Author in* [4] *has not the assumption that the function* ψ *is a non-decreasing. However, from the proof of both Theorems follows that he use this fact (page 3, lines 22-25; page 6, lines 15-18).*

(**b**) *Further, from* (ψ_2) *and* $(\psi_3), (a)$ *we follows that* $\psi^n(t) \to 0$ (as $n \to \infty$) *for all* $t \in [0, \infty)$.

Proof. Indeed, there exists $\lim_{n\to\infty} \psi^n(t) = \gamma \geq 0$. If $\gamma > 0$, then

$$\gamma = \lim_{n\to} \psi^{n+1}(t) = \lim_{n\to\infty} \psi(\psi^n(t)) < \gamma \tag{2.1}$$

(by $(\psi_3), (a)$). A contradiction. □

(**c**) *Since, must non-decreasing and* $\psi^n(t) \downarrow 0$ *as* $n \to \infty$ *for all* $t \in [0, \infty)$ *we easy obtain that* $\psi(t) < t$ *for* $t > 0$.

(**d**) *Further, we have that* $d(x_{n+1}, x_n) \to 0$ (as $n \to \infty$)*without using the notion of a weaker Meir-Keeler function. That is, lines 26-33 on page 3 are superfluous.*

(**e**) *Now, according to Lemma 1.7. one can obtain much shorter proof of Theorem 1.3. Namely, we do not use the property* (φ_2) *of the function* φ.

Proof. Indeed, putting $x = x_{m(k)-j(k),} y = x_{n(k)}$ in (1.1) we obtain a contradiction:

$$\varphi\left(d\left(fx_{m(k)-j(k),}, fx_{n(k)}\right)\right) \leq \psi\left(\varphi\left(d\left(x_{m(k)-j(k),}, x_{n(k)}\right)\right)\right) \tag{2.2}$$

that is.,

$$\varphi\left(d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right)\right) \leq \psi\left(\varphi\left(d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right)\right). \tag{2.3}$$

Now, passing to limit as $k \to \infty$ and using the properties of φ and ψ, follows

$$\varphi(\varepsilon) \leq \lim_{k\to\infty} \psi\left(\varphi\left(d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right)\right) < \varphi(\varepsilon). \tag{2.4}$$

Hence, $\{x_n\}$ is a Cauchy sequence. □

(**e**') *Similarly, putting* $x = x_{m(k)-j(k),} y = x_{n(k)}$ *in (1.2) we obtain again a contradiction:*

$$d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right) \leq \psi\left(d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right) - \varphi\left(d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right). \tag{2.5}$$

Letting to limit as $k \to \infty$ and using again the properties of φ and ψ, we have

$$\varepsilon \leq \lim_{k\to\infty} \psi\left(d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right) - \varphi(\varepsilon) < \varepsilon - \varphi(\varepsilon). \tag{2.6}$$

This means that $\{x_n\}$ is a Cauchy sequence. □

By the same method as in [16]-[18] one can prove the following two results:

Theorem 2.1. *Theorem 1.3. is a equivalent with the following:*

- *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *be a weaker* $(\psi \diamond \varphi)$ −contraction, that is.,

$$\varphi(d(fx, fy)) \leq \psi(\varphi(d(x, y))), \tag{2.7}$$

for all $x, y \in X$. *Then,* f *has a unique fixed point* $z \in X$.

Theorem 2.2. *Theorem 1.6. is a equivalent with the following:*

• *Let* (X,d) *be a complete metric space and let* $f : X \to X$ *be a weaker* (ψ,φ) −contraction, that is.,

$$d(fx,fy) \le \psi((d(x,y))) - \varphi(d(x,y)), \tag{2.8}$$

for all $x,y \in X$. *Then,* f *has a unique fixed point* $z \in X$.

Conclusion: In all previous results, that is in Theorems 3 and 4 of [4] it is sufficient that the functions ψ and φ satisfy the following conditions:

1. $\psi : [0,\infty) \to [0,\infty)$ is a non-decreasing function satisfying $(\psi_1), (\psi_2)$and (ψ_3);

2. $\varphi : [0,+\infty) \to [0,+\infty)$ is a non-decreasing and continuous function satisfying (φ_1) and (φ_3).

Hence, without weaker Meir-Keeler property for ψ as well as without the subadditivity for φ.

In the sequel we announce the following two results generalizing Theorems 1.3. and 1.6. above, that is., Theorems 3 and 4 from [4]. Firstly, we define:

Definition 2.3. Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1,...,A_m$ be nonempty subsets of X and $X = \cup_{i=1}^m A_i$. An operator $f : X \to X$ is called a cyclic generalized $(\psi \diamond \varphi)$ −contraction (resp. cyclic generalized (ψ,φ) −contraction) if:

(i) $X = \cup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;

(ii) for any $x \in A_i, y \in A_{i+1}, i \in \{1,2,...,m\}$,

$$\varphi(d(fx,fy)) \le \psi(\varphi(M(x,y))), \tag{2.9}$$

where $A_{m+1} = A_1$

$$(\text{resp. } d(fx,fy) \le \psi(M(x,y)) - \varphi(M(x,y))), \tag{2.10}$$

where $M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\}$

(iii) $\psi,\varphi : [0,\infty) \to [0,\infty)$ are functions satisfying **1**. and **2**. from above **Conclusion**.

Theorem 2.4. *Let* (X,d) *be a complete metric space,* $m \in \mathbb{N}, A_1,...,A_m$ *be nonempty closed subsets of* X *and* $X = \cup_{i=1}^m A_i$. *Let* $f : X \to X$ *be a cyclic generalized* $(\psi \diamond \varphi)$ *−contraction (resp.* cyclic generalized (ψ,φ) −contraction). *Then,* f *has a unique fixed point* $z \in \cap_{i=1}^m A_i$.

Proof. Given $x_0 \in X$ and let $x_{n+1} = fx_n$, for $n \in \{0,1,..\}$. Picard sequence. If there exists $n_0 \in \{0,1,...\}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Therefore, let $x_{n+1} \ne x_n$ for all $n \in \{0,1,...\}$. It is clear, that for any $n \in \{1,2,...\}$ there exists $i_n \in \{1,2,...,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since $f : X \to X$ is a cyclic generalized $(\psi \diamond \varphi)$ −contraction, we have that for all $n \in \{0,1,...\}$

$$\varphi(d(x_n,x_{n+1})) = \varphi(d(fx_{n-1},fx_n)) \le \psi(\varphi(M(x_{n-1},x_n))), \tag{2.11}$$

where

$$M(x_{n-1},x_n) = \max\left\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{2}\right\}$$

(2.12)

$$= \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2}\right\} \le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}.$$

If $d(x_n,x_{n+1}) > d(x_{n-1},x_n)$ then from (2.11) follows (because $\psi(t) < t, t > 0$):

$$\varphi(d(x_n,x_{n+1})) \le \psi(\varphi(d(x_n,x_{n+1}))) < \varphi(d(x_n,x_{n+1})). \tag{2.13}$$

A contradiction.

Therefore, for all $n \in \{0, 1, ...\}$ we obtain (because ψ is nondecreasing):

$$\varphi(d(x_n, x_{n+1})) \leq \psi(\varphi(d(x_{n-1}, x_n))). \tag{2.14}$$

That is., we have that

$$\varphi(d(x_n, x_{n+1})) \leq \psi(\varphi(d(x_{n-1}, x_n))) \leq ... \leq \psi^n(\varphi(d(x_0, x_1))). \tag{2.15}$$

Hence, $\varphi(d(x_n, x_{n+1})) \to 0$, i.e., $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence. If this is not case, then according to Lemma 1.7. by putting in $x = x_{m(k)-j(k)}, y = x_{n(k)}$ in (2.9) we have:

$$\varphi\left(d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right)\right) \leq \psi\left(\varphi\left(M\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right)\right), \tag{2.16}$$

where

(2.17)

$$\begin{aligned} & M\left(x_{m(k)-j(k)}, x_{n(k)}\right) \\ = \; & \max\left\{d\left(x_{m(k)-j(k)}, x_{n(k)}\right), d\left(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right),\right. \\ & \left.\frac{d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right) + d\left(x_{m(k)-j(k)+1}, x_{n(k)}\right)}{2}\right\}. \end{aligned}$$

First of all, we have

$$\lim_{k\to\infty} M\left(x_{m(k)-j(k)}, x_{n(k)}\right) = \max\left\{\varepsilon, 0, 0, \frac{\varepsilon+\varepsilon}{2}\right\} = \varepsilon, \tag{2.18}$$

that is.,

$$\lim_{k\to\infty} \varphi\left(M\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right) = \varphi\left(\lim_{k\to\infty} M\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right) = \varphi(\varepsilon). \tag{2.19}$$

Further from (2.17) as well as by the properties of the functions ψ and φ follows:

(2.20)

$$0 < \varphi(\varepsilon) \leq \lim_{k\to\infty} \psi\left(\varphi\left(M\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right)\right) < \lim_{k\to\infty} M\left(x_{m(k)-j(k)}, x_{n(k)}\right) = \varphi(\varepsilon).$$

A contradiction.

Hence $\{x_n\}$ is a Cauchy sequence.

The rest of the proof is further as in any of papers [16]-[18].

The proof for the case of cyclic generalized (ψ, φ) −contraction is very similar.

□

Finaly, we announce the following important and significant remark regarding several proofs that Picard sequence $\{x_n\}$ is a Cauchy:

Remark 2.5. Using our Lemma 1.7. we can obtain much shorter proofs that Picard sequence $x_{n+1} = fx_n$, in each of the papers [2], [3], [6], [7], [9], [12], [11] and [13] is a Cauchy. For this, it is sufficient putting $x = x_{m(k)-j(k)}, y = x_{n(k)}$ in the contractive condition of cyclic type theorem in each of the papers.

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