Int. J. Anal. Appl. (2026), 24:67

International Journal of Analysis and Applications

A Study on Stability for Stochastic Differential Equations

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Abstract. This paper contributes a non-linear stability analysis for a class of stochastic Runge-Kutta algorithms by developing mean-square contractive solutions. This paper illustrates how the stochastic perturbation of a (k, l)- algebraically stable deterministic Runge-Kutta technique takes over this method and the solutions obtained by it. The numerical examples back up the validity of the conclusions.

1. Introduction

We can represent systems that function in the influence of random disturbances by applying stochastic differential equations (SDEs). When the random occurrences in a differential equation are substantially smooth, the majority of problems may be handled using methods similar to those used in deterministic differential equation theory; these equations are referred to as regular SDEs. If the equations contain a highly irregular random process, the situation changes. Commonly, these inconsistent components are generalized random processes of the white noise type. Models of dynamical systems exposed to frequently varying unpredictable excitations can be created

Received: Sep. 26, 2025.

2020 *Mathematics Subject Classification.* 26A33; 34A12; 34K40; 47H08.

Key words and phrases. stochastic differential equation; Runge-Kutta methods; algebraic Stability; mean-square contractive solutions.

https://doi.org/10.28924/2291-8639-24-2026-67

ISSN: 2291-8639

using these equations. The need to investigate the functioning of numerous engineering systems exposed to random environmental resonances has sparked interest in the systematic study of stochastic differential equations of various sorts. In addition to that, control theory, filtering, signal processing, physics, chemistry, biology, economics and finance, all deal with such systems. Applications of SDEs has been mentioned in many papers, Refer in [1], [2] and [3].

Because of the difficulty in solving stochastic effects, many models that have been constructed to describe physical phenomena have neglected them. This is due to a lack of acceptable numerical methods as well as the absence of suitably powerful computers. However, there has recently been a surge of attention in developing numerical methods for the numerical solution of SDEs, which has resulted in the ability to solve more realistic models. In [6], Hairer and Wanner has discussed about numerical methods for the ordinary differential equations. A fundamental property of a successful numerical approach is that the solution it generates converges in some way. That is, The error of approximation and the rate of convergence are the other properties of any numerical approach. There are a multitude of studies in the literature that apply basic numerical techniques for deterministic equations for SDEs. Maruyama was most likely the first to propose a method for SDE's approximate solutions; it is a suitable stochastic analogue of Euler's approach for deterministic differential equations which converges in the mean-square sense. Euler method for SDE has been explained by Jacod and others in [17]. Among more recent contributions,In [8] Platen and Niclo have constructed a Taylor type formula for developing the solution of a SDE about the points of a time partition. The most significant class of numerical methods for SDEs are the Runge - Kutta methods. Runge - Kutta approach has been examined by lot of authors, refer in [7], [9], [10], [11], [19] and [18].

Mean-square approaches are effective for direct modeling of SDEs, which can provide information on a stochastic model's general behaviour. They serve as the foundation for the creation of weak approximation methods, which are useful in a variety of applications. Weak approximation methods are sufficient for evaluating mean values and solving mathematical physics problems. Our primary aim is to analyze the strong convergence questions for numerical approximations in the case where the drift and diffusion functions are not necessarily satisfies the globally Lipschitz functions. Authors Desmond and Chao yue has examined about the Storng convergence for Euler-type methods in [15] and [16]. Most of the existing convergence theory for numerical methods requires the functions are globally Lipschitz; Recent research has looked at probability convergence under more relaxed function constraints.

In recent years stability theory plays an important role in stochastic differential equations. Stability, Pth moment stability and exponential stability has been discussed by many authors in [20] - [25], for different types of stochastic differential equations. The theory of A-Stability is based on the autonomous linear system $x' = ax$. This theory is introduced by the author Dahlquist [4] to represent the linear multi-step methods. Later on in [13], the authors has extended this stability for SDEs. B-Stability theory is concerned with the general nonlinear systems $x' = f(y, x)$. Between these two stability theories, there is a reasonable stability concepts for a linear and nonlinear nonautonomous system $x^{'} = a(x)x, Re\ a(x) \leq 0$. The study of algebraic stability is based on two aspects, one is quadrature weights and the another one is the non-negative definiteness of the matrix.

In [13], The authors have focused on A-Stability for the following SDEs

$$dY(t) = f(Y(t))dt + g(Y(t))dW(t),\ t \in [0,T],$$
$$Y(0) = Y_0 \in \mathscr{R}^d,$$

They have proved the linear stability properties of stochastic Runge-Kutta methods and examined the mean square A-stability.

In [27], The authors has discussed about the nonlinear system of stochastic differential equations of the below form

$$dX(t) = f(X(t))dt + g(X(t))dW(t),\ t \in [0,T],$$
$$X(0) = X_0,$$

In this paper, they have examined the mean square contractivity between the two solutions of nonlinear SDEs.

In [5], Raffaele and Stefano, has discussed about the algebraic stability for the following SDEs,

$$dX(t) = f(X(t))dt + g(X(t))dW(t),\ t \in [0,T],$$
$$X(0) = X_0,$$

where the function satisfies the one-sided Lipschitz condition and they have assumed the Lipschitz constant $\mu \leq 0$ and proved the non linear stability properties of stochastic Runge-Kutta methods and examined the mean square algebraic stability.

Motivated by the above works, we consider the one sided Lipschitz condition $\mu_1 > 0$ and going to prove the $(k,l)-$ Algebraic stability for the same system. This stability is purely based on the k, d_i and l values.

The paper is coordinated as follows: Section 2 contains the preliminaries, which contains the needed definitions of this paper. The main theorem and the proof is given in the Section 3. In Section 4, some numerical examples and it illustration has given. Conclusion of this paper is given in Section 5.

2. Preliminaries

Consider the following non-linear system of SDEs,

$$dU(t) = f(U(t))dt + g(U(t))dW(t),\ t \in [0,T], \tag{2.1}$$
$$U(0) = U_0,$$

where $f, g : \mathscr{R}^d \to \mathscr{R}^d$. Consider the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with filtration $(\mathscr{F}_t)_{t\in[0,T]}$ and $W : [0,T]X\Omega \to \mathscr{R}$ be a standard $(\mathscr{F}_t)_{t\in[0,T]}$- Brownian motion with continuous sample paths on $(\Omega, \mathscr{F}, \mathscr{P})$. Consider the discretized domain

$$\mathscr{I}_h = \{t_n = nh, n = 0, 1, 2, ...N, N = T/h\}.$$

Consider the Stochastic Runge - Kutta (SRK) techniques in the following form ,

$$U_n = U_{n-1} + h\sum_{i=1}^{s} a_i f(\widehat{U}_i^{[n]}) + \Delta W_n \sum_{i=1}^{s} \gamma_i g(\widehat{U}_i^{[n]}) \; n = 1, 2, ...N, \tag{2.2}$$

$$\widehat{U}_i^{[n]} = U_{n-1} + h\sum_{i=1}^{s} b_{ij} f(\widehat{U}_j^{[n]}) + \Delta W_n \sum_{i=1}^{s} q_{ij} g(\widehat{U}_i^{[n]}) \; i = 1, 2, ...s.$$

$U(t_n)$ is approximated by U_n while $U(t_n + c_i h), i = 1, 2, ...s$ is approximated by $\widehat{U}_i^{[n]}$. ΔW_n is a gaussian random variable with zero mean and variance $h$that represents the discretized Wiener increment. The Butcher tableau for (2.2) is

$$\begin{array}{c|c|c} c & B & Q \\ \hline & a^T & \gamma^T \end{array}$$

$$\begin{array}{c|cccccc|cccccc} c_1 & b_{11} & b_{12} & . & . & . & b_{1s} & q_{11} & q_{12} & . & . & . & q_{1s} \\ c_2 & b_{21} & b_{22} & . & . & . & b_{2s} & q_{21} & q_{22} & . & . & . & q_{2s} \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \\ c_s & b_{s1} & b_{s2} & . & . & . & b_{ss} & q_{s1} & q_{s2} & . & . & . & q_{ss} \\ \hline & a_1 & a_2 & . & . & . & a_s & \gamma_1 & \gamma_2 & . & . & . & \gamma_s \end{array} \tag{2.3}$$

Consider the deterministic differential equation with the autonomous system

$$u^{'}(t) = f(u(t)), \; t \in [0, T]$$
$$u(0) = u_0$$

The stochastic perturbation of the well-known deterministic Runge-kutta technique is

$$u_n = u_{n-1} + h\sum_{i=1}^{s} a_i f(\widehat{u}_i^{[n]})$$

$$\widehat{u}_i^{[n]} = u_{n-1} + h\sum_{j=1}^{s} b_{ij} f(\widehat{u}_j^{[n]}), \; i = 1, 2, ..., s \tag{2.4}$$

where $a_i \geq 0$ and $b_{ij} \geq 0$

Definition 2.1. *[13] A Stochastic Runge - Kutta method (2.2) is said to be mean-square A-Stable if*

$$D_{SRK} \supseteq D_{SDE}$$

Definition 2.2. *[6] The mentioned Butcher tableau (2.3) is algebraically stable if*

(1) $M^* = AB + B^T - aa^T$ *is non-negative definite*

(2) $a_i \geq 0$ *for* $i = 1, 2, ...s.$

Definition 2.3. *If there exist* $d_1, d_2, ..., d_s \geq 0$ *such that the matrix*

$$\begin{bmatrix} \alpha_1 & \beta_1^T & \beta_2^T \\ \beta_1 & \alpha_2 & \beta_3^T \\ \beta_2 & \beta_3 & \alpha_3 \end{bmatrix}$$

is non negative definite, where

$$\alpha_1 = k - 1 - 2l \sum_{i=1}^{s} d_i$$

$$\alpha_2 = d_i b_{ij} + d_j b_{ji} - a_i a_j - 2l \sum_{m=1}^{s} d_m b_{mi} b_{mj}$$

$$\alpha_3 = d_i q_{ij} + d_j q_{ji} - \gamma_i \gamma_j - 2l \sum_{m=1}^{s} d_m q_{mi} q_{mj}$$

$$\beta_1 = d_i - a_i - 2l \sum_{i=1}^{s} d_i b_{ij}$$

$$\beta_2 = d_i - \gamma_i - 2l \sum_{i=1}^{s} d_i q_{ij}$$

$$\beta_3 = a_i \gamma_i - 2l \sum_{i=1}^{s} d_i (b_{ij} q_{ij} + q_{ji} b_{ji})$$

then,

$$\mathbb{E}||U_n - V_n||^2 \leq k \, \mathbb{E}||U_{n-1} - V_{n-1}||^2$$

$$\varphi_B(l) \leq \sqrt{k}$$

the corresponding RK-method is called (k, l) *- algebraically stable.*

3. Main Results

We need the following properties which need to prove the main results. [5] Consider the following hypothesis where the function drift f and the diffusion g holds the following properties for the given non-linear SDE with $||.||$ the norm in $\mathscr{R}^d$ and with the expectation operator $\mathbb{E}$

(H1) $f(0) = 0 = g(0)$

(H2) satisfies the one-sided Lipschitz condition, i.e.,

$$\langle f(u) - f(v), u - v \rangle \leq \mu_1 |u - v|^2, \text{for every } u, v \in \mathscr{R}^d \tag{3.1}$$

where μ_1 is the one-sided Lipschtiz constant

(H3) g satisfies the global Lipschitz condition,

$$\left|g(u) - g(v)\right|^2 \leq L\,|u - v|^2\,, \textit{for every } u, v \in \mathscr{R}^d, \textit{where } L > 0. \tag{3.2}$$

(H4) Let $U(t)$ and $V(t)$ be any two solutions

$$||U_n - V_n|| \leq \varphi_B(l)||U_{n-1} - V_{n-1}|| \tag{3.3}$$

Then the solutions $U(t)$ and $V(t)$ of (2.1) and $\mathbb{E}\,|U_0|^2 < \infty$ and $\mathbb{E}\,|V_0|^2 < \infty$ satisfies

$$\mathbb{E}\left|U(t) - V(t)\right|^2 \leq \mathbb{E}\,|U_0 - V_0|^2\, e^{\mu t} \textit{ where } \mu = 2\mu_1 + L \tag{3.4}$$

Here, if the diffusion function g is precisely zero, then the given system is converted into non-linear autonomous ordinary differential equation which is already explained in the reference [6]. If $\mu > 0$ and $\varphi_B(l) < \sqrt{k}$, we have

$$\mathbb{E}\left|U(t) - V(t)\right|^2 \leq \mathbb{E}\left|U(s) - V(s)\right|^2 \tag{3.5}$$

Then the given nonlinear SDE is said to generate mean-square contractive solution.

Theorem 3.1. *[5] Assume the stochastic differential equation (2.1) satisfying the hypothesis (H1)-(H3). Moreover,assume the (C, B, Q, a, γ)-SRK method (2.2), emerging from the stochastic perturbation of an algebraically stable deterministic Runge - Kutta method (2.4). We indicate*

$$\widehat{M} = \Gamma Q + Q^T\Gamma - \gamma\gamma^T$$

where $\Gamma = dia(\gamma)$. If $\widehat{M}$ is a symmetric matrix which is positive semi definite

$$AQ + B^T\Gamma = a\gamma^T$$

with $Q = dia(a)$ then any two solutions U_n and V_n to (2.1), generated by (2.2) with U_0 and V_0 are initial conditions respectively with $\mathbb{E}\,|U_0|^2 \leq \infty$ and $\mathbb{E}\,|V_0|^2 \leq \infty$, satisfy the following inequality

$$\mathbb{E}\,|U_n - V_n|^2 \leq \mathbb{E}\,|U_{n-1} - V_{n-1}|^2 + \phi(h)$$

, where

$$\phi_n(h) = 2\sum_{i=1}^{s} \gamma_i \mathbb{E}\left(\Delta W_n \left\langle \widehat{U}_i^{[n]} - \widehat{V}_i^{[n]}, g\left(\widehat{U}_i^{[n]}\right) - g\left(\widehat{V}_i^{[n]}\right)\right\rangle\right), n = 1, 2, ...N$$

Lemma 3.1. *By using the hypothesis (H3) and for any fixed $h^* > 0$, we get*

$$\lim_{h^* \to 0} \max_n \phi_n(h^*) = 0$$

$$\lim_{h^* \to 0} \phi_n(h^*) = 0$$

Theorem 3.2. *Consider the SRK-method of the non-linear autonomous system (2.1), and assume that its satisfies the hypothesis (H1)-(H4), If there exist $d_1, d_2, ..., d_s \geq 0$, $l, k > 0$ such that the matrix*

$$\begin{bmatrix} \alpha_1 & \beta_1^T & \beta_2^T \\ \beta_1 & \alpha_2 & \beta_3^T \\ \beta_2 & \beta_3 & \alpha_3 \end{bmatrix}$$

is non negative definite, where

$$\alpha_1 = k - 1 - 2l \sum_{i=1}^{s} d_i$$

$$\alpha_2 = d_i b_{ij} + d_j b_{ji} - a_i a_j - 2l \sum_{m=1}^{s} d_m b_{mi} b_{mj}$$

$$\alpha_3 = d_i q_{ij} + d_j q_{ji} - \gamma_i \gamma_j - 2l \sum_{m=1}^{s} d_m q_{mi} q_{mj}$$

$$\beta_1 = d_i - a_i - 2l \sum_{i=1}^{s} d_i b_{ij}$$

$$\beta_2 = d_i - \gamma_i - 2l \sum_{i=1}^{s} d_i q_{ij}$$

$$\beta_3 = a_i \gamma_i - 2l \sum_{i=1}^{s} d_i (b_{ij} q_{ij} + q_{ji} b_{ji})$$

then,

$$\mathbb{E}\|\Delta Z_n\|^2 \leq k\, \mathbb{E}\|\Delta Z_{n-1}\|^2$$
$$\varphi_B(l) \leq \sqrt{k}$$

the corresponding RK-method is called (k, l) *- algebraically stable.*

Proof.

$$Z_n = U_n - V_n$$
$$\widehat{Z}_i^{[n]} = \widehat{U}_i^{[n]} - \widehat{V}_i^{[n]}$$
$$\Delta f_i^{[n]} = f(\widehat{U}_i^{[n]}) - f(\widehat{V}_i^{[n]})$$
$$\Delta g_i^{[n]} = g(\widehat{U}_i^{[n]}) - g(\widehat{V}_i^{[n]})$$
$$Z_n = Z_{n-1} + h \sum_{i=1}^{s} a_i \Delta f_i^{[n]} + \Delta W_n \sum_{i=1}^{s} \gamma_i \Delta g_i^{[n]}$$
$$\widehat{Z}_i^{[n]} = Z_{n-1} + h \sum_{i=1}^{s} b_{ij} \Delta f_j^{[n]} + \Delta W_n \sum_{j=1}^{s} q_{ij} \Delta g_j^{[n]} \ i = 1, 2, ...s.$$

Now,

$$\|Z_n\|^2 - k\|Z_{n-1}\|^2 - 2h \sum_{i=1}^{s} d_i \langle \Delta f_i, \widehat{Z}_i^{[n]} \rangle =$$
$$\langle Z_{n-1} - h \sum_{i=1}^{s} a_i \Delta f_i^{[n]} + \Delta W_n \sum_{i=1}^{s} \gamma_i \Delta g_i^{[n]}, Z_{n-1} - h \sum_{i=1}^{s} a_i \Delta f_i^{[n]} + \Delta W_n \sum_{i=1}^{s} \gamma_i \Delta g_i^{[n]} \rangle$$
$$- k \langle Z_{n-1}, Z_{n-1} \rangle - 2h \sum_{i=1}^{s} d_i \langle \Delta f_i, \widehat{Z}_i^{[n]} \rangle \tag{3.6}$$

$$2h\sum_{i=1}^{s} d_i\langle \Delta f_i, \widehat{Z}_i^{[n]}\rangle \le 2l\sum_{i=1}^{s} d_i\langle \widehat{Z}_i^{[n]}, \widehat{Z}_i^{[n]}\rangle$$

$$\le 2l\sum_{i=1}^{s} d_i\langle Z_{n-1} - h\sum_{i=1}^{s} b_{ij}\Delta f_j^{[n]} + \Delta W_n\sum_{j=1}^{s} q_{ij}\Delta g_j^{[n]},$$

$$Z_{n-1} - h\sum_{i=1}^{s} b_{ij}\Delta f_j^{[n]} + \Delta W_n\sum_{j=1}^{s} q_{ij}\Delta g_j^{[n]}\rangle \tag{3.7}$$

Substitute (3.7) in the above equation (3.6)

$$\|Z_n\|^2 - k\|Z_{n-1}\|^2$$

$$\le \langle Z_{n-1} - h\sum_{i=1}^{s} a_i\Delta f_i^{[n]} + \Delta W_n\sum_{i=1}^{s}\gamma_i\Delta g_i^{[n]}, Z_{n-1} - h\sum_{i=1}^{s} a_i\Delta f_i^{[n]} + \Delta W_n\sum_{i=1}^{s}\gamma_i\Delta g_i^{[n]}\rangle$$

$$- k\langle Z_{n-1}, Z_{n-1}\rangle - \le 2l\sum_{i=1}^{s} d_i\Big\langle Z_{n-1} - h\sum_{i=1}^{s} b_{ij}\Delta f_j^{[n]} + \Delta W_n\sum_{j=1}^{s} q_{ij}\Delta g_j^{[n]},$$

$$Z_{n-1} - h\sum_{i=1}^{s} b_{ij}\Delta f_j^{[n]} + \Delta W_n\sum_{j=1}^{s} q_{ij}\Delta g_j^{[n]}\Big\rangle$$

$$\le \{1 - k + 2l\sum_{i=1}^{s} d_i\}\langle Z_{n-1}, Z_{n-1}\rangle - 2h\sum_{i=1}^{s}\{d_i - a_i - 2l\sum_{i=1}^{s} d_i b_{ij}\}\left\langle Z_{n-1}, \Delta f_i^n\right\rangle$$

$$- 2\Delta W_n\sum_{i=1}^{s}\{d_i - \gamma_i - 2l\sum_{i=1}^{s} d_i q_{ij}\}\left\langle Z_{n-1}, \Delta g_i^n\right\rangle$$

$$- h^2\sum_{i,j=1}^{s}\{d_i b_{ij} + d_j b_{ji} - a_i a_j - 2l\sum_{m=1}^{s} d_m b_{mi} b_{mj}\}\left\langle \Delta f_i^n, \Delta f_j^n\right\rangle$$

$$- \Delta W_n^2\sum_{i,j=1}^{s}\{d_i q_{ij} + d_j q_{ji} - \gamma_i\gamma_j - 2l\sum_{m=1}^{s} d_m q_{mi} q_{mj}\}\left\langle \Delta g_i^n, \Delta g_j^n\right\rangle$$

$$- h\Delta W_n\sum_{i,j=1}^{s}\{a_i\gamma_i - 2l\sum_{i=1}^{s} d_i(b_{ij}q_{ij} + q_{ji}b_{ji}\}\left\langle \Delta f_i^n, \Delta f_g^n\right\rangle$$

$$\le \alpha_1\langle Z_{n-1}, Z_{n-1}\rangle - 2h\sum_{i=1}^{s}\beta_1\left\langle Z_{n-1}, \Delta f_i^n\right\rangle - 2\Delta W_n\sum_{i=1}^{s}\beta_2\left\langle Z_{n-1}, \Delta g_i^n\right\rangle$$

$$- h^2\sum_{i,j=1}^{s}\alpha_2\left\langle \Delta f_i^n, \Delta f_j^n\right\rangle - \Delta W_n^2\sum_{i,j=1}^{s}\alpha_3\left\langle \Delta g_i^n, \Delta g_j^n\right\rangle$$

$$- h\Delta W_n\sum_{i,j=1}^{s}\alpha_3\left\langle \Delta f_i^n, \Delta f_g^n\right\rangle$$

Where,

$$\alpha_1 = k - 1 - 2l \sum_{i=1}^{s} d_i$$

$$\alpha_2 = d_i b_{ij} + d_j b_{ji} - a_i a_j - 2l \sum_{m=1}^{s} d_m b_{mi} b_{mj}$$

$$\alpha_3 = d_i q_{ij} + d_j q_{ji} - \gamma_i \gamma_j - 2l \sum_{m=1}^{s} d_m q_{mi} q_{mj}$$

$$\beta_1 = d_i - a_i - 2l \sum_{i=1}^{s} d_i b_{ij}$$

$$\beta_2 = d_i - \gamma_i - 2l \sum_{i=1}^{s} d_i q_{ij}$$

$$\beta_3 = a_i \gamma_i - 2l \sum_{i=1}^{s} d_i (b_{ij} q_{ij} + q_{ji} b_{ji})$$

Then,

$$\mathbb{E}\|\Delta Z_n\|^2 \leq k\, \mathbb{E}\|\Delta Z_{n-1}\|^2$$

Then the system is (k, l)-algebraically stable. □

Remark 3.1. *In this above theorem, the mentioned matrix should be non negative. The minor of the matrix* $\begin{bmatrix} \alpha_2 & \beta_3^T \\ \beta_3 & \alpha_3 \end{bmatrix}$ *must also be non negative, This non negativity is depend on the values of* l *and* d_i*. By choosing the values of* k, l *and* d_i *we can fix the non negativity of the matrix.*

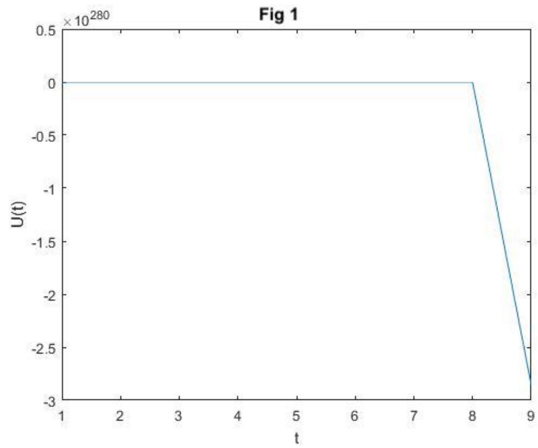
4. Numerical Examples

Example 1

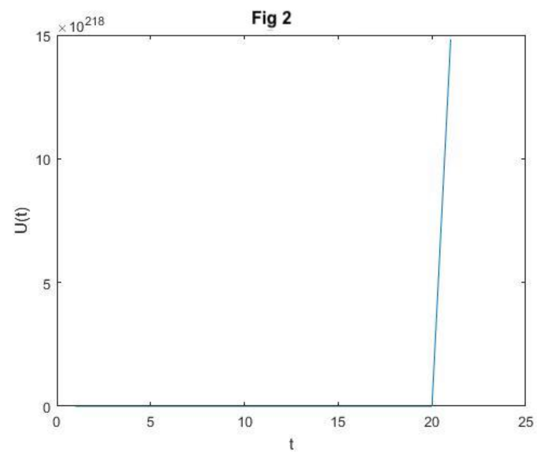
1. Assume the SDE (2.1) with

$$f(U(t)) = 4U(t) + U(t)^3, g(U(t)) = U(t), \tag{4.1}$$

for $t \in [0, 100], U_0 = 1$ and $V_0 = 0$. In this example, the values mentioned in hypothesis (H2) and (H3) are provided by $l = 1, \mu_1 = 4$ and $\mu = 7$.



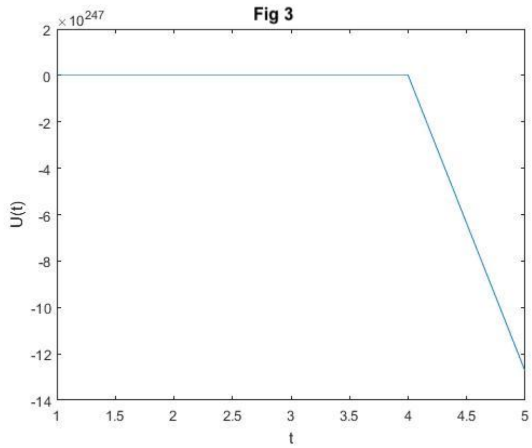
With the hypothesis, by choosing appropriate k and d_i values we get the constants are $\alpha_1 = 20, \alpha_2 = 112, \alpha_3 = 80, \beta_1 = -32, \beta_2 = -5, \beta_3 = -60$ and the given matrix is non negative definite and the given system is (k,l)algebraically stable.



Suppose if we change the k and d_i values, the matrix is not non negative definite. The eigenvalues are both positive and negative. So in this case, the matrix is indefinite (see fig 1.3).
Example 2 2. Consider the SDE (2.1) with

$$f(U(t)) = 5U(t), g(U(t)) = \sin U(t), \tag{4.2}$$

for $t \in [0, 100], U_0 = 1$ and $V_0 = 0$. In this example, the values mentioned in the hypothesis (H2) and (H3) are provided by $l = 2, \mu_1 = 2$ and $\mu = 11$.



With the hypothesis by choosing appropriate k and d_i values we get the constants are $l = 1, \alpha_1 = 25, \alpha_2 = 225, \alpha_3 = 112, \beta_1 = -50, \beta_2 = -6, \beta_3 = -95$ and the given matrix is non negative definite and the given system is (k,l)algebraically stable. Suppose if we change the k and d_i values, the matrix is not non negative definite. The eigenvalues are both positive and negative. So in this case, the matrix is indefinite.

5. Conclusions

In this Paper, we have studied the (k,l)-Algebraic Stability for Stochastic Runge - Kutta methods which focus on the one sided Lipschitz condition $\mu > 0$ where as in [5], authors have discussed about the one sided Lipschitz condition for $\mu < 0$. The numerical solutions also discussed for the same. In the numerical examples, We could note that based on the k, d_i and l values the algebraic stability of the system is decided. For the future studies, we can prove the Algebraic stability and (k,l)-Algebraic stability for the different types of numerical solutions for the stochastic differential equations and also we can concentrate the same for multidimensional systems with multiple Wiener processes.

Acknowledgments: This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-54.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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