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The Rainbow Mean Coloring of Some Operations of Graphs

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Abstract. In a connected graph G with a minimum of three vertices, an edge coloring c allocates positive numbers to the edges. The chromatic mean of a vertex v is calculated by averaging the colors of all incident edges, provided that the result remains a positive integer. A coloring c is a rainbow mean coloring if each vertex in G has a unique chromatic mean. The rainbow mean index of c is the highest chromatic mean assigned to any vertex, while the rainbow mean index of G is the smallest possible maximum chromatic mean for all valid rainbow mean colorings. This study calculates the rainbow mean index of tensor product graphs, specifically $G_1 \times G_2$, where $G_1 \in \{C_q, K_q\}$ and $G_2 \in \{C_t, K_t\}$; $P_q \times H$, where $H \in \{C_t, K_t, W_t, F_t\}$ and $\chi_{rm}(H) = t$. We also compute the rainbow mean index for the rooted product of two graphs, the join of two graphs, and the caterpillar graph.

1. Introduction

For a connected graph G, let $V(G)$ and $E(G)$ represent the vertex and edge sets, respectively. Definitions and terminology not addressed in this article can be found in [1]. The degree of a vertex $v \in V(G)$ is defined as the number of edges incident to v, represented as $d(v)$. If $d(v) = \ell$ for all v in G, then G is classified as ℓ-regular. A cycle $C_q \in G$ is defined as a spanning cycle if $V(C_q) = V(G)$. Let $P_q, F_q, K_q, S_q, K_{q,t}$, and O_q denote the path graph, fan graph, complete graph, star, complete bipartite graph, and null graph, respectively.

In graph theory, a well-known result states that every connected graph contains two vertices with the same degree. This concept is even referenced indirectly in David Wells' [7], which highlights it as one of the most beautiful among the 24 theorems. Initially, graphs with all vertices of distinct degrees were called perfect; later they became known as irregular. Consequently, it is established that no nontrivial graph can be perfect.

Numerous researchers have concentrated on the notion of irregular graphs throughout the

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years (see [2–4,6]). Although no nontrivial simple graph is irregular, irregular multigraphs can be constructed for every order $q \geq 3$. Interpreting a multigraph M as a labeled graph G_M, every edge uv in G_M is assigned a label representing the number of parallel edges connecting u and v in M. The degree of a vertex v in M is determined by summing the labels of the edges in G_M that are incident to v. Later, these edge labels were treated as edge colors, and the total number of labels incident to a vertex was known as its chromatic sum, which determined the vertex's color.

In 1986, Gary Chartrand introduced the concept of irregularity strength during the 250th Anniversary of the Graph Theory Conference at Indiana University–Purdue University Fort Wayne (now Purdue University Fort Wayne). This is the lowest integer k that can be used to color the edges of a graph from the set $[k] = 1, 2, \ldots, k$ in a way ensures that all vertex colors (the sum of the colors of edges incident to v) are unique [5]. This forms a vertex coloring, often called rainbow coloring. However, in many cases, the greatest vertex color exceeds the graph's order. This behavior raised the question of whether edge coloring might generate separate integer vertex colors while minimizing the maximum vertex color. While minimizing k under distinct vertex colors has been widely studied, another approach focused on defining vertex colors as integer chromatic averages and minimizing the biggest vertex color. Here, the emphasis shifts from the edge colors to the resulting vertex colors, ensuring that they remain distinct and as small as possible.

An edge coloring $c : E(G) \to \mathbb{N}$ of a connected graph G of order at least 3 with positive integers (where adjacent edges may be colored the same) is called a *mean coloring* of G if the chromatic mean $cm(v)$ of a vertex v of G, defined as $cm(v) = \frac{\sum_{e \in E_v} c(e)}{d(v)}$, where E_v is the set of edges incident with v, is an integer. If the distinct vertices receives distinct chromatic means, then the edges-coloring c is called a *Rainbow Mean Coloring* (RMC) of G. For a rainbow mean coloring c of G, the maximum vertex coloring c is the *Rainbow Mean Index* (RMI) $\chi_{rm}(c)$ of c. The rainbow mean index $\chi_{rm}(G)$ of G is defined as

$$\chi_{rm}(G) = \min\{\chi_{rm}(c) : c \text{ is a RMC of } G\}.$$

Chartrand et al. [8] introduced the notion of RMI and made the following conjecture and theorems.

Conjecture 1.1. *For every connected graph G with q vertices $q \geq 3$, $q \leq \chi_{rm}(G) \leq q + 2$.*

Observation 1.1. *For any connected graph G with q vertices, $\chi_{rm}(G) \geq q$.*

Theorem 1.1. *Let G be a graph with $q \geq 6$ vertices, $q = 4\xi + 2$ where $\xi \geq 1$, and $d(v)$ is odd for all $v \in V(G)$. It follows that $\chi_{rm}(G) \geq q + 1$.*

Theorem 1.2. *For $q \geq 4$,*

$$\chi_{rm}(C_q) = \begin{cases} q \text{ if } q \equiv 0, 1 \pmod 4, \\ q + 1 \text{ if } q \equiv 2, 3 \pmod 4. \end{cases}$$

Theorem 1.3. *For $q \geq 4$,*

$$\chi_{rm}(K_q) = \begin{cases} q \text{ if } q \equiv 0, 1, 3 \pmod 4, \\ q + 1 \text{ if } q \equiv 2 \pmod 4. \end{cases}$$

Later, in 2021 the same authors [9] studied the RMI of the path graphs. In 2020, Hallas et al. [10] studied the RMI of several bipartite graphs. In the same year, the same authors [11] computed the RMI for double stars, cubic caterpillars of even order, and subdivisions of stars.

In 2022, Anantharaman et al. [12] computed the RMI of several graphs, and proved the following Theorems.

Theorem 1.4. *For* $q \geq 4$, $\chi_{rm}(P_q \vee K_1) = q + 1$.

Theorem 1.5. *For* $q \geq 4$ *and* $q \neq 1 \pmod 4$, $\chi_{rm}(C_q \vee K_1) = q + 1$.

Theorem 1.6. *For* $q \geq 5$ *and* $q \equiv 1 \pmod 4$, $\chi_{rm}(C_q \vee K_1) = q + 2$.

In 2023, Garciano et al. [13] determined the RMI of brooms and double brooms.

In 2024, Maheswari and Rajasekaran [14] studied the RMI of the Cartesian product of two graphs, chains of cycles, join of the n wheel, and transformations of path graphs. Later, in 2025 [15], the same authors determined the RMI of the corona product of two graphs.

Applications of the rainbow mean index include network optimization, resource scheduling and allocation, data clustering and categorization, fault-tolerant system design, cryptography, and secure communication.

In this work, we present the RMI of the tensor product of two graphs; Rooted product of two graphs; Join of two graphs, and caterpillar graph.

2. Tensor Product

Definition 2.1. *The Tensor Product* $G \times H$ *[16], of graphs* G *and* H*, is defined as a simple graph with vertex set* $V(G) \times V(H)$*. The vertices* (u_1, v_1) *and* (u_2, v_2) *in* $G \times H$ *are adjacent if* $u_1 \sim u_2$ *in* G *and* $v_1 \sim v_2$ *in* H*. Consequently,* $G \times H$ *is connected if and only if at least one of* G *or* H *is non-bipartite.*

In the following, throughout this article, the graph G is regular and contains a Hamiltonian cycle with $q \geq 4$ vertices. Let $u_1 - u_2 - u_3 - \cdots - u_q - u_1$ be a Hamiltonian cycle in G whose RMI is q. Similarly, let $H \in \{C_t, W_t, K_t, F_t\}$ and $\chi_{rm}(H) = t$.

Theorem 2.1. *For* $q \geq 5$, $\chi_{rm}(P_q \times H) = qt$.

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_t\}$, $V(P_q) = \{u_1, u_2, \ldots, u_q\}$ and $V(P_q \times H) = \{(u_i, v_j) : i \in \{1, 2, \ldots, q\}, j \in \{1, 2, \ldots, t\}\}$, $E(P_q \times H) = \{(u_i v_j, u_{i+1} v_{j'}) : u_i u_{i+1} \in E(P_q), v_j v_{j'} \in E(H)\}$. Clearly, $|V(P_q \times H)| = qt$, Define $c' : E(P_q \times H) \to \mathbb{N}$ as:

We distinguish two situations, depending on whether q is even or odd.

Case 1. q is odd.

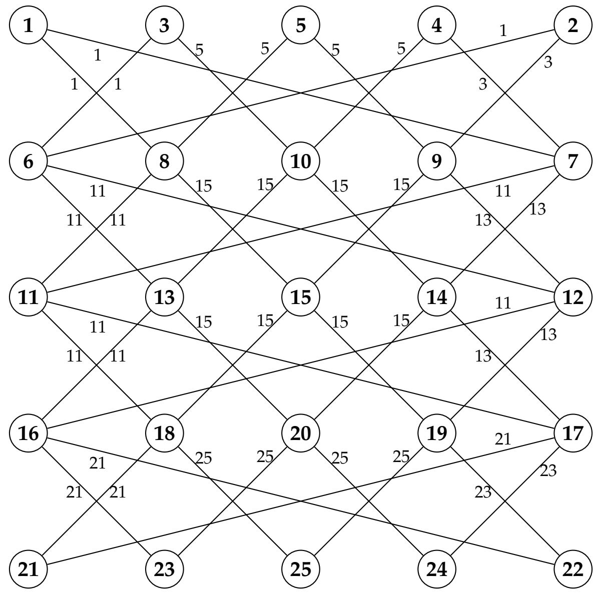

FIGURE 1. $\chi_{rm}(P_5 \times C_5) = 25$.

Recall that: c is a RMC of H.
For $j, j' \in \{1, 2, \ldots, t\}$,

$$c'(u_i v_j, u_{i+1} v_{j'}) = c(v_j v_{j'}) + t(i-1) \text{ if } 1 \le i \le q-2 \text{ is odd,}$$
$$c'(u_i v_j, u_{i+1} v_{j'}) = c(v_j v_{j'}) + ti \text{ if } 2 \le i \le q-1 \text{ is even.}$$

Therefore, the vertices of $P_q \times H$ have the following chromatic mean:
$cm(u_i v_j) = cm(v_j) + t(i-1)$ if $1 \le i \le q, 1 \le j \le t$.

Case 2. q is even.

$$c'(u_{q-3} v_j, u_{q-2} v_{j'}) = c(v_j v_{j'}) + t(q-2);$$
$$c'(u_{q-2} v_j, u_{q-1} v_{j'}) = c(v_j v_{j'}) + qt;$$
$$c'(u_{q-1} v_j, u_q v_{j'}) = c(v_j v_{j'}) + t(q-4);$$

Fix the colors to the leftover edges of $P_q \times H$ as shown in *case 1.*
Therefore, the vertices of $P_q \times H$ have the following chromatic mean:

$$cm(u_{q-3} v_j) = cm(v_j) + t(q-3);$$
$$cm(u_{q-2} v_j) = cm(v_j) + (q-1)t;$$
$$cm(u_{q-1} v_j) = cm(v_j) + (q-2)t;$$
$$cm(u_q v_j) = cm(v_j) + t(q-4);$$

and color the leftover vertices in $P_q \times H$ as in *case 1*.
Thus, in both cases $\chi_{rm}(P_q \times H) \leq qt$. By observation 1.1, $\chi_{rm}(P_q \times H) \geq qt$.
Hence $\chi_{rm}(P_q \times H) = qt$ (see Figure 1, the RMI of $P_5 \times C_5$ is 25). □

Theorem 2.2. *Let q and t be integers such that both are greater than or equal to 3, and at least one of them is odd. Let $G_1 = \{C_q, K_q\}$ and $G_2 = \{C_t, K_t\}$ be two graphs.*
1. $\chi_{rm}(G_1 \times G_2) = qt$ *if* $qt = 4\xi,\ 4\xi + 1,$ *and* $\xi \geq 1$.
2. $\chi_{rm}(G_1 \times G_2) \leq qt + 1$ *if* $qt = 4\xi + 2,\ 4\xi + 3,$ *and* $\xi \geq 1$.

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_q\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_t\}$. $V(G_1 \times G_2) = \{(u_i, v_j) : 1 \leq i \leq q, 1 \leq j \leq t\}$. WLOG, we assume that the number of vertices in G_2 is odd. Recall that if G_1 is r_1-regular and G_2 is r_2-regular, then the tensor product $G_1 \times G_2$ is $r_1 r_2$-regular; denote this degree by $\ell = r_1 r_2$. Therefore, $G_1 \times G_2$ is ℓ-regular.
From the construction of $G_1 \times G_2$ we obtain the following ordered listing of vertices (to be used as a Hamiltonian cycle): $(u_1, v_1) - (u_2, v_2) - (u_1, v_3) - (u_2, v_4) - \cdots - (u_1, v_y) - (u_2, v_1) - (u_3, v_2) - \cdots - (u_2, v_y) - (u_3, v_1) - \cdots - (u_{q-1}, v_t) - (u_q, v_1) - (u_1, v_2) - (u_q, v_3) - \cdots - (u_q, v_t) - (u_1, v_1)$. For convenience, we rename vertices along this listing by $(u_1, v_1) = z_1,\ (u_2, v_2) = z_2,\ (u_1, v_3) = z_3,\ (u_2, v_4) = z_4,\ \ldots, (u_q, v_t) = z_{qt}$.
Case 1. $qt = 4\xi,\ 4\xi + 1,$ *and* $\xi \geq 1$.
For $1 \leq k \leq \left\lceil \frac{qt}{2} \right\rceil$,

$$c(z_k z_{k+1}) = \begin{cases} 1 + \ell(k-1) \text{ if } k \text{ is odd,} \\ 1 + \ell k \text{ if } k \text{ is even;} \end{cases}$$

$c(z_k z_{k+1}) = 1 + \ell(qt - k)$ if $\lceil \frac{qt}{2} \rceil + 1 \leq k \leq qt - 1$,
$c(z_{qt} z_1) = 1$; Assign color 1 to all edges remaining in $G_1 \times G_2$.
Therefore, the vertices of $G_1 \times G_2$ have the following chromatic mean:

$$\begin{aligned} cm(z_k) &= \frac{1}{\ell}\left[1 + \ell(k-2) + 1 + \ell k + \ell - 2\right] \\ &= 2k - 1 \text{ if } 1 \leq k \leq \left\lceil \frac{qt}{2} \right\rceil; \\ cm(z_k) &= \frac{1}{\ell}\left[1 + \ell qt - k\ell + \ell + 1 + \ell qt - k\ell + \ell - 2\right] \\ &= 2(qt - k + 1) \text{ if } \left\lceil \frac{qt}{2} \right\rceil + 1 \leq k \leq qt. \end{aligned}$$

Thus, $\chi_{rm}(G_1 \times G_2) \leq qt$. By observation 1.1, $\chi_{rm}(G_1 \times G_2) \geq qt$.
Hence $\chi_{rm}(G_1 \times G_2) = qt$.
Case 2.
Case 2(a). $qt = 4\xi + 2,\ \xi \geq 1$.
For $\frac{qt}{2} + 1 \leq k \leq qt - 1$,

$$c(z_k z_{k+1}) = \begin{cases} 1 + \ell(qt - k) \text{ if } k \text{ is odd,} \\ 1 + \ell(qt + 2 - k) \text{ if } k \text{ is even;} \end{cases}$$

and color the leftover edges in $G_1 \times G_2$ follows as in *Case 1.*
Therefore, the vertices of $G_1 \times G_2$ have the following chromatic mean:

$$\begin{aligned} cm(z_{\frac{qt+2}{2}}) &= qt+1; \\ cm(z_k) &= \frac{1}{\ell}[1+\ell qt - k\ell + \ell + 1 + \ell qt + 2\ell - k\ell + \ell - 2] \\ &= 2(qt-k+2) \text{ if } \frac{qt+4}{2} \leq k \leq qt-1; \\ cm(z_{qt}) &= 2. \end{aligned}$$

and the chromatic mean of the leftover vertices in $G_1 \times G_2$ is obtained as in *Case 1.*
Case 2(b). $qt = 4\xi + 3,\ \xi \geq 1.$

$$c(z_1z_2) = 1, c(z_2z_3) = 1 + 2\ell;$$

For $3 \leq k \leq \frac{qt-1}{2}$,

$$c(z_kz_{k+1}) = \begin{cases} 1+\ell(k+1) \text{ if } k \text{ is odd,} \\ 1+k\ell \text{ if } k \text{ is even;} \end{cases}$$

$$\begin{aligned} c(z_kz_{k+1}) &= 1+\ell(qt-k) \text{ if } \frac{qt+1}{2} \leq k \leq qt-1; \\ c(z_{qt}z_1) &= 1; \end{aligned}$$

Assign color 1 to all edges remaining in $G_1 \times G_2$.
Therefore, the vertices of $G_1 \times G_2$ have the following chromatic mean:

$$\begin{aligned} cm(z_k) &= 2k-1 \text{ if } 1 \leq k \leq 2, \\ cm(z_k) &= \frac{1}{\ell}[1+\ell k + 1 + \ell k + \ell - 2] \\ &= 2k+1 \text{ if } 3 \leq k \leq \frac{qt-1}{2}; \\ cm(z_k) &= \frac{1}{\ell}[1+\ell qt - k\ell + \ell + 1 + \ell qt - k\ell + \ell - 2] \\ &= 2(qt-k+1) \text{ if } \frac{qt+1}{2} \leq k \leq qt. \end{aligned}$$

Hence, $\chi_{rm}(G_1 \times G_2) \leq qt+1$. □

Theorem 2.3. *Let $H \in \{F_t,\ W_t\}$ and $\chi_{rm}(H) = t$. For $q, t \geq 4$, $\chi_{rm}(G \times H) = qt$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_q\}$ and $V(H) = \{v_1, v_2, \ldots, v_t\}$. Clearly, $V(G \times H) = \{(u_i, v_j) : 1 \leq i \leq q, 1 \leq j \leq t\}$, $E(G \times H) = \{(u_iv_j, u_{i'}v_{j'}) : u_iu_{i'} \in E(G), v_jv_{j'} \in E(H)\}$. $|V(G \times H)| = qt$. Define $c' : E(G \times H) \to \mathbb{N}$ as:
For $1 \leq i \leq \lceil \frac{q}{2} \rceil$ and $1 \leq j, j' \leq t$,

$$c'(u_iv_j, u_{i+1}v_{j'}) = \begin{cases} \ell t(i-1) + c(v_jv_{j'}) \text{ if odd } i, \\ \ell ti + c(v_jv_{j'}) \text{ if even } i; \end{cases}$$

$$c'(u_iv_j, u_{i+1}v_{j'}) = \ell t(q-i) + c(v_jv_{j'}) \text{ if } \left\lceil \frac{q}{2} \right\rceil + 1 \leq i \leq q-1;$$

$$c'(u_qv_j, u_1v_{j'}) = c(v_jv_{j'});$$

and assign colors to all remaining edges as in $c(v_jv_{j'})$.
Therefore, the vertices of $G \times H$ have the following chromatic mean:
For $1 \leq j \leq t$,

$$cm(u_iv_j) = t(2i-2) + cm(v_j) \text{ if } 1 \leq i \leq \left\lceil \frac{q}{2} \right\rceil;$$

$$cm(u_iv_j) = t(2q-2i+1) + cm(v_j) \text{ if } \left\lceil \frac{q}{2} \right\rceil + 1 \leq i \leq q.$$

Therefore, $\chi_{rm}(G \times H) \leq qt$. By observation 1.1 $\chi_{rm}(G \times H) \geq qt$.
Hence $\chi_{rm}(G \times H) = qt$. □

Theorem 2.4. *For $q \geq 5$ and q is odd, $\chi_{rm}(S_q \times H) = qt$.*

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_t\}$, $V(S_q) = \{u_0, u_1, u_2 \ldots, u_{q-1}\}$, $E(S_q) = \{u_0u_i : 1 \leq i \leq q-1\}$ and $V(S_q \times H) = \{(u_i, v_j) : 0 \leq i \leq q-1, 1 \leq j \leq t\}$, $E(S_q \times H) = \{(u_iv_j, u_0v_{j'}) : u_iu_0 \in E(S_q), v_jv_{j'} \in E(G)\}$.
Define $c' : E(S_q \times H) \to \mathbb{N}$ as follows:
For $1 \leq j, j' \leq t$,

$$c'(u_0v_j, u_iv_{j'}) = c(v_jv_{j'}) + t(i-1) \text{ if } i \in \left\{1, 2, \ldots, \frac{q-1}{2}\right\};$$

$$c'(u_0v_j, u_iv_{j'}) = c(v_jv_{j'}) + it \text{ if } i \in \left\{\frac{q+1}{2}, \frac{q+3}{2}, \ldots, q-1\right\}.$$

Therefore, the vertices of $S_q \times H$ have the following chromatic mean:
For $1 \leq j \leq t$,

$$cm(u_iv_j) = cm(v_j) + t(i-1) \text{ if } 1 \leq i \leq \frac{q-1}{2};$$

$$cm(u_0v_j) = cm(v_j) + t\left(\frac{q-1}{2}\right);$$

$$cm(u_iv_j) = cm(v_j) + ti \text{ if } \frac{q+1}{2} \leq i \leq q-1.$$

Clearly, $\chi_{rm}(S_q \times H) \leq qt$. By observation 1.1, $\chi_{rm}(S_q \times H) \geq qt$.
Hence $\chi_{rm}(S_q \times H) = qt$. □

3. Caterpillar

The corona product $P_q \circ O_t$ is referred to as a caterpillar graph. In 2022 Anantharaman et al. computed the RMI of caterpillar (odd path with even leaves). In this section, we extend this analysis to calculate the RMI for the remaining cases of caterpillars as follows.

Theorem 3.1. *For $q \geq 5$ and $t \geq 1$,*

$$\chi_{rm}(P_q \circ O_t) = \begin{cases} q(t+1)+1 \text{ if } q \text{ is odd and } t = 4\xi+1,\ \xi \geq 1, \\ q(t+1) \text{ otherwise.} \end{cases}$$

Proof. Let $V(P_q \circ O_t) = V(P_q) \cup \{v_j^i : i \in \{1,2,\ldots,q\}, j \in \{1,2,\ldots,t\}\}$ and $E(P_q \circ O_t) = E(P_q) \cup \{u_i v_j^i : 1 \leq i \leq q,\ 1 \leq j \leq t\}$. Define $c' : E(P_q \circ O_t) \to \mathbb{N}$ as:

Case 1. q is even.

For $1 \leq j \leq t$,

$$c'(u_i v_j^i) = j + (t+1)(i-1) \text{ if } 1 \leq i \leq q-4;$$
$$c'(u_i v_j^i) = j + (t+1)(2i-q+3) \text{ if } i \in \{q-3, q-2\};$$
$$c'(u_i v_j^i) = j + (t+1)(3q-4-2i) \text{ if } i \in \{q-1, q\};$$

Next, assign the colors to the edges of P_q as:

$$c'(u_i u_{i+1}) = \frac{(t+1)(t+2i)}{2} \text{ if } 1 \leq i \leq q-5 \text{ is odd};$$
$$c'(u_i u_{i+1}) = (t+1)(1+i) \text{ if } 2 \leq i \leq q-4 \text{ is even};$$
$$c'(u_{q-3} u_{q-2}) = \frac{(t+1)(t+2q-2)}{2};$$
$$c'(u_{q-2} u_{q-1}) = (q+1)(t+1);$$
$$c'(u_{q-1} u_q) = \frac{t^2+2q(t+1)-5t-6}{2}.$$

Therefore, the vertices of $P_q \circ O_t$ have the following chromatic mean:

$$cm(v_j^i) = c(u_i v_j^i) \text{ if } 1 \leq j \leq t \text{ and } 1 \leq i \leq q;$$
$$cm(u_1) = \frac{1}{(t+1)}\left[\frac{t^2+t}{2} + \frac{t^2+3t+2}{2}\right] = t+1;$$
$$cm(u_i) = \frac{1}{(t+2)}\left[\frac{t^2+t}{2}(2i-1) + \frac{t^2+2i(1+t)-2-t}{2} + t + it + 1 + i\right]$$
$$= i(t+1) \text{ if } 2 \leq i \leq q-4;$$
$$cm(u_{q-3}) = \frac{1}{(t+2)}\left[\frac{t^2+t}{2}(2q-5) + qt - 3t + q - 3 + \frac{t^2+2qt+2q-(t+2)}{2}\right]$$
$$= (t+1)(q-2);$$
$$cm(u_{q-2}) = \frac{1}{(t+2)}\left[\frac{t^2+t}{2}(2q+1) + qt + t + q + 1 + \frac{t^2+2q(t+1)-(t+2)}{2}\right]$$
$$= q(t+1);$$
$$cm(u_{q-1}) = \frac{1}{t+2)}\left[\frac{t^2+t}{2}(2q-3) + t + qt + 1 + q + \frac{t^2+2qt+2q-5t-6}{2}\right]$$
$$= (t+1)(q-1);$$

$$cm(u_q) = \frac{1}{(t+1)}\left[\frac{t^2+t}{2}(2q-7) + \frac{t^2+2qt+2q-5t-6}{2}\right]$$
$$= (t+1)(q-3).$$

Thus, $\chi_{rm}(P_q \circ O_t) \leq q(t+1)$. By observation 1.1, $\chi_{rm}(P_q \circ O_t) \geq q(t+1)$.
Hence $\chi_{rm}(P_q \circ O_t) = q(t+1)$ (see Figure2, RMI of $P_6 \circ O_2$ is 18).

Case 2. q is odd and $t = 4\xi + 3,\ \xi \geq 1$.

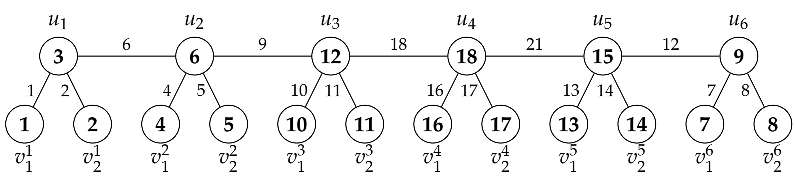

FIGURE 2. $\chi_{rm}(P_6 \circ O_2) = 18$.

Color the pendant edges of $P_q \circ O_t$ as:

$$c'\left(u_1 v_t^1\right) = \frac{5t+5}{4},$$
$$c'\left(u_2 v_{\frac{t+1}{4}}^2\right) = t+1,$$

For $1 \leq i \leq q$,

$$c'(u_i v_j^i) = \begin{cases} j + ti - t + i - 1 \text{ if } 1 \leq j \leq \frac{t+1}{2}; \\ j + it + i - t \text{ if } \frac{t+3}{2} \leq j \leq t; \end{cases}$$

Next, color the edges of P_q as follows:

$$c'(u_1 u_2) = \frac{3t+7}{4};$$

For $2 \leq i \leq q-1$,

$$c'(u_i u_{i+1}) = \begin{cases} \frac{t(2i-1)+2i+1}{2} \text{ if odd } i; \\ i(t+1) + 2 + t \text{ if even } i. \end{cases}$$

Therefore, the vertices of $P_q \circ O_t$ have the following chromatic mean:

$$cm(v_j^i) = c(u_i v_j^i) \text{ if } 1 \leq i \leq q \text{ and } 1 \leq j \leq t;$$
$$cm(u_1) = \frac{t+3}{2};\ cm(u_2) = \frac{3t+5}{2};$$
$$cm(u_i) = \frac{1}{t+2}\Bigg[3t + 4 + it - 3t + i - 3 + \frac{t+3+(2t+2)(i-1)}{2} +$$
$$\frac{(t+1)(t+2)-(t+3)}{2})(2i-1) - (t-1)(i-1)\Bigg]$$
$$= \frac{2t^2 i - t^2 - t + 6ti + 4i + 2}{2(t+2)}.$$

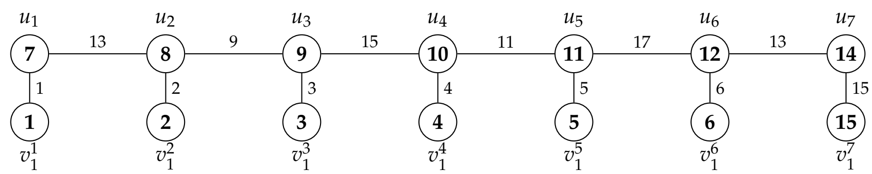

FIGURE 3. $\chi_{rm}(P_7 \circ O_1) = 15$.

Hence $\chi_{rm}(P_q \circ O_t) = q(t+1)$.
Case 3. q is odd and $t = 4\xi + 1,\ \xi \geq 1$.
Color the pendant edges of $P_q \circ O_t$ as follows:

$$c'\left(u_1 v_t^1\right) = \tfrac{5t+7}{4};\ c'\left(u_2 v_{\frac{t+3}{4}}^2\right) = t+1,$$

$$c(u_q v_j^q) = \begin{cases} j + q(t+1) - t \text{ if } j \in \{1, 2, \dots, \frac{t+1}{2}\}; \\ j + 1 + q(t+1) - t \text{ if } j \in \{\frac{t+3}{2}, \frac{t+5}{2}, \dots, t\}; \end{cases}$$

Next, color the edges of P_q as follows:

$$c'(u_1 u_2) = \tfrac{3t+5}{4};$$

For $2 \leq i \leq q-1$,

$$c'(u_i u_{i+1}) = \begin{cases} \frac{(t+1)(2i-1)}{2} \text{ if odd } i; \\ i(t+1) + t + 3 \text{ if even } i; \end{cases}$$

and color the leftover edges in $P_q \circ O_t$ as in *Case 2.*
Therefore, the vertices of $P_q \circ O_t$ have the following chromatic mean:

$$cm(v_j^q) = c'(u_q v_j^q) \text{ if } 1 \leq j \leq t;$$

$$cm(u_q) = \frac{2t^2 q - t^2 + t + 6qt + 4q + 6}{2(t+2)};$$

and color the leftover vertices of $P_q \circ O_t$ as in *Case 2.*
Thus, $\chi_{rm}(P_q \circ O_t) \leq q(t+1) + 1$. By Theorem 1.1, $\chi_{rm}(P_q \circ O_t) \geq q(t+1) + 1$.
Hence $\chi_{rm}(P_q \circ O_t) = q(t+1) + 1$ (see Figure 3, RMI of $P_7 \circ O_1$ is 15). □

4. ROOT PRODUCT

Definition 4.1. *The Rooted Product of the graphs is constructed as: Given a graph G with $n(G)$ vertices and another graph H that has a root vertex v, the root product graph, denoted $G \circ_v H$, is formed by taking a single copy of G and $n(G)$ copies of H. Each vertex i in G is then identified with the root vertex v in the corresponding i^{th} copy of H, for all i in $\{1, 2, \dots, n(G)\}$.*

Theorem 4.1. *Let the order of H_1 be t and $\chi_{rm}(H_1) = t$. Then $\chi_{rm}(G \circ_{v_1} H_1) = qt$.*

Proof. Let $V(G) = \{u_i : i \in \{1, 2, \ldots, q\}\}$ and $V(H_1) = \{v_j : 1 \le j \le t\}$. Thus, $V(G \circ_{v_1} H_1) = V(G) \cup \{v_j^i : 1 \le i \le q, 1 \le j \le t\}$. Consider v_1 to be the root vertex in $G \circ_{v_1} H_1$. Recall that g is the RMC of H_1, where $\chi_{rm}(H_1) = t$. Define $c : E(G \circ_{v_1} H_1) \to \mathbb{N}$ as:
For $1 \le j, j' \le t$,

$$c(v_j^i v_{j'}^i) = g(v_j^1 v_{j'}^1) + 2t(i-1) \text{ if } 1 \le i \le \left\lceil \frac{q}{2} \right\rceil,$$

$$c(v_j^i v_{j'}^i) = g(v_j^1 v_{j'}^1) + t(2q-2i+1) \text{ if } \left\lceil \frac{q}{2} \right\rceil + 1 \le i \le q.$$

$$cm(v_j^i) = cm(v_j^1) + t(2i-2) \text{ if } 1 \le i \le \left\lceil \frac{q}{2} \right\rceil,$$

$$cm(v_j^i) = cm(v_j^1) + t(2q-2i+1) \text{ if } \left\lceil \frac{q}{2} \right\rceil + 1 \le i \le q.$$

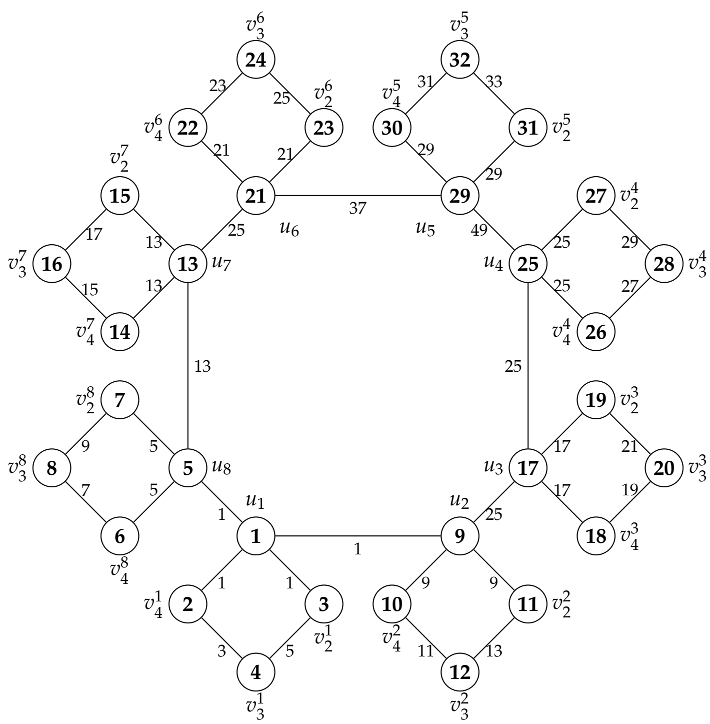

Figure 4. $\chi_{rm}(C_8 \circ_{v_1} C_4) = 32$.

Next, fix the colors to $u_i u_{i+1}$ as follows:
For $i \in \{1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$,

$$c(u_i u_{i+1}) = \begin{cases} cm(v_1^1) + t\ell(i-1) \text{ if odd } i, \\ cm(v_1^1) + t\ell i \text{ if even } i; \end{cases}$$

$c(u_i u_{i+1}) = cm(v_1^1) + t\ell(q-i)$ if $\lceil \frac{q}{2} \rceil + 1 \le i \le q$; and fix color $cm(v_1^1)$ to all remaining edges in $G \circ_{v_1} H_1$.

Therefore, the vertices of $G \circ_{v_1} H_1$ have the following chromatic mean:

$$cm(u_i) = cm(v_1^1) + t(2i-2) \text{ if } i \in \left\{1, 2, \dots, \left\lceil \frac{q}{2} \right\rceil \right\};$$
$$cm(u_i) = cm(v_1^1) + t(2q-2i+1) \text{ if } \left\lceil \frac{q}{2} \right\rceil + 1 \le i \le q.$$

Clearly, $\chi_{rm}(G \circ_{v_1} H_1) \le qt$. By observation 1.1, $\chi_{rm}(G \circ_{v_1} H_1) \ge qt$.
Hence $\chi_{rm}(G \circ_{v_1} H_1) = qt$ (see Figure 4, RMI of $C_8 \circ_{v_1} C_4$ is 32). □

Theorem 4.2. *Let the order of H_1 be t and $\chi_{rm}(H_1) = t$. For $q \ge 5$, $\chi_{rm}(P_q \circ_{v_1} H_1) = qt$.*

Proof. Let $V(P_q) = \{u_i : i \in \{1, 2, \dots, q\}\}$ and $V(H_1) = \{v_j : 1 \le j \le t\}$. Then $V(P_q \circ_{v_1} H_1) = V(P_q) \cup \{v_j^i : 1 \le i \le q,\ 1 \le j \le t\}$. Let g be the RMC of H_1 with $\chi_{rm}(H_1) = t$. Clearly, $|V(P_x \circ_{v_1} H_1)| = qt$. Define $c : E(P_q \circ_{v_1} H_1) \to \mathbb{N}$ as: Color the edges of H_1^i as:

Case 1. q is odd.

For $1 \le i \le q, 1 \le j, j' \le t$,

$$c(v_j^i v_{j'}^i) = g(v_j^1 v_{j'}^1) + t(i-1).$$
$$cm(v_j^i) = cm(v_j^1) + t(i-1).$$

Assign colors to the edges of P_q as:

For $1 \le i \le q-1$,

$$c(u_i u_{i+1}) = \begin{cases} cm(v_1^1) + i(t-1) \text{ if odd } i, \\ cm(v_1^1) + ti \text{ if even } i; \end{cases}$$

Therefore, the vertices of $P_q \circ_{v_1} H_1$ have the following chromatic mean:

$cm(u_i) = cm(v_1^1) + ti - t$ if $i \in \{1, 2, 3, \dots, q\}$;

Hence $\chi_{rm}(P_q \circ_{v_1} H_1) = qt$.

Case 2. q is even.

For $1 \le j, j' \le t$,

$$c(v_j^{q-3} v_{j'}^{q-3}) = g(v_j^1 v_{j'}^1) + t(q-3);$$
$$c(v_j^{q-2} v_{j'}^{q-2}) = g(v_j^1 v_{j'}^1) + t(q-1);$$
$$c(v_j^{q-1} v_{j'}^{q-1}) = g(v_j^1 v_{j'}^1) + t(q-2);$$
$$c(v_j^{q} v_{j'}^{q}) = g(v_j^1 v_{j'}^1) + t(q-4);$$

Next, color the edges of P_q as follows:

$$c(u_{q-3} u_{q-2}) = cm(v_1^1) + t(q-2);$$
$$c(u_{q-2} u_{q-1}) = cm(v_1^1) + qt;$$
$$c(u_{q-1} u_q) = cm(v_1^1) + t(q-4);$$

Assign colors to all leftover edges of $P_q \circ_{v_1} H_1$ as shown in *Case 1.*
Therefore, the vertices of $P_q \circ_{v_1} H_1$ have the following chromatic mean:

$$cm(u_{q-3}) = cm(v_1^1) + t(q-3);$$
$$cm(u_{q-2}) = cm(v_1^1) + t(q-1);$$
$$cm(u_{q-1}) = cm(v_1^1) + t(q-2);$$
$$cm(u_q) = cm(v_1^1) + t(q-4);$$

The chromatic mean of the remaining vertices in $P_q \circ_{v_1} H_1$ is calculated in the same manner as in *case 1*.
Thus, in both case $\chi_{rm}(P_q \circ_{v_1} H_1) \leq qt$. By observation 1.1, $rm(P_q \circ_{v_1} H_1) \geq qt$.
Hence $\chi_{rm}(p_q \circ_{v_1} H_1) = qt$. □

5. Join Graph

Definition 5.1. *The join graph of two graphs, H_1 and H_2 [16], denoted as $H_1 \vee H_2$, is formed by making every vertex in H_1 being adjacent to all vertex in H_2.*

Theorem 5.1. *For $q \geq 3$,*

$$\chi_{rm}(G \vee O_q) = \begin{cases} 2q+1 \text{ if } q \text{ is odd,} \\ 2q \text{ otherwise.} \end{cases}$$

Proof. Let $V(G) = \{u_i : 1 \leq i \leq q\}\}$ and $V(O_q) = \{v_j : 1 \leq j \leq q\}$. Clearly, $V(G \vee O_q) = V(G) \cup V(O_q)$, $E(G \vee O_q) = E(G) \cup \{u_i v_j : 1 \leq i, j \leq q\}$. Define $c : E(G \vee O_q) \to \mathbb{N}$ as:
Case 1. $q = 4\xi,\ \xi \geq 1$.
Color the edges lies between u_i and v_j as follows:
For $1 \leq i \leq q$,

$$c(u_i v_j) = j \text{ if } 1 \leq j \leq q-4;$$
$$c(u_i v_{q-3}) = q-2;$$
$$c(u_i v_{q-2}) = q;$$
$$c(u_i v_{q-1}) = q-1;$$
$$c(u_i v_q) = q-3;$$

Next, we assign colors to $u_i u_{i+1}$ as follows:
For $1 \leq i \leq \frac{q}{2}$,

$$c(u_i u_{i+1}) = \begin{cases} \frac{q(q+1)+4i(q+\ell)-4\ell+4}{4} \text{ if odd } i, \\ \frac{q^2+5q+4+4(q+\ell)i}{4} \text{ if even } i; \end{cases}$$

$$c(u_i u_{i+1}) = \frac{5q(q+1)+4-4i(q+\ell)+4\ell q}{4} \text{ if } \frac{q}{2}+1 \leq i \leq q-1;$$
$$c(u_q u_1) = c(u_1 u_2);$$

and fix a color $q+1$ to all leftover edges in $G \vee O_q$.
Therefore, the vertices of $G \vee O_q$ have the following chromatic mean:
For $1 \leq j \leq q$,

$$cm(v_j) = c(u_i v_j);$$
$$cm(u_i) = q - 1 + 2i \text{ if } 1 \leq i \leq \frac{q}{2};$$
$$cm(u_i) = 3q - 2(i-1) \text{ if } \frac{q}{2} + 1 \leq i \leq q.$$

Clearly, $\chi_{rm}(G \vee O_q) \leq 2q$. By observation 1.1, $\chi_{rm}(G \vee O_q) \geq 2q$.
Hence $\chi_{rm}(G \vee O_q) = 2q$.
Case 2. $q = 4\xi + 1,\ \xi \geq 1$.
$c(u_i v_j) = j$ if $1 \leq i, j \leq q$;
Next, we assign colors to $u_i u_{i+1}$ as:

$$c(u_i u_{i+1}) = \begin{cases} \frac{q^2+8+3q+4i(q+\ell)-4\ell}{4} \text{ if } 1 \leq i \leq \frac{q+1}{2} \text{ is odd,} \\ \frac{q^2+8+7q+4i(q+\ell)}{4} \text{ if } 2 \leq i \leq \frac{q-1}{2} \text{ is even,} \\ \frac{5q^2+7q+4q\ell+8-4iq-4i\ell}{4} \text{ if } \frac{q+3}{2} \leq i \leq q-1. \end{cases}$$

$c(v_q v_1) = c(v_1 v_2)$; and give color $q+2$ to all edges remaining in $G \vee O_q$.
Therefore, the vertices of $G \vee O_q$ have the following chromatic mean:

$$cm(v_j) = j \text{ if } 1 \leq j \leq q;$$
$$cm(u_i) = q + 2i \text{ if } 1 \leq i \leq \frac{q+1}{2};$$
$$cm(u_i) = 3q + 3 - 2i \text{ if } \frac{q+3}{2} \leq i \leq q.$$

Clearly, $\chi_{rm}(G \vee O_q) \leq 2q+1$. By Theorem 1.1, $\chi_{rm}(G \vee O_q) \geq 2q+1$.
Hence $\chi_{rm}(G \vee O_q) = 2q+1$ (see Figure 5, RMI of $K_5 \vee O_5$ is 11.)

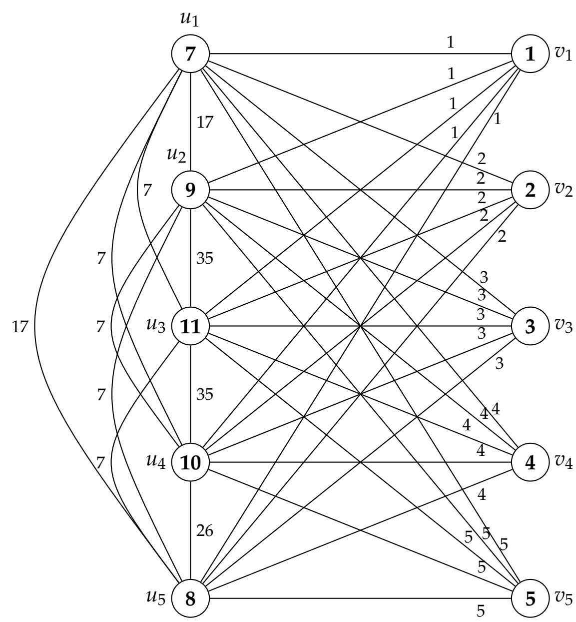

Figure 5. $\chi_{rm}(K_5 \vee O_5) = 11$.

Case 3. $q = 4\xi + 2,\ \xi \geq 1$.

For $1 \leq i \leq q$,

$c(u_i v_j) = j$ if $1 \leq j \leq q-1$;

$c(u_i v_q) = q+3$;

Next, we assign colors to $u_i u_{i+1}$ as:

$$c(u_i u_{i+1}) = \begin{cases} \frac{q^2-q-6+4i(q+\ell)-4\ell}{4} & \text{if } 1 \leq i \leq \frac{q}{2} \text{ is odd,} \\ \frac{q^2+3q-6+4i(q+\ell)}{4} & \text{if } 2 \leq i \leq \frac{q+2}{2} \text{ is even,} \\ \frac{5q^2+11q-6+4\ell(q+2)-4i(q+\ell)}{4} & \text{if } \frac{q+6}{2} \leq i \leq q-2 \text{ is even,} \\ \frac{5q^2+3q-6+4q\ell-4i(q+\ell)}{4} & \text{if } \frac{q+4}{2} \leq i \leq q-1 \text{ is odd;} \end{cases}$$

$c(u_q u_1) = c(u_1 u_2)$.

Therefore, the vertices of $G \vee O_q$ have the following chromatic mean:

$$cm(v_j) = j \text{ if } 1 \leq j \leq q-1;$$

$$cm(v_q) = q+3;$$

$$cm(u_i) = q + 2(i-1) \text{ if } 1 \leq i \leq \frac{q+2}{2};$$

$$cm(u_i) = 3(q+1) - 2i \text{ if } \frac{q+4}{2} \leq i \leq q-1;$$

$$cm(u_q) = q+1.$$

Clearly, $\chi_{rm}(G \vee O_q) \leq 2q$. By observation 1.1, $\chi_{rm}(G \vee O_q) \geq 2q$.
Hence $\chi_{rm}(G \vee O_q) = 2q$.
Case 4. $q = 4\xi + 3,\ \xi \geq 1$.
For $1 \leq i \leq q$,
$c(u_i v_j) = j$ if $1 \leq j \leq q$;
Assign colors to $u_i u_{i+1}$ as follows:

$$c(u_i u_{i+1}) = \begin{cases} \frac{q^2+5q+4}{4} \text{ if } i = 1, \\ \frac{q^2+9q+4\ell+4+4i(q+\ell)}{4} \text{ if } i \in \left\{3, 5, \ldots, \frac{q-1}{2}\right\}, \\ \frac{q^2+5q+4+4iq+4i\ell}{4} \text{ if } i \in \left\{2, 4, \ldots, \frac{q-3}{2}\right\}, \\ \frac{5q^2+5q+4(\ell q+1)-4i(q+\ell)}{4} \text{ if } i \in \left\{\frac{q+1}{2}, \frac{q+3}{2}, \ldots, q-1\right\}; \end{cases}$$

$c(u_q u_1) = c(u_1 u_2)$, and give color $q+1$ to all edges remaining in $G \vee O_q$.
Therefore, the vertices of $G \vee O_q$ have the following chromatic mean:

$$cm(v_j) = j \text{ if } 1 \leq j \leq q;$$
$$cm(u_i) = q - 1 + 2i \text{ if } i = 1, 2;$$
$$cm(u_i) = q + 1 + 2i \text{ if } 3 \leq i \leq \frac{q-1}{2};$$
$$cm(u_i) = 3q - 2(i-1) \text{ if } \frac{q+1}{2} \leq i \leq q;$$

Clearly, $\chi_{rm}(G \vee O_q) \leq 2q + 1$. By Theorem 1.1, $\chi_{rm}(G \vee O_q) \geq 2q + 1$.
Hence $\chi_{rm}(G \vee O_q) = 2q + 1$. □

Corollary 5.1. *For $q \geq 3$, $\chi_{rm}(K_2 \vee O_q) = q + 2$.*

Theorem 5.2. *For $q \geq 3$,*

$$\chi_{rm}(G \vee P_q) = \begin{cases} 2q+1 \text{ if } q \text{ is odd}, \\ 2q \text{ otherwise}. \end{cases}$$

Proof. Let $V(G) = \{u_i : i \in \{1, 2, \ldots, q\}\}$, $V(P_q) = \{v_j : 1 \leq j \leq q\}$. Thus, $V(G \vee P_q) = \{u_i : i \in \{1, 2, \ldots, q\}\} \cup \{v_j : j \in \{1, 2, \ldots, q\}\}$, $E(G \vee P_q) = E(G) \cup E(P_q) \cup \{u_i v_j : 1 \leq i, j \leq q\}$. Define $c' : E(G \vee P_q) \to \mathbb{N}$ as:
First, assign colors to the edges of $G \vee P_q$ as:
Case 1. $q = 4\xi,\ \xi \geq 1$,

$$c'(v_j v_{j+1}) = j \text{ if } 1 \leq j \leq q - 5 \text{ is odd};$$
$$c'(v_j v_{j+1}) = j + 1 \text{ if } 2 \leq j \leq q - 4 \text{ is even};$$
$$c'(v_{q-3} v_{q-2}) = q - 1;$$
$$c'(v_{q-2} v_{q-1}) = q + 1;$$
$$c'(v_{q-1} v_q) = q - 3.$$

Case 2. $q = 4\xi + 1$, $q = 4\xi + 3$ $\xi \geq 1$,

$$c'(v_jv_{j+1}) = j \text{ if } 1 \leq j \leq q-2 \text{ is odd;}$$
$$c'(v_jv_{j+1}) = j+1 \text{ if } 2 \leq j \leq q-1 \text{ is even.}$$

Case 3. $q = 4\xi + 2$, $\xi \geq 1$,

$$c'(v_jv_{j+1}) = j \text{ if } 1 \leq j \leq q-3 \text{ is odd;}$$
$$c'(v_jv_{j+1}) = j+1 \text{ if } 2 \leq j \leq q-2 \text{ is even;}$$
$$c'(v_{q-1}v_q) = q+3;$$
$$c'(v_{q-1}u_{q-1}) = q-3;$$
$$c'(v_{q-1}u_q) = q-3;$$
$$c'(u_{q-1}u_q) = \frac{q^2+7q+2+4\ell}{4}.$$

In all cases, the color of the remaining edges and vertices follows as in Theorem 5.1
Hence,

$$\chi_{rm}(G \vee P_q) = \begin{cases} 2q+1 \text{ if } q \text{ is odd,} \\ 2q \text{ otherwise.} \end{cases}$$

□

Theorem 5.3. *For* $q \geq 3$, $rm(P_q \vee O_q) = 2q$.

Proof. Let $V(P_q) = \{u_i : i \in \{1,2,\ldots,q\}\}$ and $V(O_q) = \{v_j : j \in \{1,2,\ldots,q\}\}$. Clearly, $V(P_q \vee O_q) = V(P_q) \cup V(O_q)$, $E(P_q \vee O_q) = \{u_iv_j : 1 \leq i, j \leq q\} \cup \{u_iu_{i+1} : 1 \leq i \leq q-1\}$.
Define $c : E(P_q \vee O_q) \to \mathbb{N}$ as:
Case 1. q is odd.
$c(u_iv_j) = j$ if $i, j \in \{1,2,\ldots,q\}$;
For $1 \leq i \leq q-1$,

$$c(u_iu_{i+1}) = \begin{cases} \frac{q^2+2q+2i+qi}{2} \text{ if odd } i, \\ \frac{4q+6+i(q+2)}{2} \text{ if even } i. \end{cases}$$

Therefore, the vertices of $P_q \vee O_q$ have the following chromatic mean:

$$cm(v_j) = j \text{ if } 1 \leq j \leq q,$$
$$cm(u_1) = q+1$$
$$cm(u_q) = q+2,$$
$$cm(u_i) = i+q+1 \text{ if } 2 \leq i \leq q-1.$$

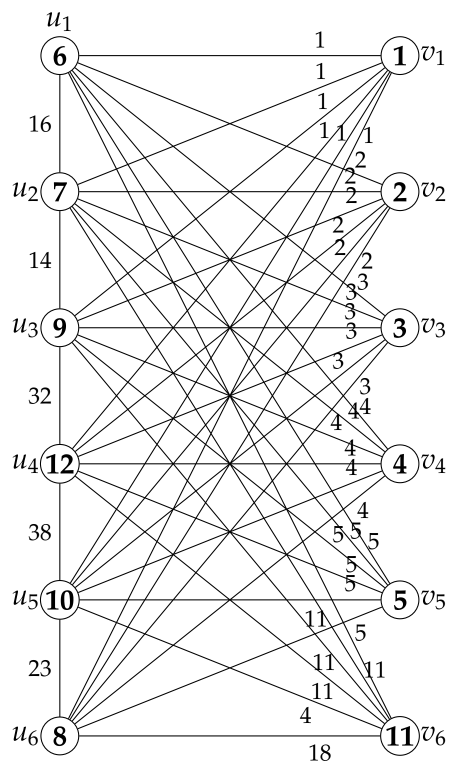

FIGURE 6. $\chi_{rm}(P_6 \vee O_6) = 12$.

Case 2. q is even.
$c(u_iv_j) = j$ if $i \in \{1,2,\ldots,q\},\ j \in \{1,2,\ldots,q-1\}$,
$c(u_iv_q) = 2q-1$ if $1 \le i \le q-2$,

$$c(u_{q-1}v_q) = \begin{cases} \frac{5q+2}{2} \text{ if } q = 4\xi,\ \xi \ge 1, \\ q-2 \text{ if } q = 4\xi+2,\ \xi \ge 1. \end{cases}$$

$$c(u_qv_q) = \begin{cases} \frac{3q-6}{2} \text{ if } q = 4\xi,\ \xi \ge 1, \\ 3q \text{ if } q = 4\xi+2,\ \xi \ge 1. \end{cases}$$

Next, we assign colors to the edges of P_q as:
For $1 \le i \le \frac{q}{2}$,

$$c(u_iu_{i+1}) = \begin{cases} \frac{q(q-3)+2i(q+2)-2}{2} \text{ if odd } i, \\ iq+2i-2 \text{ if even } i; \end{cases}$$

If $q = 4\xi+2$, $\xi \ge 1$,

$$c(u_iu_{i+1}) = \begin{cases} q^2+5q+4-iq-2i \text{ if } \frac{q+2}{2} \le i \le q-2 \text{ is even}, \\ \frac{3q^2+q-2-2iq-4i}{2} \text{ if } \frac{q+4}{2} \le i \le q-3 \text{ is odd}; \end{cases}$$

If $q = 4\xi$, $\xi \geq 1$,

$$c(u_i u_{i+1}) = \begin{cases} q^2 + 2q - 2 - iq + 2i \text{ if } \frac{q+4}{2} \leq i \leq q-2 \text{ is even}, \\ \frac{3q^2+7q+10-2i(q+2)}{2} \text{ if } \frac{q+2}{2} \leq i \leq q-3 \text{ is odd}; \end{cases}$$

$$c(u_{q-1} u_q) = \begin{cases} \frac{q^2+q+4}{2} \text{ if } q = 4\xi + 2,\ \xi \geq 1; \\ \frac{q^2+4q+10}{2} \text{ if } q = 4\xi,\ \xi \geq 1; \end{cases}$$

Therefore, the vertices of $P_q \vee O_q$ have the following chromatic mean:

$$cm(v_j) = j \text{ if } 1 \leq j \leq q-1;$$

$$cm(v_q) = 2q - 1;$$

$$cm(u_1) = q;$$

$$cm(u_i) = q + 2i - 3 \text{ if } 2 \leq i \leq \frac{q}{2};$$

$$cm(u_i) = 3q - 2i + 2 \text{ if } \frac{q+2}{2} \leq i \leq q.$$

Thus, in both cases $\chi_{rm}(P_q \vee O_q) \leq 2q$. By observation 1.1, $\chi_{rm}(P_q \vee O_q) \geq 2q$.
Hence $rm(P_q \vee O_q) = 2q$ (see Figure 6, RMI of $P_6 \vee O_6$ is 12). □

Corollary 5.2. *For $q \geq 4$, $\chi_{rm}(P_q \vee K_1) = q + 1$.*

Theorem 5.4. *If $q = t \geq 3$, then $rm(P_q \vee P_t) = 2q$*

Proof. Let $V(P_q) = \{u_i : 1 \leq i \leq q\}$, $V(P_t) = \{v_j : 1 \leq j \leq t\}$ and the $V(P_q \vee P_t) = \{u_i : 1 \leq i \leq q\} \cup \{v_j : 1 \leq j \leq t\}$, $E(P_q \vee P_t) = \{u_i v_j : 1 \leq i, j \leq q\} \cup \{u_i u_{i+1} : 1 \leq i \leq q-1\} \cup \{v_j v_{j+1} : 1 \leq j \leq t-1\}$.
Define $c' : E(P_q \vee P_t) \to \mathbb{N}$ as:

Case 1. q is odd.

For $1 \leq j \leq t-1$,

$$c'(v_j v_{j+1}) = \begin{cases} j \text{ if odd } j, \\ j+1 \text{ if even } j; \end{cases}$$

and assign colors to the leftover edges and vertices in $P_q \vee P_t$ same as in Theorem 5.3.

Case 2. q is even.

For $1 \leq j \leq t-2$,

$$c'(v_j v_{j+1}) = \begin{cases} j \text{ if odd } j, \\ j+1 \text{ if even } j; \end{cases}$$

$c'(v_{t-1} v_t) = 2q - 1$; and $c'(v_{q-1} u_{q-1}) = c'(t_{q-1} u_q) = \frac{q-2}{2}$;

$$c'(u_{q-1} u_q) = \begin{cases} \frac{q^2+2q+4}{2} \text{ if } q = 4\xi + 2,\ \xi \geq 1, \\ \frac{q^2+5q+10}{2} \text{ if } q = 4\xi,\ \xi \geq 1; \end{cases}$$

and fix colors to the leftover edges and vertices in $P_q \vee P_t$ same as in Theorem 5.3.
Thus, in both cases, $\chi_{rm}(P_q \vee P_t) \leq 2q$. By observation 1.1, $\chi_{rm}(P_q \vee P_t) \geq 2q$.
Hence $\chi_{rm}(P_q \vee P_t) = 2q$. □

6. Conclusion.

Since Rainbow mean index is a new advancement in graph coloring, many graph classes still have unresolved RMC. This work discusses the RMI of the tensor product of two graphs, the root product of two graphs, and the join of two graphs. The findings articulated in this work support the conjecture made in [8]. In the future, we will examine the RMI of other product graphs.

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