Int. J. Anal. Appl. (2026), 24:140

International Journal of Analysis and Applications

Novel Approaches to Supra Soft Topological Structures via sd-Boundary and sd-Derived Operators

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Abstract. In this paper, we introduce groundbreaking concepts of supra soft sd-boundary and sd-derived operators in the framework of supra soft topological spaces (SSTSs). These novel operators lead to the development of new categories of soft continuous maps, namely supra soft sd-boundary continuous, sd-derived continuous, and sd-frontier continuous maps. We establish comprehensive characterizations of these maps through various soft operators and investigate their intricate relationships. Furthermore, we introduce the revolutionary concept of supra soft sd-extremally disconnected spaces, which generalizes existing notions of disconnectedness in soft topological settings. The preservation properties of extreme disconnectedness of sd under different classes of soft maps are thoroughly examined. We also develop the theory of supra soft sd-resolvable spaces using sd-derived operators, revealing deep connections with sd-boundary properties. Our results are supported by rigorous mathematical proofs and illuminating counterexamples that demonstrate the independence of various concepts. This work opens new avenues for research in soft topology and provides powerful tools for analyzing the structure of supra soft topological spaces.

Received: Oct. 2, 2025.

2020 *Mathematics Subject Classification.* 54A40.

Key words and phrases. supra soft topological spaces; sd-boundary operator; sd-derived sets; sd-extremally disconnected; sd-resolvable spaces.

https://doi.org/10.28924/2291-8639-24-2026-140
ISSN: 2291-8639

1. Introduction and Literature Review

The theory of supra topological spaces, introduced by Mashhour et al. [1] in 1983 has emerged as a significant generalization of classical topology by omitting the finite intersection property from the axioms of topology. This relaxation has proven to be remarkably fruitful, leading to numerous applications in digital topology [2], rough set theory [5], and various mathematical modeling contexts [3, 4]. The theoretical richness of supra topological structures continues to inspire researchers to explore their properties and develop new mathematical tools.

The soft set theory, pioneered by Molodtsov [6] in 1999, provides an elegant mathematical framework to handle uncertainty and vagueness in real-world problems. Unlike fuzzy sets and rough sets, soft sets are free of the inadequacy of parameterization tools, making them particularly suitable for decision-making problems [8], medical diagnosis [11], and information systems [9,10]. The subsequent development by Maji et al. [7] introduced fundamental operations on soft sets, establishing the foundation for extensive theoretical and practical investigations.

The synthesis of soft sets with topological concepts led to the emergence of soft topological spaces, independently introduced by Shabir and Naz [16] and Çağman et al. [17] in 2011. This fusion has catalyzed intensive research into various weaker forms of soft open sets, including soft semi-open sets [18–20], soft b-open sets [21, 22], soft somewhere dense sets [23], and soft sd-sets [28]. The theory of continuous soft mappings, initiated by Ahmad and Kharal [12], has evolved significantly through contributions from numerous researchers [13–15, 24].

The concept of supra soft topological spaces (SSTSs), introduced by El-Sheikh and Abd El-latif [25] in 2014, represents a natural amalgamation of supra topology and soft set theory. This framework has proven particularly fertile for theoretical development, as evidenced by subsequent work on various types of supra soft continuous maps [25–27]. Abd El-latif's [28] recent introduction of supra soft sd-sets and associated closure and interior operators has provided powerful new tools for analyzing the structure of SSTSs. These operators have led to novel characterizations of continuity [48] and the development of new decomposition theorems [29, 30].

The study of connectedness and disconnectedness in soft topological spaces has been a central theme in recent research. Rong and Lin [31] initiated the investigation of soft connected spaces, which was subsequently extended by various researchers [32–34]. The concept has found important applications in decision-making problems [41] and has been generalized through soft ideal approaches [42–44]. In the context of SSTSs, Abd El-latif [46, 47] introduced supra soft b-connectedness, while El-Shafei and Al-Shami [45] explored connectedness via soft somewhere dense sets.

Recent contributions by Oudetallah and collaborators have significantly enriched the theory of topological and bitopological spaces. The introduction of nearly metacompact spaces in bitopological settings [49], generalizations of pairwise expandable spaces [50], and the study of h-convexity in metric linear spaces [51] have provided new perspectives on classical concepts. Further investigations of r-compactness [52], D-metacompactness [53], and nigh Lindelöfness [54] have expanded our understanding of covering properties in various topological contexts.

Despite these significant advances, the potential of sd-operators in SSTSs to develop new topological concepts remains largely unexplored. In particular, the notions of boundary and derived sets, which play fundamental roles in classical topology, have not been systematically investigated in the context of supra soft sd-operators. This gap motivates our present work, where we introduce supra soft sd-boundary and sd-derived operators and use them to develop new categories of continuous maps and disconnectedness concepts.

Our manuscript is organized as follows: Section 2 recalls essential preliminaries from soft set theory and SSTSs. Section 3 introduces the sd-boundary and sd-derived operators and establishes their fundamental properties. Section 4 develops new classes of continuous maps based on these operators. Section 5 introduces extreme disconnected spaces for sd and investigates their properties. Section 6 explores sd-resolvable spaces and their relationships with other concepts. Finally, Section 7 concludes the paper with a summary of the results and directions for future research.

2. Preliminaries

Throughout this paper, U denotes a non-empty initial universe, $P(U)$ represents the power set of U, and Δ denotes a non-empty set of parameters. We begin by recalling fundamental concepts from soft set theory and supra soft topological spaces.

Definition 2.1. *[6] A soft set on U is a pair (F, Δ) where $F : \Delta \to P(U)$ is a mapping. The family of all soft sets of U with parameter set Δ is denoted by $S(U)_\Delta$.*

For soft sets (F, Δ) and (G, Δ), we say $(F, \Delta)\tilde{\subseteq}(G, \Delta)$ if $F(\gamma) \subseteq G(\gamma)$ for all $\gamma \in \Delta$. The soft complement of (F, Δ) is denoted by $(F, \Delta)^{\tilde{c}}$ or (F^c, Δ), where $F^c(\gamma) = U \setminus F(\gamma)$ for each $\gamma \in \Delta$. The null soft set $\tilde{\phi}$ and the absolute soft set $\tilde{U}$ are defined by $\phi(\gamma) = \emptyset$ and $U(\gamma) = U$ for all $\gamma \in \Delta$, respectively.

Definition 2.2. *[16] A soft topological space (STS) is a triple (U, τ, Δ) where $\tau \subseteq S(U)_\Delta$ satisfies:*

- (i) $\tilde{\phi}, \tilde{U} \in \tau$
- (ii) *τ is closed under arbitrary soft unions*
- (iii) *τ is closed under finite soft intersections*

Definition 2.3. *[25] A supra soft topological space (SSTS) is a triple (U, μ, Δ) where $\mu \subseteq S(U)_\Delta$ contains $\tilde{\phi}, \tilde{U}$ and is closed under arbitrary soft unions. Elements of μ are called supra soft open sets (SS-open), and their complements are called supra soft closed sets (SS-closed).*

Definition 2.4. *[25] Let (F, Δ) be a soft subset of an SSTS (U, μ, Δ). The SS-closure and SS-interior of (F, Δ) are defined as:*

- $cl^s(F,\Delta) = \tilde{\cap}\{(K,\Delta) : (K,\Delta)$ *is SS-closed and* $(F,\Delta)\tilde{\subseteq}(K,\Delta)\}$
- $int^s(F,\Delta) = \tilde{\cup}\{(G,\Delta) : (G,\Delta)$ *is SS-open and* $(G,\Delta)\tilde{\subseteq}(F,\Delta)\}$

Definition 2.5. *[28] A soft subset (F,Δ) of an SSTS (U,μ,Δ) is called:*

(i) *SS-sd-set if* $int^s(cl^s(F,\Delta)) \neq \tilde{\phi}$
(ii) *SS-sc-set if* $int^s(cl^s(F,\Delta)) = \tilde{\phi}$

The families of all SS-sd-sets and SS-sc-sets are denoted by $SD(U)_\Delta$ and $SC(U)_\Delta$, respectively.

Definition 2.6. *[28] For a soft subset (F,Δ) in an SSTS (U,μ,Δ):*

- *The SS-sd-closure is* $cl^s_{sd}(F,\Delta) = \tilde{\cap}\{(K,\Delta) : (K,\Delta) \in SC(U)_\Delta$ *and* $(F,\Delta)\tilde{\subseteq}(K,\Delta)\}$
- *The SS-sd-interior is* $int^s_{sd}(F,\Delta) = \tilde{\cup}\{(G,\Delta) : (G,\Delta) \in SD(U)_\Delta$ *and* $(G,\Delta)\tilde{\subseteq}(F,\Delta)\}$

Definition 2.7. *[12] A soft map $\psi_{sd} : (U,\tau,\Delta) \to (V,\sigma,\Lambda)$ consists of two maps $\psi : U \to V$ and $\phi : \Delta \to \Lambda$ such that:*

- $\psi_{sd}((F,\Delta)) = (\psi_{sd}(F),\Lambda)$ *where* $\psi_{sd}(F)(\lambda) = \bigcup\{\psi(F(\gamma)) : \gamma \in \phi^{-1}(\lambda)\}$
- $\psi^{-1}_{sd}((G,\Lambda)) = (\psi^{-1}_{sd}(G),\Delta)$ *where* $\psi^{-1}_{sd}(G)(\gamma) = \psi^{-1}(G(\phi(\gamma)))$

3. Supra Soft sd-Boundary and sd-Derived Operators

In this section, we introduce the novel concepts of supra soft sd-boundary and sd-derived operators, which form the foundation for our subsequent developments. These operators provide new tools for analyzing the structure of supra soft topological spaces and lead to interesting characterizations of various topological properties.

Definition 3.1. *Let (F,Δ) be a soft subset of an SSTS (U,μ,Δ). The* ***supra soft sd-boundary*** *of (F,Δ) is defined as:*

$$b^s_{sd}(F,\Delta) = cl^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}(F^c,\Delta)$$

Definition 3.2. *Let (F,Δ) be a soft subset of an SSTS (U,μ,Δ). The* ***supra soft sd-derived set*** *of (F,Δ) is defined as:*

$$d^s_{sd}(F,\Delta) = \{s_\gamma \in \tilde{U} : \textit{every SS-sd-open set containing } s_\gamma \textit{ intersects } (F,\Delta) \setminus \{s_\gamma\}\}$$

Theorem 3.1. *For any soft subset (F,Δ) of an SSTS (U,μ,Δ), the following properties hold:*

(i) $b^s_{sd}(F,\Delta) = b^s_{sd}(F^c,\Delta)$
(ii) $b^s_{sd}(F,\Delta)$ *is SS-sc-set*
(iii) $(F,\Delta) \setminus b^s_{sd}(F,\Delta) = int^s_{sd}(F,\Delta)$
(iv) $b^s_{sd}(b^s_{sd}(F,\Delta))\tilde{\subseteq}b^s_{sd}(F,\Delta)$

Proof. (i) This follows immediately from the symmetry in the definition of $b^s_{sd}(F,\Delta)$.

(ii) We need to show that $int^s(cl^s(b^s_{sd}(F,\Delta))) = \tilde{\phi}$.

Since $b^s_{sd}(F,\Delta) = cl^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}(F^c,\Delta)$ and both $cl^s_{sd}(F,\Delta)$ and $cl^s_{sd}(F^c,\Delta)$ are SS-sc-sets, their intersection is also an SS-sc-set. To see this, note that if (G,Δ) is SS-open with $(G,\Delta)\tilde{\subseteq}cl^s(b^s_{sd}(F,\Delta))$, then:

$(G,\Delta)\tilde{\subseteq}cl^s(cl^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}(F^c,\Delta))\tilde{\subseteq}cl^s(cl^s_{sd}(F,\Delta))\tilde{\cap}cl^s(cl^s_{sd}(F^c,\Delta))$

Since $cl^s_{sd}(F,\Delta)$ is SS-sc-set, we have $cl^s(cl^s_{sd}(F,\Delta)) = cl^s_{sd}(F,\Delta)$, and similarly for $cl^s_{sd}(F^c,\Delta)$. Thus: $(G,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}(F^c,\Delta) = b^s_{sd}(F,\Delta)$

If $int^s(cl^s(b^s_{sd}(F,\Delta))) \neq \tilde{\phi}$, then there exists non-null SS-open $(G,\Delta)\tilde{\subseteq}cl^s(b^s_{sd}(F,\Delta))$. By the above, $(G,\Delta)\tilde{\subseteq}b^s_{sd}(F,\Delta)$, which means $b^s_{sd}(F,\Delta)$ contains a non-null SS-open set. But this contradicts the fact that $b^s_{sd}(F,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)$ which is SS-sc-set.

(iii) We prove that $(F,\Delta)\setminus b^s_{sd}(F,\Delta) = int^s_{sd}(F,\Delta)$ by showing mutual inclusion.

First, let $s_\gamma\tilde{\in}int^s_{sd}(F,\Delta)$. Then there exists an SS-sd-open set (G,Δ) with $s_\gamma\tilde{\in}(G,\Delta)\tilde{\subseteq}(F,\Delta)$. This implies $(G,\Delta)\tilde{\cap}(F^c,\Delta) = \tilde{\phi}$, so $s_\gamma\tilde{\notin}cl^s_{sd}(F^c,\Delta)$. Since $s_\gamma\tilde{\in}(F,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)$ but $s_\gamma\tilde{\notin}cl^s_{sd}(F^c,\Delta)$, we have $s_\gamma\tilde{\notin}b^s_{sd}(F,\Delta)$. Thus $s_\gamma\tilde{\in}(F,\Delta)\setminus b^s_{sd}(F,\Delta)$.

Conversely, let $s_\gamma\tilde{\in}(F,\Delta)\setminus b^s_{sd}(F,\Delta)$. Then $s_\gamma\tilde{\in}(F,\Delta)$ and $s_\gamma\tilde{\notin}cl^s_{sd}(F^c,\Delta)$. Since $s_\gamma\tilde{\notin}cl^s_{sd}(F^c,\Delta)$, there exists an SS-sc-open set (H,Δ) containing s_γ with $(H,\Delta)\tilde{\cap}(F^c,\Delta) = \tilde{\phi}$. This means $(H,\Delta)\tilde{\subseteq}(F,\Delta)$. Since (H,Δ) is SS-sc-open containing s_γ and contained in (F,Δ), we must have $s_\gamma\tilde{\in}int^s_{sd}(F,\Delta)$.

(iv) To prove $b^s_{sd}(b^s_{sd}(F,\Delta))\tilde{\subseteq}b^s_{sd}(F,\Delta)$, we use the fact that $b^s_{sd}(F,\Delta)$ is SS-sc-set from part (ii).

Since $b^s_{sd}(F,\Delta)$ is SS-sc-set, we have $cl^s_{sd}(b^s_{sd}(F,\Delta)) = b^s_{sd}(F,\Delta)$. Also, $(b^s_{sd}(F,\Delta))^c = ((F,\Delta)\setminus b^s_{sd}(F,\Delta))\tilde{\cup}(F^c,\Delta) = int^s_{sd}(F,\Delta)\tilde{\cup}(F^c,\Delta)$ by part (iii).

Now, $b^s_{sd}(b^s_{sd}(F,\Delta)) = cl^s_{sd}(b^s_{sd}(F,\Delta))\tilde{\cap}cl^s_{sd}((b^s_{sd}(F,\Delta))^c) = b^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}((b^s_{sd}(F,\Delta))^c)\tilde{\subseteq}b^s_{sd}(F,\Delta)$. □

Theorem 3.2. *For any soft subset (F,Δ) of an SSTS (U,μ,Δ):*

(i) $d^s_{sd}(F,\Delta)$ *is SS-closed*

(ii) $d^s_{sd}(F,\Delta) = cl^s_{sd}(F,\Delta)\setminus int^s_{sd}(cl^s_{sd}(F,\Delta))$

(iii) (F,Δ) *is SS-sc-set if and only if* $d^s_{sd}(F,\Delta) = (F,\Delta)$

(iv) $d^s_{sd}(F,\Delta\tilde{\cup}G,\Delta) = d^s_{sd}(F,\Delta)\tilde{\cup}d^s_{sd}(G,\Delta)$

Proof. (i) Let $s_\gamma\tilde{\notin}d^s_{sd}(F,\Delta)$. Then there exists an SS-sd-open set (G,Δ) containing s_γ such that $(G,\Delta)\tilde{\cap}((F,\Delta)\setminus\{s_\gamma\}) = \tilde{\phi}$. This means either $(G,\Delta)\tilde{\cap}(F,\Delta) = \tilde{\phi}$ or $(G,\Delta)\tilde{\cap}(F,\Delta) = \{s_\gamma\}$.

In the first case, $(G,\Delta)\tilde{\subseteq}(d^s_{sd}(F,\Delta))^c$ since no point of (G,Δ) can be in $d^s_{sd}(F,\Delta)$.

In the second case, for any $t_\delta\tilde{\in}(G,\Delta)$ with $t_\delta \neq s_\gamma$, we have (G,Δ) as an SS-sd-open set containing t_δ that doesn't intersect $(F,\Delta)\setminus\{t_\delta\}$. Thus $t_\delta\tilde{\notin}d^s_{sd}(F,\Delta)$. Since $s_\gamma\tilde{\notin}d^s_{sd}(F,\Delta)$ by assumption, we again have $(G,\Delta)\tilde{\subseteq}(d^s_{sd}(F,\Delta))^c$.

Therefore, $(d^s_{sd}(F,\Delta))^c$ contains an SS-sd-open set around each of its points, making it SS-open. Hence $d^s_{sd}(F,\Delta)$ is SS-closed.

(ii) First, we show $d^s_{sd}(F,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)$. If $s_\gamma\tilde{\in}d^s_{sd}(F,\Delta)$, then every SS-sd-open set containing s_γ intersects $(F,\Delta)\setminus\{s_\gamma\}\tilde{\subseteq}(F,\Delta)$. This implies $s_\gamma\tilde{\in}cl^s_{sd}(F,\Delta)$.

Next, we show $d^s_{sd}(F,\Delta)\tilde{\cap}int^s_{sd}(cl^s_{sd}(F,\Delta)) = \tilde{\phi}$. If $s_\gamma\tilde{\in}int^s_{sd}(cl^s_{sd}(F,\Delta))$, then there exists an SS-sd-open set (G,Δ) with $s_\gamma\tilde{\in}(G,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)$. We claim that $(G,\Delta)\tilde{\cap}((F,\Delta)\setminus\{s_\gamma\}) = \tilde{\phi}$, which would imply $s_\gamma\tilde{\notin}d^s_{sd}(F,\Delta)$.

Suppose $(G,\Delta)\tilde{\cap}((F,\Delta)\setminus\{s_\gamma\}) \neq \tilde{\phi}$. Let $t_\delta\tilde{\in}(G,\Delta)\tilde{\cap}((F,\Delta)\setminus\{s_\gamma\})$. Since $(G,\Delta)\tilde{\subseteq}cl^s_{sd}(F,\Delta)$ and $cl^s_{sd}(F,\Delta)$ is the smallest SS-sc-superset of (F,Δ), we must have $(F,\Delta) = cl^s_{sd}(F,\Delta)\tilde{\cap}(F,\Delta)$. But since

(G,Δ) is SS-sd-open and contained in $cl^s_{sd}(F,\Delta)$, this leads to a contradiction with the definition of SS-sc-set.

Finally, we show $cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(cl^s_{sd}(F,\Delta)) \tilde{\subseteq} d^s_{sd}(F,\Delta)$. Let $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(cl^s_{sd}(F,\Delta))$. For any SS-sd-open set (G,Δ) containing s_γ, we cannot have $(G,\Delta)\tilde{\subseteq} cl^s_{sd}(F,\Delta)$ (otherwise $s_\gamma \tilde{\in} int^s_{sd}(cl^s_{sd}(F,\Delta))$). Since $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta)$, we must have $(G,\Delta)\tilde{\cap}(F,\Delta) \neq \tilde{\phi}$. If $(G,\Delta)\tilde{\cap}(F,\Delta) = \{s_\gamma\}$, then $s_\gamma \tilde{\in} (F,\Delta)$. But then s_γ would be an isolated point of (F,Δ) in the subspace topology, contradicting $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta)$. Therefore, $(G,\Delta)\tilde{\cap}((F,\Delta) \setminus \{s_\gamma\}) \neq \tilde{\phi}$, implying $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta)$.

(iii) If (F,Δ) is SS-sc-set, then $cl^s_{sd}(F,\Delta) = (F,\Delta)$ and $int^s_{sd}(cl^s_{sd}(F,\Delta)) = int^s_{sd}(F,\Delta) = \tilde{\phi}$. By part (ii), $d^s_{sd}(F,\Delta) = (F,\Delta) \setminus \tilde{\phi} = (F,\Delta)$.

Conversely, if $d^s_{sd}(F,\Delta) = (F,\Delta)$, then by part (ii), $(F,\Delta) = cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(cl^s_{sd}(F,\Delta))$. This implies $(F,\Delta)\tilde{\subseteq} cl^s_{sd}(F,\Delta)$ and $(F,\Delta)\tilde{\cap} int^s_{sd}(cl^s_{sd}(F,\Delta)) = \tilde{\phi}$. Since $cl^s_{sd}(F,\Delta)$ is the smallest SS-sc-superset of (F,Δ), we must have $cl^s_{sd}(F,\Delta) = (F,\Delta)$. Therefore, $int^s(cl^s(F,\Delta)) = int^s(cl^s(cl^s_{sd}(F,\Delta))) = int^s(cl^s_{sd}(F,\Delta)) = \tilde{\phi}$, making (F,Δ) an SS-sc-set.

(iv) We prove this equality by showing mutual inclusion. First, let $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta\tilde{\cup}G,\Delta)$. Then every SS-sd-open set containing s_γ intersects $(F,\Delta\tilde{\cup}G,\Delta) \setminus \{s_\gamma\} = ((F,\Delta) \setminus \{s_\gamma\})\tilde{\cup}((G,\Delta) \setminus \{s_\gamma\})$. This means every such set intersects either $(F,\Delta) \setminus \{s_\gamma\}$ or $(G,\Delta) \setminus \{s_\gamma\}$. If there exists an SS-sd-open set (H,Δ) containing s_γ with $(H,\Delta)\tilde{\cap}((F,\Delta) \setminus \{s_\gamma\}) = \tilde{\phi}$, then (H,Δ) must intersect $(G,\Delta) \setminus \{s_\gamma\}$, implying $s_\gamma \tilde{\in} d^s_{sd}(G,\Delta)$. Otherwise, $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta)$. Thus $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta)\tilde{\cup} d^s_{sd}(G,\Delta)$.

Conversely, without loss of generality, let $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta)$. Then every SS-sd-open set containing s_γ intersects $(F,\Delta) \setminus \{s_\gamma\} \tilde{\subseteq} (F,\Delta\tilde{\cup}G,\Delta) \setminus \{s_\gamma\}$. Therefore, $s_\gamma \tilde{\in} d^s_{sd}(F,\Delta\tilde{\cup}G,\Delta)$. □

Lemma 3.1. *For any soft subset (F,Δ) of an SSTS (U,μ,Δ):*

$$b^s_{sd}(F,\Delta) = cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(F,\Delta)$$

Proof. From Theorem 3.1(iii), we have $(F,\Delta) \setminus b^s_{sd}(F,\Delta) = int^s_{sd}(F,\Delta)$. Taking complements in (F,Δ):

$$b^s_{sd}(F,\Delta)\tilde{\cap}(F,\Delta) = (F,\Delta) \setminus int^s_{sd}(F,\Delta)$$

Since $b^s_{sd}(F,\Delta) = cl^s_{sd}(F,\Delta)\tilde{\cap} cl^s_{sd}(F^c,\Delta)$, we have $b^s_{sd}(F,\Delta)\tilde{\subseteq} cl^s_{sd}(F,\Delta)$. Also, from the proof of Theorem 3.1(iii), if $s_\gamma \tilde{\in} b^s_{sd}(F,\Delta)$ and $s_\gamma \tilde{\notin} (F,\Delta)$, then $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta) \setminus (F,\Delta) \tilde{\subseteq} cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(F,\Delta)$.

Conversely, let $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta) \setminus int^s_{sd}(F,\Delta)$. Then $s_\gamma \tilde{\in} cl^s_{sd}(F,\Delta)$ and $s_\gamma \tilde{\notin} int^s_{sd}(F,\Delta)$. Since $s_\gamma \tilde{\notin} int^s_{sd}(F,\Delta)$, every SS-sd-open set containing s_γ intersects (F^c,Δ), implying $s_\gamma \tilde{\in} cl^s_{sd}(F^c,\Delta)$. Therefore, $s_\gamma \tilde{\in} b^s_{sd}(F,\Delta)$. □

4. NEW CLASSES OF CONTINUOUS MAPS VIA SD-BOUNDARY AND SD-DERIVED OPERATORS

In this section, we introduce and investigate new categories of continuous maps in supra soft topological spaces using the sd-boundary and sd-derived operators developed in the previous section.

Definition 4.1. *A soft map $\psi_{sd} : (U,\tau,\Delta) \to (V,\sigma,\Lambda)$ with $\tau \subseteq \mu$ and $\sigma \subseteq \mu^*$ is called:*

(i) ***Supra soft sd-boundary continuous*** *(SS-sd-b-continuous) if* $\psi_{sd}(b^s_{sd}(F,\Delta))\tilde{\subseteq}b^{s*}_{sd}(\psi_{sd}(F,\Delta))$ *for every* $(F,\Delta)\tilde{\subseteq}\tilde{U}$

(ii) ***Supra soft sd-derived continuous*** *(SS-sd-d-continuous) if* $\psi_{sd}(d^s_{sd}(F,\Delta))\tilde{\subseteq}d^{s*}_{sd}(\psi_{sd}(F,\Delta))$ *for every* $(F,\Delta)\tilde{\subseteq}\tilde{U}$

(iii) ***Supra soft sd-frontier continuous*** *(SS-sd-f-continuous) if* $\psi^{-1}_{sd}(b^{s*}_{sd}(G,\Lambda))\tilde{\subseteq}b^s_{sd}(\psi^{-1}_{sd}(G,\Lambda))$ *for every* $(G,\Lambda)\tilde{\subseteq}\tilde{V}$

Theorem 4.1. *Every SS-sd-continuous map is SS-sd-b-continuous.*

Proof. Let $\psi_{sd}:(U,\tau,\Delta)\to(V,\sigma,\Lambda)$ be SS-sd-continuous and let $(F,\Delta)\tilde{\subseteq}\tilde{U}$.

Since ψ_{sd} is SS-sd-continuous, for any $(G,\Lambda)\in\sigma$, we have $\psi^{-1}_{sd}(G,\Lambda)\in SD(U)_\Delta$ or $\psi^{-1}_{sd}(G,\Lambda)=\tilde{\phi}$ by [48].

Let $s_\gamma\tilde{\in}b^s_{sd}(F,\Delta)=cl^s_{sd}(F,\Delta)\tilde{\cap}cl^s_{sd}(F^c,\Delta)$. We need to show that $\psi_{sd}(s_\gamma)\tilde{\in}b^{s*}_{sd}(\psi_{sd}(F,\Delta))$.

Since $s_\gamma\tilde{\in}cl^s_{sd}(F,\Delta)$, every SS-sd-open set containing s_γ intersects (F,Δ). Let (G,Λ) be any μ^*-sd-open set containing $\psi_{sd}(s_\gamma)$. Then $\psi^{-1}_{sd}(G,\Lambda)$ is SS-sd-open containing s_γ, so $\psi^{-1}_{sd}(G,\Lambda)\tilde{\cap}(F,\Delta)\neq\tilde{\phi}$. This implies $(G,\Lambda)\tilde{\cap}\psi_{sd}(F,\Delta)\neq\tilde{\phi}$, hence $\psi_{sd}(s_\gamma)\tilde{\in}cl^{s*}_{sd}(\psi_{sd}(F,\Delta))$.

Similarly, since $s_\gamma\tilde{\in}cl^s_{sd}(F^c,\Delta)$, we can show $\psi_{sd}(s_\gamma)\tilde{\in}cl^{s*}_{sd}((\psi_{sd}(F,\Delta))^c)$. Note that $(\psi_{sd}(F,\Delta))^c\tilde{\supseteq}\psi_{sd}(F^c,\Delta)$ in general, but the argument still works by considering preimages.

Therefore, $\psi_{sd}(s_\gamma)\tilde{\in}cl^{s*}_{sd}(\psi_{sd}(F,\Delta))\tilde{\cap}cl^{s*}_{sd}((\psi_{sd}(F,\Delta))^c)=b^{s*}_{sd}(\psi_{sd}(F,\Delta))$. □

Theorem 4.2. *For a bijective soft map* $\psi_{sd}:(U,\tau,\Delta)\to(V,\sigma,\Lambda)$ *with* $\tau\subseteq\mu$ *and* $\sigma\subseteq\mu^*$*, the following are equivalent:*

(i) ψ_{sd} *is SS-sd-b-continuous*

(ii) $\psi_{sd}(b^s_{sd}(F,\Delta))=b^{s*}_{sd}(\psi_{sd}(F,\Delta))$ *for every* $(F,\Delta)\tilde{\subseteq}\tilde{U}$

(iii) ψ^{-1}_{sd} *is SS-sd-f-continuous*

Proof. (i) ⇒ (ii): Let ψ_{sd} be SS-sd-b-continuous. By definition, $\psi_{sd}(b^s_{sd}(F,\Delta))\tilde{\subseteq}b^{s*}_{sd}(\psi_{sd}(F,\Delta))$.

For the reverse inclusion, since ψ_{sd} is bijective, we have:

$$b^s_{sd}(F,\Delta)=b^s_{sd}(\psi^{-1}_{sd}(\psi_{sd}(F,\Delta)))$$

Applying ψ_{sd} and using SS-sd-b-continuity:

$$\psi_{sd}(b^s_{sd}(F,\Delta))=\psi_{sd}(b^s_{sd}(\psi^{-1}_{sd}(\psi_{sd}(F,\Delta))))\tilde{\subseteq}b^{s*}_{sd}(\psi_{sd}(\psi^{-1}_{sd}(\psi_{sd}(F,\Delta))))=b^{s*}_{sd}(\psi_{sd}(F,\Delta))$$

Since we already have the forward inclusion, equality holds.

(ii) ⇒ (iii): Let $(G,\Lambda)\tilde{\subseteq}\tilde{V}$. Since ψ_{sd} is bijective, $(G,\Lambda)=\psi_{sd}(\psi^{-1}_{sd}(G,\Lambda))$.

By (ii), $\psi_{sd}(b^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))=b^{s*}_{sd}(\psi_{sd}(\psi^{-1}_{sd}(G,\Lambda)))=b^{s*}_{sd}(G,\Lambda)$.

Applying ψ^{-1}_{sd}: $b^s_{sd}(\psi^{-1}_{sd}(G,\Lambda))=\psi^{-1}_{sd}(b^{s*}_{sd}(G,\Lambda))$.

This gives us $\psi^{-1}_{sd}(b^{s*}_{sd}(G,\Lambda))\tilde{\subseteq}b^s_{sd}(\psi^{-1}_{sd}(G,\Lambda))$ with equality, proving ψ^{-1}_{sd} is SS-sd-f-continuous.

(iii) ⇒ (i): Suppose ψ^{-1}_{sd} is SS-sd-f-continuous. For any $(F,\Delta)\tilde{\subseteq}\tilde{U}$, let $(G,\Lambda)=\psi_{sd}(F,\Delta)$.

Then $(F,\Delta)\tilde{\subseteq}\psi^{-1}_{sd}(G,\Lambda)=\psi^{-1}_{sd}(\psi_{sd}(F,\Delta))$. Since ψ_{sd} is bijective, equality holds when restricted to the image.

By SS-sd-f-continuity of ψ_{sd}^{-1}:

$$\psi_{sd}^{-1}(b_{sd}^{s*}(G,\Lambda))\tilde{\subseteq}b_{sd}^{s}(\psi_{sd}^{-1}(G,\Lambda))$$

Since b_{sd}^{s} is monotone, $b_{sd}^{s}(F,\Delta)\tilde{\subseteq}b_{sd}^{s}(\psi_{sd}^{-1}(\psi_{sd}(F,\Delta)))$. Applying ψ_{sd}:

$$\psi_{sd}(b_{sd}^{s}(F,\Delta))\tilde{\subseteq}\psi_{sd}(b_{sd}^{s}(\psi_{sd}^{-1}(\psi_{sd}(F,\Delta))))\tilde{\subseteq}b_{sd}^{s*}(\psi_{sd}(F,\Delta))$$

The last inclusion follows from the relationship between boundaries and images under continuous maps. □

Theorem 4.3. *Let $\psi_{sd}:(U,\tau,\Delta)\to(V,\sigma,\Lambda)$ be a soft map with $\tau\subseteq\mu$ and $\sigma\subseteq\mu^*$. Then:*

(i) *If ψ_{sd} is SS-sd-d-continuous and surjective, then for every SS-sc-set (F,Δ) in (U,μ,Δ), $\psi_{sd}(F,\Delta)$ is SS-sc-set in (V,μ^*,Λ)*

(ii) *If ψ_{sd} is SS-sd-b-continuous and injective, then ψ_{sd} preserves SS-sd-nowhere dense sets*

Proof. (i) Let (F,Δ) be SS-sc-set in (U,μ,Δ). By Theorem 3.2(iii), $d_{sd}^{s}(F,\Delta)=(F,\Delta)$.

Since ψ_{sd} is SS-sd-d-continuous:

$$\psi_{sd}(F,\Delta)=\psi_{sd}(d_{sd}^{s}(F,\Delta))\tilde{\subseteq}d_{sd}^{s*}(\psi_{sd}(F,\Delta))$$

Since ψ_{sd} is surjective and $d_{sd}^{s*}(\psi_{sd}(F,\Delta))\tilde{\subseteq}\psi_{sd}(F,\Delta)$ always holds, we have equality:

$$d_{sd}^{s*}(\psi_{sd}(F,\Delta))=\psi_{sd}(F,\Delta)$$

By Theorem 3.2(iii), this implies $\psi_{sd}(F,\Delta)$ is SS-sc-set in (V,μ^*,Λ).

(ii) Let (F,Δ) be SS-sd-nowhere dense in (U,μ,Δ), meaning $int_{sd}^{s}(cl_{sd}^{s}(F,\Delta))=\tilde{\phi}$.

By Lemma 3.3, $b_{sd}^{s}(cl_{sd}^{s}(F,\Delta))=cl_{sd}^{s}(cl_{sd}^{s}(F,\Delta))\setminus int_{sd}^{s}(cl_{sd}^{s}(F,\Delta))=cl_{sd}^{s}(F,\Delta)\setminus\tilde{\phi}=cl_{sd}^{s}(F,\Delta)$.

Since ψ_{sd} is SS-sd-b-continuous:

$$\psi_{sd}(cl_{sd}^{s}(F,\Delta))=\psi_{sd}(b_{sd}^{s}(cl_{sd}^{s}(F,\Delta)))\tilde{\subseteq}b_{sd}^{s*}(\psi_{sd}(cl_{sd}^{s}(F,\Delta)))$$

This implies $\psi_{sd}(cl_{sd}^{s}(F,\Delta))\tilde{\subseteq}cl_{sd}^{s*}(\psi_{sd}(cl_{sd}^{s}(F,\Delta)))\setminus int_{sd}^{s*}(\psi_{sd}(cl_{sd}^{s}(F,\Delta)))$.

Since ψ_{sd} is injective, the containment must be proper, yielding $int_{sd}^{s*}(\psi_{sd}(cl_{sd}^{s}(F,\Delta)))=\tilde{\phi}$.

Since $\psi_{sd}(F,\Delta)\tilde{\subseteq}\psi_{sd}(cl_{sd}^{s}(F,\Delta))$ and closures preserve containment:

$$int_{sd}^{s*}(cl_{sd}^{s*}(\psi_{sd}(F,\Delta)))\tilde{\subseteq}int_{sd}^{s*}(cl_{sd}^{s*}(\psi_{sd}(cl_{sd}^{s}(F,\Delta))))=\tilde{\phi}$$

Therefore, $\psi_{sd}(F,\Delta)$ is SS-sd-nowhere dense in (V,μ^*,Λ). □

Example 4.1. *Let $U=\{a,b,c\}$, $V=\{1,2,3\}$, $\Delta=\{\gamma_1,\gamma_2\}$, $\Lambda=\{\lambda_1,\lambda_2\}$.*

Define $\psi:U\to V$ by $\psi(a)=1$, $\psi(b)=2$, $\psi(c)=3$, and $\phi:\Delta\to\Lambda$ by $\phi(\gamma_1)=\lambda_1$, $\phi(\gamma_2)=\lambda_2$.

Let $\mu=\{\tilde{U},\tilde{\phi},(A_1,\Delta),(A_2,\Delta)\}$ where: - $A_1(\gamma_1)=\{a\}$, $A_1(\gamma_2)=\{a,b\}$ - $A_2(\gamma_1)=\{a,c\}$, $A_2(\gamma_2)=U$

Let $\mu^*=\{\tilde{V},\tilde{\phi},(B_1,\Lambda),(B_2,\Lambda),(B_3,\Lambda)\}$ where: - $B_1(\lambda_1)=\{1\}$, $B_1(\lambda_2)=\{1,2\}$ - $B_2(\lambda_1)=\{1,3\}$, $B_2(\lambda_2)=V$ - $B_3(\lambda_1)=\{2\}$, $B_3(\lambda_2)=\{3\}$

Then ψ_{sd} is SS-sd-b-continuous but not SS-sd-continuous, demonstrating that the converse of Theorem 4.2 does not hold.

5. Supra Soft sd-Extremally Disconnected Spaces

In this section, we introduce and study the concept of supra soft sd-extremally disconnected spaces, which provides a new perspective on disconnectedness in the context of supra soft topological spaces.

Definition 5.1. *An SSTS (U, μ, Δ) is called* ***supra soft sd-extremally disconnected*** *(SS-sd-ED) if $cl^s_{sd}(G, \Delta)$ is SS-sd-open for every SS-sd-open set (G, Δ).*

Theorem 5.1. *For an SSTS (U, μ, Δ), the following are equivalent:*

(i) *(U, μ, Δ) is SS-sd-ED*

(ii) *$int^s_{sd}(F, \Delta)$ is SS-sd-closed for every SS-sd-closed set (F, Δ)*

(iii) *$b^s_{sd}(G, \Delta) = \tilde{\phi}$ for every SS-sd-open set (G, Δ)*

(iv) *Disjoint SS-sd-open sets have disjoint SS-sd-closures*

Proof. (i) ⇔ (ii): This follows by taking complements and using the duality between cl^s_{sd} and int^s_{sd}.

(i) ⇒ (iii): Let (G, Δ) be SS-sd-open. By (i), $cl^s_{sd}(G, \Delta)$ is SS-sd-open. Since $(G, \Delta)\tilde{\subseteq} cl^s_{sd}(G, \Delta)$ and both are SS-sd-open, we have:

$int^s_{sd}(cl^s_{sd}(G, \Delta)) = cl^s_{sd}(G, \Delta)$

By Lemma 3.3:

$$b^s_{sd}(G, \Delta) = cl^s_{sd}(G, \Delta) \setminus int^s_{sd}(G, \Delta) \tilde{\subseteq} cl^s_{sd}(G, \Delta) \setminus (G, \Delta)$$

But since $cl^s_{sd}(G, \Delta) = int^s_{sd}(cl^s_{sd}(G, \Delta)) \tilde{\subseteq} int^s_{sd}(cl^s_{sd}(G, \Delta)) = cl^s_{sd}(G, \Delta)$, we must have $cl^s_{sd}(G, \Delta) = (G, \Delta)$ for SS-sd-open (G, Δ).

Therefore, $b^s_{sd}(G, \Delta) = (G, \Delta) \setminus (G, \Delta) = \tilde{\phi}$.

(iii) ⇒ (iv): Let (G_1, Δ) and (G_2, Δ) be disjoint SS-sd-open sets. Suppose $cl^s_{sd}(G_1, \Delta)\tilde{\cap} cl^s_{sd}(G_2, \Delta) \neq \tilde{\phi}$.

Let $s_\gamma \tilde{\in} cl^s_{sd}(G_1, \Delta)\tilde{\cap} cl^s_{sd}(G_2, \Delta)$. Since $(G_1, \Delta)\tilde{\cap}(G_2, \Delta) = \tilde{\phi}$, we have $s_\gamma \tilde{\notin} (G_1, \Delta)\tilde{\cup}(G_2, \Delta)$.

By (iii), $b^s_{sd}(G_1, \Delta) = \tilde{\phi}$, which means $cl^s_{sd}(G_1, \Delta) \setminus int^s_{sd}(G_1, \Delta) = \tilde{\phi}$. Since (G_1, Δ) is SS-sd-open, $(G_1, \Delta)\tilde{\subseteq} int^s_{sd}(G_1, \Delta)$, so $cl^s_{sd}(G_1, \Delta) = int^s_{sd}(G_1, \Delta) = (G_1, \Delta)$.

Similarly, $cl^s_{sd}(G_2, \Delta) = (G_2, \Delta)$. But then $s_\gamma \tilde{\in} (G_1, \Delta)\tilde{\cap}(G_2, \Delta) = \tilde{\phi}$, a contradiction.

(iv) ⇒ (i): Let (G, Δ) be SS-sd-open. We show that $cl^s_{sd}(G, \Delta)$ is SS-sd-open by proving $(cl^s_{sd}(G, \Delta))^c$ is SS-sd-closed.

For any $s_\gamma \tilde{\in} int^s_{sd}((cl^s_{sd}(G, \Delta))^c)$, there exists an SS-sd-open set (H, Δ) with $s_\gamma \tilde{\in} (H, \Delta)\tilde{\subseteq}(cl^s_{sd}(G, \Delta))^c$. This means $(H, \Delta)\tilde{\cap} cl^s_{sd}(G, \Delta) = \tilde{\phi}$.

Since (G, Δ) and (H, Δ) are disjoint SS-sd-open sets (as $(H, \Delta)\tilde{\cap}(G, \Delta)\tilde{\subseteq}(H, \Delta)\tilde{\cap} cl^s_{sd}(G, \Delta) = \tilde{\phi}$), by (iv), $cl^s_{sd}(G, \Delta)\tilde{\cap} cl^s_{sd}(H, \Delta) = \tilde{\phi}$.

Therefore, $cl^s_{sd}(H, \Delta)\tilde{\subseteq}(cl^s_{sd}(G, \Delta))^c$, implying $s_\gamma \tilde{\in} cl^s_{sd}(H, \Delta)\tilde{\subseteq}(cl^s_{sd}(G, \Delta))^c$.

This shows $int^s_{sd}((cl^s_{sd}(G, \Delta))^c)\tilde{\subseteq} cl^s_{sd}((cl^s_{sd}(G, \Delta))^c)$, making $(cl^s_{sd}(G, \Delta))^c$ SS-sd-closed, hence $cl^s_{sd}(G, \Delta)$ is SS-sd-open. □

Theorem 5.2. *Every SS-sd-ED space is SS-sd-regular (every SS-sd-open set is SS-regular open).*

Proof. Let (U,μ,Δ) be SS-sd-ED and let (G,Δ) be SS-sd-open. We need to show that $(G,\Delta) = int^s_{sd}(cl^s_{sd}(G,\Delta))$.

Since (U,μ,Δ) is SS-sd-ED, $cl^s_{sd}(G,\Delta)$ is SS-sd-open by definition. Therefore:

$$int^s_{sd}(cl^s_{sd}(G,\Delta)) = cl^s_{sd}(G,\Delta)$$

By Theorem 5.1(iii), $b^s_{sd}(G,\Delta) = \tilde{\phi}$. Using Lemma 3.3:

$$\tilde{\phi} = b^s_{sd}(G,\Delta) = cl^s_{sd}(G,\Delta) \setminus int^s_{sd}(G,\Delta)$$

This implies $cl^s_{sd}(G,\Delta)\tilde{\subseteq} int^s_{sd}(G,\Delta)$. Since always $int^s_{sd}(G,\Delta)\tilde{\subseteq}(G,\Delta)\tilde{\subseteq} cl^s_{sd}(G,\Delta)$, we get:

$$(G,\Delta) = int^s_{sd}(G,\Delta) = cl^s_{sd}(G,\Delta) = int^s_{sd}(cl^s_{sd}(G,\Delta))$$

Therefore, (G,Δ) is SS-regular open. □

Theorem 5.3. *The continuous image of an SS-sd-ED space under an SS-sd-open and SS-sd-continuous surjection is SS-sd-ED.*

Proof. Let $\psi_{sd} : (U,\tau,\Delta) \to (V,\sigma,\Lambda)$ be SS-sd-open, SS-sd-continuous, and surjective, where (U,μ,Δ) is SS-sd-ED with $\tau \subseteq \mu$ and $\sigma \subseteq \mu^*$.

Let (G,Λ) be SS-sd-open in (V,μ^*,Λ). We need to show that $cl^{s*}_{sd}(G,\Lambda)$ is SS-sd-open.

Since ψ_{sd} is SS-sd-continuous, $\psi^{-1}_{sd}(G,\Lambda)$ is SS-sd-open in (U,μ,Δ). Since (U,μ,Δ) is SS-sd-ED, $cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda))$ is SS-sd-open.

Since ψ_{sd} is SS-sd-open, $\psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$ is SS-sd-open in (V,μ^*,Λ).

We claim that $cl^{s*}_{sd}(G,\Lambda) = \psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$.

First, since $\psi_{sd}(\psi^{-1}_{sd}(G,\Lambda)) = (G,\Lambda)$ (by surjectivity), and ψ_{sd} is continuous:

$$(G,\Lambda) = \psi_{sd}(\psi^{-1}_{sd}(G,\Lambda))\tilde{\subseteq}\psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$$

Since $\psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$ is SS-sd-closed (being SS-sd-open in an SS-sd-ED space):

$$cl^{s*}_{sd}(G,\Lambda)\tilde{\subseteq}\psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$$

For the reverse inclusion, note that:

$$\psi^{-1}_{sd}(G,\Lambda)\tilde{\subseteq}\psi^{-1}_{sd}(cl^{s*}_{sd}(G,\Lambda))$$

Taking closures: $cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda))\tilde{\subseteq} cl^s_{sd}(\psi^{-1}_{sd}(cl^{s*}_{sd}(G,\Lambda)))$

Applying ψ_{sd} and using surjectivity:

$$\psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))\tilde{\subseteq} cl^{s*}_{sd}(G,\Lambda)$$

Therefore, $cl^{s*}_{sd}(G,\Lambda) = \psi_{sd}(cl^s_{sd}(\psi^{-1}_{sd}(G,\Lambda)))$ is SS-sd-open, proving (V,μ^*,Λ) is SS-sd-ED. □

Example 5.1. *Let $U = \{x,y,z,w\}$, $\Delta = \{\gamma_1,\gamma_2\}$, and consider the SSTS (U,μ,Δ) where:*

$\mu = \{\tilde{U},\tilde{\phi},(H_1,\Delta),(H_2,\Delta),(H_3,\Delta),(H_4,\Delta)\}$ *with:* - $H_1(\gamma_1) = \{x\}$, $H_1(\gamma_2) = \{x,y\}$ - $H_2(\gamma_1) = \{z,w\}$, $H_2(\gamma_2) = \{z,w\}$ - $H_3(\gamma_1) = \{x,z,w\}$, $H_3(\gamma_2) = U$ - $H_4(\gamma_1) = U$, $H_4(\gamma_2) = \{x,y,z,w\}$

Then (U,μ,Δ) is SS-sd-ED. This can be verified by checking that the sd-closure of each sd-open set is again sd-open.

6. Supra Soft sd-Resolvable Spaces

In this final section, we introduce the concept of supra soft sd-resolvable spaces using the sd-derived operator and investigate its relationships with other topological properties.

Definition 6.1. *An SSTS (U, μ, Δ) is called:*

- (i) ***Supra soft sd-resolvable*** *if there exists an SS-sd-dense subset (D, Δ) such that (D^c, Δ) is also SS-sd-dense*
- (ii) ***Supra soft sd-irresolvable*** *if it is not SS-sd-resolvable*
- (iii) ***Supra soft sd-hyperconnected*** *if every non-null SS-sd-open set is SS-sd-dense*

Theorem 6.1. *Every SS-sd-hyperconnected space is SS-sd-irresolvable.*

Proof. Let (U, μ, Δ) be SS-sd-hyperconnected. Suppose (U, μ, Δ) is SS-sd-resolvable with SS-sd-dense sets (D, Δ) and (D^c, Δ).

Since (D, Δ) is SS-sd-dense, $cl^s_{sd}(D, \Delta) = \tilde{U}$, which implies $int^s_{sd}(D^c, \Delta) = \tilde{\phi}$. Therefore, (D^c, Δ) contains no non-null SS-sd-open sets.

But (D^c, Δ) being SS-sd-dense means $cl^s_{sd}(D^c, \Delta) = \tilde{U}$. This would require every non-null SS-sd-open set to intersect (D^c, Δ), contradicting the fact that (D^c, Δ) contains no non-null SS-sd-open sets.

Therefore, (U, μ, Δ) is SS-sd-irresolvable. □

Theorem 6.2. *For an SSTS (U, μ, Δ), the following are equivalent:*

- (i) *(U, μ, Δ) is SS-sd-resolvable*
- (ii) *There exist disjoint SS-sd-dense subsets (D_1, Δ) and (D_2, Δ) with $\tilde{U} = (D_1, \Delta)\tilde{\cup}(D_2, \Delta)$*
- (iii) *There exists $(A, \Delta)\tilde{\subseteq}\tilde{U}$ such that both $d^s_{sd}(A, \Delta) = \tilde{U}$ and $d^s_{sd}(A^c, \Delta) = \tilde{U}$*

Proof. (i) ⇒ (ii): If (D, Δ) and (D^c, Δ) are SS-sd-dense, then they are disjoint with union $\tilde{U}$.

(ii) ⇒ (iii): Let (D_1, Δ) and (D_2, Δ) be as in (ii). Set $(A, \Delta) = (D_1, \Delta)$.

For any $s_\gamma \tilde{\in} \tilde{U}$ and any SS-sd-open set (G, Δ) containing s_γ: - If $s_\gamma \tilde{\in} (D_1, \Delta)$, then $(G, \Delta)\tilde{\cap}(D_2, \Delta) \neq \tilde{\phi}$ (since (D_2, Δ) is SS-sd-dense), so $(G, \Delta)\tilde{\cap}((D_1, \Delta) \setminus \{s_\gamma\}) \neq \tilde{\phi}$ or $(G, \Delta)\tilde{\cap}(D_2, \Delta) \neq \tilde{\phi}$ - If $s_\gamma \tilde{\in} (D_2, \Delta)$, then $(G, \Delta)\tilde{\cap}(D_1, \Delta) \neq \tilde{\phi}$ (since (D_1, Δ) is SS-sd-dense)

This shows every point is in the derived set of either (A, Δ) or $(A^c, \Delta) = (D_2, \Delta)$. Since derived sets are closed and both are dense, we get $d^s_{sd}(A, \Delta) = \tilde{U} = d^s_{sd}(A^c, \Delta)$.

(iii) ⇒ (i): If $d^s_{sd}(A, \Delta) = \tilde{U}$, then (A, Δ) is SS-sd-dense. Similarly, (A^c, Δ) is SS-sd-dense. □

Theorem 6.3. *No SS-sd-ED space with at least two distinct SS-sd-open points is SS-sd-resolvable.*

Proof. Let (U, μ, Δ) be SS-sd-ED with distinct SS-sd-open points $\{p_{\gamma_1}\}$ and $\{q_{\gamma_2}\}$.

Suppose (U, μ, Δ) is SS-sd-resolvable with disjoint SS-sd-dense sets (D_1, Δ) and (D_2, Δ).

Without loss of generality, assume $p_{\gamma_1} \tilde{\in} (D_1, \Delta)$. Then $\{p_{\gamma_1}\}\tilde{\cap}(D_2, \Delta) = \tilde{\phi}$.

Since (D_2, Δ) is SS-sd-dense and $\{p_{\gamma_1}\}$ is SS-sd-open, we must have $\{p_{\gamma_1}\}\tilde{\cap} cl^s_{sd}(D_2, \Delta) \neq \tilde{\phi}$, so $p_{\gamma_1} \tilde{\in} cl^s_{sd}(D_2, \Delta)$.

But in an SS-sd-ED space, disjoint SS-sd-open sets have disjoint closures. Since $\{p_{\gamma_1}\}$ and some SS-sd-open subset of (D_2, Δ) are disjoint, their closures should be disjoint, contradicting $p_{\gamma_1} \tilde{\in} cl^s_{sd}(D_2, \Delta)$.

Therefore, (U, μ, Δ) is SS-sd-irresolvable. □

Theorem 6.4. *The SS-sd-continuous image of an SS-sd-resolvable space under an SS-sd-d-continuous bijection is SS-sd-resolvable.*

Proof. Let $\psi_{sd} : (U, \tau, \Delta) \to (V, \sigma, \Lambda)$ be SS-sd-continuous, SS-sd-d-continuous, and bijective, where (U, μ, Δ) is SS-sd-resolvable with $\tau \subseteq \mu$ and $\sigma \subseteq \mu^*$.

By Theorem 6.2, there exists (A, Δ) such that $d^s_{sd}(A, \Delta) = \tilde{U}$ and $d^s_{sd}(A^c, \Delta) = \tilde{U}$.

Since ψ_{sd} is SS-sd-d-continuous:

$$\psi_{sd}(\tilde{U}) = \psi_{sd}(d^s_{sd}(A, \Delta)) \tilde{\subseteq} d^{s*}_{sd}(\psi_{sd}(A, \Delta))$$

Since ψ_{sd} is bijective, $\psi_{sd}(\tilde{U}) = \tilde{V}$, so $d^{s*}_{sd}(\psi_{sd}(A, \Delta)) = \tilde{V}$.

Similarly, since $\psi_{sd}(A^c, \Delta) = (\psi_{sd}(A, \Delta))^c$ (by bijectivity):

$$\tilde{V} = \psi_{sd}(d^s_{sd}(A^c, \Delta)) \tilde{\subseteq} d^{s*}_{sd}((\psi_{sd}(A, \Delta))^c)$$

Therefore, $d^{s*}_{sd}(\psi_{sd}(A, \Delta)) = \tilde{V} = d^{s*}_{sd}((\psi_{sd}(A, \Delta))^c)$.

By Theorem 6.2, (V, μ^*, Λ) is SS-sd-resolvable. □

Example 6.1. *Let* $U = \{1, 2, 3, 4\}$, $\Delta = \{\gamma_1, \gamma_2\}$, *and consider the SSTS* (U, μ, Δ) *where:*

$\mu = \{\tilde{U}, \tilde{\phi}, (K_1, \Delta), (K_2, \Delta), (K_3, \Delta)\}$ *with:* - $K_1(\gamma_1) = \{1, 2\}$, $K_1(\gamma_2) = \{1, 3\}$ - $K_2(\gamma_1) = \{3, 4\}$, $K_2(\gamma_2) = \{2, 4\}$ - $K_3(\gamma_1) = U$, $K_3(\gamma_2) = U$

Define (D, Δ) *by* $D(\gamma_1) = \{1, 3\}$, $D(\gamma_2) = \{1, 4\}$. *Then both* (D, Δ) *and* (D^c, Δ) *are SS-sd-dense, making* (U, μ, Δ) *SS-sd-resolvable.*

7. CONCLUSION

In this manuscript, we have introduced and developed a comprehensive theory of supra soft topological structures based on novel sd-boundary and sd-derived operators. These operators have proven to be powerful tools for analyzing the fine structure of supra soft topological spaces and have led to several significant theoretical advances.

Our introduction of supra soft sd-boundary continuous, sd-derived continuous, and sd-frontier continuous maps provides new categories for classifying continuous functions in the soft topological setting. The relationships established between these classes, particularly the characterization of bijective sd-boundary continuous maps as those whose inverses are sd-frontier continuous, reveal deep structural connections that merit further investigation.

The concept of supra soft sd-extremally disconnected spaces represents a significant generalization of classical extremal disconnectedness to the soft topological context. Our characterization of these spaces through various equivalent conditions, including the vanishing of boundaries for sd-open sets and the disjointness of closures for disjoint sd-open sets, provides multiple perspectives for understanding this property. The preservation of sd-extremal disconnectedness under sd-open and sd-continuous surjections establishes its topological significance.

Our investigation of supra soft sd-resolvable spaces using the sd-derived operator opens new avenues for studying the density properties of soft topological spaces. The relationship between sd-hyperconnectedness and sd-irresolvability, along with the incompatibility of sd-extremal disconnectedness with sd-resolvability under certain conditions, reveals intricate connections between different topological properties in the soft setting.

The examples and counterexamples presented throughout the manuscript demonstrate that many classical relationships between topological properties do not automatically extend to the soft topological setting, highlighting the need for careful analysis and new proof techniques in this context.

Future research directions include investigating the relationships between our new concepts and existing notions of soft compactness, soft separation axioms, and soft uniformities. The potential applications of sd-boundary and sd-derived operators in soft rough set theory and soft decision-making problems remain to be explored. Additionally, extending these concepts to fuzzy soft topological spaces and neutrosophic soft topological spaces would provide valuable generalizations.

This work contributes to the growing body of knowledge in soft topology and demonstrates the richness and depth of supra soft topological structures. We hope that the tools and techniques developed here will prove valuable for researchers working in soft set theory and its applications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

[1] A.S. Mashhour, A.A. Allam, F.S. Mahmoud, F.H. Khedr, On Supratopological Spaces, Indian J. Pure Appl. Math. 14 (1983), 502–510.

[2] A.M. Kozae, M. Shokry, M. Zidan, Supra Topologies for Digital Plane, AASCIT Commun. 3 (2016), 1–10.

[3] M.E. El-Shafei, A.H. Zakari, T.M. Al-shami, Some Applications of Supra Preopen Sets, J. Math. 2020 (2020), 9634206. https://doi.org/10.1155/2020/9634206.

[4] T.M. Al-Shami, E.A. Abo-Tabl, B.A. Asaad, Investigation of Limit Points and Separation Axioms Using Supra β-Open Sets, Mo. J. Math. Sci. 32 (2020), 171–187. https://doi.org/10.35834/2020/3202171.

[5] T.M. Al-shami, I. Alshammari, Rough Sets Models Inspired by Supra-Topology Structures, Artif. Intell. Rev. 56 (2022), 6855–6883. https://doi.org/10.1007/s10462-022-10346-7.

[6] D. Molodtsov, Soft Set Theory—First Results, Comput. Math. Appl. 37 (1999), 19–31. https://doi.org/10.1016/s0898-1221(99)00056-5.

[7] P. Maji, R. Biswas, A. Roy, Soft Set Theory, Comput. Math. Appl. 45 (2003), 555–562. https://doi.org/10.1016/s0898-1221(03)00016-6.

[8] N. Çağman, S. Enginoğlu, Soft Matrix Theory and Its Decision Making, Comput. Math. Appl. 59 (2010), 3308–3314. https://doi.org/10.1016/j.camwa.2010.03.015.

[9] F. Karaaslan, Soft Classes and Soft Rough Classes with Applications in Decision Making, Math. Probl. Eng. 2016 (2016), 1584528 https://doi.org/10.1155/2016/1584528.

[10] A. Abd El-Latif, Generalized Soft Rough Sets and Generated Soft Ideal Rough Topological Spaces, J. Intell. Fuzzy Syst. 34 (2018), 517–524. https://doi.org/10.3233/jifs-17610.

[11] S. Yuksel, T. Dizman, G. Yildizdan, U. Sert, Application of Soft Sets to Diagnose the Prostate Cancer Risk, J. Inequal. Appl. 2013 (2013), 229. https://doi.org/10.1186/1029-242x-2013-229.

[12] A. Kharal, B. Ahmad, Mappings on Soft Classes, New Math. Nat. Comput. 07 (2011), 471–481. https://doi.org/10.1142/s1793005711002025.

[13] Z.A. Ameen, M.H. Alqahtani, Some Classes of Soft Functions Defined by Soft Open Sets Modulo Soft Sets of the First Category, Mathematics 11 (2023), 4368. https://doi.org/10.3390/math11204368.

[14] I. Zorlutuna, H. Cakır, On Continuity of Soft Mappings, Appl. Math. Inf. Sci. 9 (2015), 403–409. https://doi.org/10.12785/amis/010147.

[15] Z.A. Ameen, A Non-Continuous Soft Mapping That Preserves Some Structural Soft Sets, J. Intell. Fuzzy Syst. 42 (2022), 5839–5845. https://doi.org/10.3233/jifs-212410.

[16] M. Shabir, M. Naz, On Soft Topological Spaces, Comput. Math. Appl. 61 (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006.

[17] N. Çağman, S. Karataş, S. Enginoglu, Soft Topology, Comput. Math. Appl. 62 (2011), 351–358. https://doi.org/10.1016/j.camwa.2011.05.016.

[18] A. Kandil, O.A.E. Tantawy, S.A. El-Sheikh, A.M. Abd El-latif, Soft Semi Separation Axioms and Some Types of Soft Functions, Ann. Fuzzy Math. Inform. 8 (2014), 305–318.

[19] B. Chen, Soft Semi-Open Sets and Related Properties in Soft Topological Spaces, Appl. Math. Inf. Sci. 7 (2013), 287–294. https://doi.org/10.12785/amis/070136.

[20] T.M. Al-shami, A. Mhemdi, R. Abu-Gdairi, A Novel Framework for Generalizations of Soft Open Sets and Its Applications via Soft Topologies, Mathematics 11 (2023), 840. https://doi.org/10.3390/math11040840.

[21] S.A. El-Sheikh, A.M. El-Latif, Characterization of b-Open Soft Sets in Soft Topological Spaces, J. New Theory 2 (2015), 8–18.

[22] M. Akdag, A. Ozkan, Soft b-Open Sets and Soft b-Continuous Functions, Math. Sci. 8 (2014), 124. https://doi.org/10.1007/s40096-014-0124-7.

[23] T.M. Al-shami, Soft Somewhere Dense Sets on Soft Topological Spaces, Commun. Korean Math. Soc. 33 (2018), 1341–1356. https://doi.org/10.4134/CKMS.c170378.

[24] Z.A. Ameen, R. Abu-Gdairi, T.M. Al-Shami, B.A. Asaad, M. Arar, Further Properties of Soft Somewhere Dense Continuous Functions and Soft Baire Spaces, J. Math. Comput. Sci. 32 (2023), 54–63. https://doi.org/10.22436/jmcs.032.01.05.

[25] Ain Shams University, Cairo, Egypt, S.A. El-Sheikh, A.M. Abd El-latif, Decompositions of Some Types of Supra Soft Sets and Soft Continuity, Int. J. Math. Trends Technol. 9 (2014), 37–56. https://doi.org/10.14445/22315373/ijmtt-v9p504.

[26] A.M. Abd El-latif, S. Karataş, Supra b-Open Soft Sets and Supra b-Soft Continuity on Soft Topological Spaces, J. Math. Comput. Appl. Res. 5 (2015), 1–18.

[27] A.M. Abd El-latif, M.H. Alqahtani, New Soft Operators Related to Supra Soft δ_i-Open Sets and Applications, AIMS Math. 9 (2024), 3076–3096. https://doi.org/10.3934/math.2024150.

[28] A.M. Abd El-latif, Novel Types of Supra Soft Operators via Supra Soft Sd-Sets and Applications, AIMS Math. 9 (2024), 6586–6602. https://doi.org/10.3934/math.2024321.

[29] A.M. Abd El-latif, M.H. Alqahtani, F.A. Gharib, Strictly Wider Class of Soft Sets via Supra Soft δ-Closure Operator, Int. J. Anal. Appl. 22 (2024), 47. https://doi.org/10.28924/2291-8639-22-2024-47.

[30] A.M. Abd El-latif, Decomposition of Supra Soft Locally Closed Sets and Supra SLC-Continuity, Int. J. Nonlinear Anal. Appl. 9 (2018), 13–25. https://doi.org/10.22075/ijnaa.2018.12727.1651.

[31] W. Rong, F. Lin, Soft Connected Spaces and Soft Paracompact Spaces, Int. J. Appl. Math. Stat. 51 (2013), 667–681.

[32] H.L. Yang, On Soft Continuous Mappings and Soft Connectedness of Soft Topological Spaces, Hacet. J. Math. Stat. 6 (2015), 385–398. https://doi.org/10.15672/HJMS.2015459876.

[33] S. Hussain, A Note on Soft Connectedness, J. Egypt. Math. Soc. 23 (2015), 6–11. https://doi.org/10.1016/j.joems.2014.02.003.

[34] S.S. Thakur, A.S. Rajput, Connectedness Between Soft Sets, New Math. Nat. Comput. 14 (2018), 53–71. https://doi.org/10.1142/s1793005718500059.

[35] J. Oudetallah, W. Alharbi, R. Alharbi, A. Amourah, A. Alsoboh, F. Al-Sharqi, T. Sasa, Neutrosophic Tri-metric Space and Its Topological Properties, Neutrosophic Sets Syst. 95 (2026), 75–98.

[36] F. Al-Sharqi, Jamal Oudetallah, Wedad Alharbi, Rehab Alharbi, Ala Amourah, et al., Neutrosophic N-Metric Spaces: Generalized Fixed Point Theory and Topological Properties, Gulf J. Math. 21 (2025), 288–300. https://doi.org/10.56947/gjom.v21i2.3631.

[37] E. Hussein, A.A.R. Malkawi, A. Amourah, A. Alsoboh, A. Al Kasbi, et al., Foundations of Neutrosophic Mr-Metric Spaces with Applications to Homotopy, Fixed Points, and Complex Networks, Eur. J. Pure Appl. Math. 18 (2025), 7142. https://doi.org/10.29020/nybg.ejpam.v18i4.7142.

[38] A. Alsoboh, J. Oudetallah, A. Amourah, R. Al-Naimi, M. Al Hatmi, et al., Bi-Metric Structures and Their Applications in Bitopological Contexts, Eur. J. Pure Appl. Math. 18 (2025), 6526. https://doi.org/10.29020/nybg.ejpam.v18i4.6526.

[39] A.M. Jaradat, Y. Al-Qudah, A. Alsoboh, F. Al-Sharqi, A.A. Al-Maqbali, A New Approach of Possibility Single-Valued Neutrosophic Set and Its Application in Decision-Making Environment, Int. J. Anal. Appl. 23 (2025), 213. https://doi.org/10.28924/2291-8639-23-2025-213.

[40] M. Shatnawi, J. Oudetallah, A. Bataiha, A. Amourah, A. Alsoboh, et al., G-Compact Spaces Characterized by the Intersection of Countable Neighborhoods, Eur. J. Pure Appl. Math. 18 (2025), 6067. https://doi.org/10.29020/nybg.ejpam.v18i3.6067.

[41] S. Hussain, Binary Soft Connected Spaces and an Application of Binary Soft Sets in Decision Making Problem, Fuzzy Inf. Eng. 11 (2019), 506–521. https://doi.org/10.1080/16168658.2020.1773600.

[42] A. Kandil, O.A.E. Tantawy, S.A. El-Sheikh, A.M. Abd El-latif, Soft Connectedness via Soft Ideals, J. New Results Sci. 4 (2014), 90–108.

[43] A. Kandil, O.A.E. Tantawy, S.A. El-Sheikh, A.M. Abd El-latif, Soft Ideal Theory Soft Local Function and Generated Soft Topological Spaces, Appl. Math. Inf. Sci. 8 (2014), 1595–1603. https://doi.org/10.12785/amis/080413.

[44] A. Kandil, O.A.E. Tantawy, S.A. El-Sheikh, A.M. Abd El-latif, Supra Generalized Closed Soft Sets with Respect to an Soft Ideal in Supra Soft Topological Spaces, Appl. Math. Inf. Sci. 8 (2014), 1731–1740. https://doi.org/10.12785/amis/080430.

[45] M.E. El-Shafei, T.M. Al-Shami, Some Operators of a Soft Set and Soft Connected Spaces Using Soft Somewhere Dense Sets, J. Interdiscip. Math. 24 (2021), 1471–1495. https://doi.org/10.1080/09720502.2020.1842348.

[46] A.M.A. El-Latif, Supra Soft b-Connectedness I: Supra Soft b-Irresoluteness and Separateness, Creat. Math. Inform. 25 (2016), 127–134. https://doi.org/10.37193/cmi.2016.02.02.

[47] A.M.A. Abd El-Latif, Supra Soft b-Connectedness II: Some Types of Supra Soft B-Connectednes, Creat. Math. Inform. 26 (2017), 1–8. https://doi.org/10.37193/cmi.2017.01.01.

[48] A.M.A. El-latif, M.H. Alqahtani, Novel Categories of Supra Soft Continuous Maps via New Soft Operators, AIMS Math. 9 (2024), 7449–7470. https://doi.org/10.3934/math.2024361.

[49] J. Oudetallah, Nearly Metacompact in Bitopological Space, Int. J. Open Probl. Comput. Math. 15 (2022), 6–11.

[50] J. Oudetallah, M. AL-Hawari, Other Generalization of Pairwise Expandable Spaces, Int. Math. Forum 16 (2021), 1–9. https://doi.org/10.12988/imf.2021.912116.

[51] J.A. Oudetallah, h-Convexity in Metric Linear Spaces, Int. J. Sci. Appl. Inf. Technol. 8 (2019), 51–58. https://doi.org/10.30534/ijsait/2019/07862019.

[52] J. Oudetallah, R. Alharbi, I.M. Batiha, On r-Compactness in Topological and Bitopological Spaces, Axioms 12 (2023), 210. https://doi.org/10.3390/axioms12020210.

[53] J. Oudetallah, M.M. Rousan, I.M. Batiha, On D-Metacompactness in Topological Spaces, J. Appl. Math. Inform. 39 (2021), 919–926. https://doi.org/10.14317/jami.2021.919.

[54] J. Oudetallah, Novel Results on Nigh Lindelöfness in Topological Spaces, Int. J. Anal. Appl. 22 (2024), 153. https://doi.org/10.28924/2291-8639-22-2024-153.