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On Hesitant Fuzzy Subalgebraic Systems in IUP-Algebras

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Abstract. We introduce hesitant fuzzy sets of algebraic structures in IUP-algebras, including hesitant fuzzy IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These notions are defined via relaxed set-theoretic and algebraic conditions suitable for hesitant fuzzy membership functions. Their mutual relationships are analyzed, showing how they extend and unify prior definitions in the UP-algebraic setting. Several examples and counterexamples are provided to illustrate essential differences between these classes and to demonstrate that each concept is a proper generalization of its classical or fuzzy counterpart. A diagram of class inclusions is also presented to visualize the structural hierarchy among these hesitant fuzzy sets.

1. Introduction

Following Torra's seminal work [33], hesitant fuzzy sets (HFSs) assign to each element a subset of $[0,1]$ that represents admissible membership degrees. At the same time, the classical setting uses nonempty finite subsets; in this paper, we allow the empty set and do not impose finiteness. In algebraic settings, hesitant fuzzy constructions have been developed across a wide range of algebraic systems. In BCK/BCI-algebras, several generalized forms have been explored, including hesitant fuzzy soft subalgebras and ideals [13, 16], inf-hesitant fuzzy ideals [15], hesitant p-ideals and quasi-associative ideals [14], sup-hesitant fuzzy quasi-associative ideals [8], commutative hesitant fuzzy ideals [1], and n-polar Z-hesitant complementary fuzzy soft sets [2]. In Boolean and BL-algebras, hesitant fuzzy structures have been examined through (λ, μ)-subalgebras [19], anti-type hesitant fuzzy filters [11], and hesitant fuzzy ideals in Boolean algebras [6]. Within semigroups, the development includes thresholded ideals such as $\sup_{\alpha}^{+}$- and $\sup_{\beta}^{-}$-ideals [12], hybrid interior ideals [5], and (α, β)-hesitant fuzzy approaches to ideal theory [34]. In group-theoretic

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settings, hesitant fuzzy subgroup generalizations have been proposed using α-thresholding mechanisms [7]. Related investigations in fuzzy implication algebras have addressed hesitant fuzzy filters and their structural behavior [18]. In the setting of UP-algebras, foundational notions such as hesitant fuzzy subalgebras, filters, and ideals have been introduced and expanded through characteristic, anti-type, partial-constant, and sup-hesitant variants, as well as operations on hesitant fuzzy soft sets including restricted union, extended intersection, and logical connectives [20–26]. Despite this extensive body of work, hesitant fuzzy structures on IUP-algebras have yet to be systematically developed. In this paper, we introduce hesitant fuzzy IUP-subalgebras, ideals, filters, and strong ideals under a general framework that admits empty and infinite-valued membership sets. We investigate their interrelations, closure under homomorphisms, and illustrate key distinctions through explicit examples and counterexamples that recover the fuzzy case and extend known results from UP-algebras.

The concept of IUP-algebras was introduced by Iampan et al. [9] in 2022, who axiomatized the structure and organized its principal families of distinguished subsets—IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals—establishing foundational properties and launching a research program. Subsequent work broadened both the core theory and its uncertainty-aware extensions. In 2023, Chanmanee et al. [4] analyzed direct products of (possibly infinite) families, introduced weak direct products, and developed results on (anti-)IUP-homomorphisms, thereby deepening the structural understanding. In a related development, Chanmanee et al. [3] examined external direct products in the setting of dual IUP-algebras, contributing further insight into product constructions. In 2024, Kuntama et al. [17] integrated fuzzy set (FS) theory with IUP-algebras, defining fuzzy IUP-subalgebras, ideals, filters, and strong ideals; in parallel, Suayngam et al. [32] advanced intuitionistic fuzzy set (IFS) counterparts, and later extended the framework to Fermatean fuzzy sets (FSSs) [30]. The line of generalization continued in 2025 with neutrosophic sets (NSs) [29], Pythagorean fuzzy sets [31], intuitionistic NSs [27], and Pythagorean NSs [28], yielding new characterizations of subalgebras, filters, ideals, and strong ideals under richer multi-valued semantics. Most recently, Inthachot et al. (2026) [10] proposed bipolar fuzzy IUP-subalgebras, filters, ideals, and strong ideals, further diversifying the landscape. Taken together, these developments trace a clear trajectory from the original crisp theory to a unified family of multi-valued extensions, with homomorphisms, product constructions, and subset hierarchies serving as common threads throughout.

In this paper, we systematically extend the theory of hesitant fuzzy sets to the setting of IUP-algebras. We introduce four core structures: hesitant fuzzy IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals, and establish formal relationships among them. In particular, we show that strong IUP-ideals coincide with constant hesitant fuzzy sets, and that hesitant fuzzy IUP-ideals properly generalize both subalgebras and filters. We also characterize algebraic subsets through their corresponding characteristic hesitant fuzzy sets and study the notion of primeness

TABLE 1. Comparison of fuzzy set extensions

Type	Membership	Non-membership	Indeterminacy	Hesitation	Constraint
FS	$\mu \in [0,1]$	–	–	–	None
HFS	$h \subseteq [0,1]$	–	–	✓	No size constraint
IFS	$\mu, \nu \in [0,1]$	✓	Implicit $(1-\mu-\nu)$	–	$\mu + \nu \leq 1$
PFS	$\mu, \nu \in [0,1]$	✓	Implicit	–	$\mu^2 + \nu^2 \leq 1$
FFS	$\mu, \nu \in [0,1]$	✓	Implicit	–	$\mu^3 + \nu^3 \leq 1$
NS	$T, I, F \in [0,1]$	✓	✓(explicit)	–	$T + I + F$ unrestricted

in this context. These results are supported by explicit examples and counterexamples, clarifying the structural hierarchy among hesitant fuzzy subsets in IUP-algebras.

2. Preliminaries

To establish a rigorous foundation for our study, we begin by recalling the basic structure of IUP-algebras and several auxiliary definitions essential to the hesitant fuzzy framework developed in this paper. Illustrative examples are also provided to motivate and clarify these concepts.

Definition 2.1. *[9] An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a non-empty set, $*$ is a binary operation on X, and 0 is the constant of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 * x = x) \tag{IUP-1}$$

$$(\forall x \in X)(x * x = 0) \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x * y) * (x * z) = y * z) \tag{IUP-3}$$

Throughout this paper, we denote an IUP-algebra by $X = (X, *, 0)$ unless stated otherwise, for notational clarity and consistency.

Example 2.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the Cayley table as follows:*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	3	1	5	2
2	2	5	0	4	3	1
3	5	4	1	0	2	3
4	1	3	5	2	0	4
5	3	2	4	5	1	0

Then $X = (X, *, 0)$ is an IUP-algebra.

Several foundational properties of the IUP-algebra X are listed below, based on results from [9].

$$(\forall x, y \in X)((x * 0) * (x * y) = y), \tag{2.1}$$
$$(\forall x \in X)((x * 0) * (x * 0) = 0), \tag{2.2}$$
$$(\forall x, y \in X)((x * y) * 0 = y * x), \tag{2.3}$$
$$(\forall x \in X)((x * 0) * 0 = x), \tag{2.4}$$
$$(\forall x, y \in X)(x * ((x * 0) * y) = y), \tag{2.5}$$
$$(\forall x, y \in X)(((x * 0) * y) * x = y * 0), \tag{2.6}$$
$$(\forall x, y, z \in X)(x * y = x * z \Leftrightarrow y = z), \tag{2.7}$$
$$(\forall x, y \in X)(x * y = 0 \Leftrightarrow x = y), \tag{2.8}$$
$$(\forall x \in X)(x * 0 = 0 \Leftrightarrow x = 0), \tag{2.9}$$
$$(\forall x, y, z \in X)(y * x = z * x \Leftrightarrow y = z), \tag{2.10}$$
$$(\forall x, y \in X)(x * y = y \Rightarrow x = 0), \tag{2.11}$$
$$(\forall x, y, z \in X)((x * y) * 0 = (z * y) * (z * x)), \tag{2.12}$$
$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (z * x) * (z * y) = 0), \tag{2.13}$$
$$(\forall x, y, z \in X)(x * y = 0 \Leftrightarrow (x * z) * (y * z) = 0), \tag{2.14}$$
$$\text{the right and the left cancellation laws hold.} \tag{2.15}$$

To investigate hesitant fuzzy analogues of algebraic structures in IUP-algebras, it is essential to recall the general concept of hesitant fuzzy sets. These sets, introduced to capture uncertainty with multiple degrees of membership, provide a flexible foundation for defining hesitant fuzzy IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals. The following definition serves as the cornerstone for constructing and analyzing these hesitant fuzzy structures throughout the remainder of this work.

Definition 2.2. *[9] A non-empty subset S of X is called*

(i) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x * y \in S), \tag{2.16}$$

(ii) *an IUP-filter of X if it satisfies the following conditions:*

$$\textit{the constant } 0 \textit{ of } X \textit{ is in } S, \tag{2.17}$$
$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S), \tag{2.18}$$

(iii) *an IUP-ideal of X if it satisfies the condition* (2.17) *and the following condition:*

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S), \tag{2.19}$$

(iv) a strong IUP-ideal of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x * y \in S). \tag{2.20}$$

From axiom (IUP-2), we have the following remark:

Remark 2.1. *Every IUP-subalgebra of X satisfies* (2.17).

According to [9], the notion of an IUP-filter serves as a common generalization of both IUP-ideals and IUP-subalgebras, which in turn generalize the concept of strong IUP-ideals. In any IUP-algebra X, the only possible strong IUP-ideal is X itself; no proper subset satisfies the defining conditions. These structures thus form a nested hierarchy, as illustrated in the inclusion diagram shown in Figure 1.

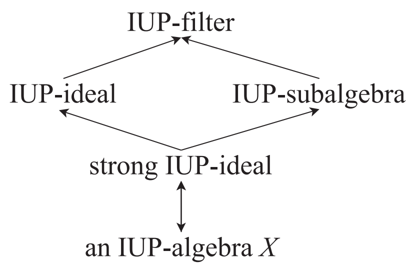

Figure 1. Special subsets of IUP-algebras

Definition 2.3. *A nonempty subset S of X is called a prime subset of X if it satisfies the following property:*

$$(\forall x, y \in X)(x * y \in S \Rightarrow x \in S \text{ or } y \in S). \tag{2.21}$$

Definition 2.4. *An IUP-subalgebra (resp., IUP-filter, IUP-ideal, strong IUP-ideal) B of X is called a prime IUP-subalgebra (resp., prime IUP-filter, prime IUP-ideal, prime strong IUP-ideal) of X if B is a prime subset of X.*

3. Main Results

In this section, we introduce and investigate several classes of hesitant fuzzy structures defined over IUP-algebras. These include hesitant fuzzy IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals—each formulated through relaxed membership conditions inspired by hesitant fuzzy logic. We explore their algebraic properties, examine the hierarchical relationships among them, and provide counterexamples to illustrate strictness in inclusion. Our results show that these classes not only generalize their counterparts in UP-algebras and fuzzy frameworks, but also reveal nuanced distinctions that arise uniquely in the IUP context.

Definition 3.1. *Let X be a nonempty reference set. A hesitant fuzzy set (HFS) on X is defined as a function*

$$h : X \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$. That is, for each element $x \in X$, the value $h(x)$ is a subset of $[0,1]$, representing all possible membership degrees that a decision-maker hesitates among when assigning a membership value to x.

We proceed by formulating four fundamental hesitant fuzzy structures within the framework of IUP-algebras: namely, hesitant fuzzy IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. Each is defined under suitable conditions reflecting the nature of hesitant membership functions.

Definition 3.2. *A hesitant fuzzy set h on X is called a hesitant fuzzy IUP-subalgebra of X if it satisfies the following property:*

$$(\forall x, y \in X)(h(x * y) \supseteq h(x) \cap h(y)) \tag{3.1}$$

Definition 3.3. *A hesitant fuzzy set h on X is called a hesitant fuzzy IUP-filter of X if it satisfies the following properties:*

$$(\forall x \in X)(h(0) \supseteq h(x)) \tag{3.2}$$

$$(\forall x, y \in X)(h(y) \supseteq h(x * y) \cap h(x)) \tag{3.3}$$

Definition 3.4. *A hesitant fuzzy set h on X is called a hesitant fuzzy IUP-ideal of X if it satisfies* (3.2) *and the following property:*

$$(\forall x, y, z \in X)(h(x * z) \supseteq h(x * (y * z)) \cap h(y)). \tag{3.4}$$

Definition 3.5. *A hesitant fuzzy set h on X is called a hesitant fuzzy strong IUP-ideal of X if it satisfies the following property:*

$$(\forall x, y \in X)(h(x * y) \supseteq h(y)). \tag{3.5}$$

Lemma 3.1. *Every hesitant fuzzy IUP-subalgebra on X satisfies* (3.2).

Proof. Assume that h is a hesitant fuzzy IUP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned} h(0) &= h(x * x) && \text{(by (IUP-2))} \\ &\supseteq h(x) \cap h(x) && \text{(by (3.1))} \\ &= h(x). \end{aligned}$$

□

Lemma 3.2. *Every hesitant fuzzy strong IUP-ideal on X satisfies* (3.2).

Proof. Assume that h is a hesitant fuzzy strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned} h(0) &= h(x * x) && \text{(by (IUP-2))} \\ &\supseteq h(x). && \text{(by (3.5))} \end{aligned}$$

□

Theorem 3.1. *Hesitant fuzzy strong IUP-ideals and constant hesitant fuzzy sets of X coincide.*

Proof. Assume that $\mathtt{h}$ is a hesitant fuzzy strong IUP-ideal of X. By Lemma 3.2, we have $\mathtt{h}$ satisfies (3.2). Let $x \in X$. Then

$$\begin{aligned} \mathtt{h}(x) &= \mathtt{h}((x * 0) * 0) && \text{(by (2.4))} \\ &\supseteq \mathtt{h}(0). && \text{(by (3.5))} \end{aligned}$$

Thus, $\mathtt{h}(x) = \mathtt{h}(0)$ for all $x \in X$, that is, $\mathtt{h}$ is constant of X. Clearly, every constant hesitant fuzzy set of X is a hesitant fuzzy strong IUP-ideal of X. Hence, hesitant fuzzy strong IUP-ideals and constant hesitant fuzzy sets of X coincide. □

Theorem 3.2. *Every hesitant fuzzy strong IUP-ideal of X is a hesitant fuzzy IUP-ideal of X.*

Proof. It is a direct consequence of Theorem 3.1. □

Example 3.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $*$ defined by the following Cayley table:*

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	3	4	0	1	5	2
3	2	3	5	0	1	4
4	4	2	1	5	0	3
5	1	5	3	4	2	0

Then $X = (X, *, 0)$ is an IUP-algebra. We define a hesitant fuzzy set $\mathtt{h}$ on X as follows:

$$\mathtt{h} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \{0,1\} & \{0\} & \{0\} & \{0\} & \{0,1\} & \{0\} \end{pmatrix}$$

Then $\mathtt{h}$ is a hesitant fuzzy IUP-ideal of X. Since $\mathtt{h}(1 * 0) = \{0\} \not\supseteq \{0, 1\} = \mathtt{h}(0)$, we have $\mathtt{h}$ is not a hesitant fuzzy strong IUP-ideal of X.

Theorem 3.3. *Every hesitant fuzzy strong IUP-ideal of X is a hesitant fuzzy IUP-subalgebra of X.*

Proof. It is a direct consequence of Theorem 3.1. □

Example 3.2. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $*$ defined by the following Cayley table:*

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	4	2	0	1	5	3
3	3	4	5	0	1	2
4	2	3	1	5	0	4
5	1	5	3	4	2	0

Then $X = (X, *, 0)$ is an IUP-algebra. We define a hesitant fuzzy set $\mathtt{h}$ *on X as follows:*

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \{0.2, 0.5, 0.6, 0.7\} & \{0.5, 0.7\} & \{0.5, 0.7\} & \{0.2, 0.5, 0.6, 0.7\} & \{0.5, 0.7\} & \{0.5, 0.7\} \end{pmatrix}$$

Then $\mathtt{h}$ *is a hesitant fuzzy IUP-subalgebra of X. Since $h(1 * 0) = \{0.5, 0.7\} \not\supseteq \{0.2, 0.5, 0.6, 0.7\} = h(0)$, we have* $\mathtt{h}$ *is not a hesitant fuzzy strong IUP-ideal of X.*

Example 3.3. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $*$ defined by the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	5	2	1	3
2	2	5	0	4	3	1
3	3	2	1	0	5	4
4	1	4	3	5	0	2
5	5	3	4	1	2	0

Then $X = (X, *, 0)$ is an IUP-algebra. We define a hesitant fuzzy set $\mathtt{h}$ *on X as follows:*

$$h = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \{0.6, 0.9\} & \{0.6\} & \{0.6, 0.9\} & \{0.6\} & \{0.6\} & \{0.6\} \end{pmatrix}$$

Then $\mathtt{h}$ *is a hesitant fuzzy IUP-subalgebra of X. Since $h(1 * 5) = \{0.6\} \not\supseteq \{0.6, 0.9\} = h(1 * (2 * 5)) \cap h(2)$, we have* $\mathtt{h}$ *is not a hesitant fuzzy IUP-ideal of X.*

Theorem 3.4. *Every hesitant fuzzy IUP-subalgebra of X is a hesitant fuzzy IUP-filter of X.*

Proof. Assume that $\mathtt{h}$ is a hesitant fuzzy IUP-subalgebra of X. By Lemma 3.1, we have $\mathtt{h}$ satisfies (3.2). Let $x, y \in X$. Then

$$\begin{aligned} \mathtt{h}(y) &= \mathtt{h}(0 * y) && \text{(by (IUP-1))} \\ &= \mathtt{h}((x * 0) * (x * y)) && \text{(by (IUP-3))} \\ &\supseteq \mathtt{h}(x * 0) \cap \mathtt{h}(x * y) && \text{(by (3.1))} \\ &\supseteq \mathtt{h}(x) \cap \mathtt{h}(0) \cap \mathtt{h}(x * y) && \text{(by (3.1))} \\ &= \mathtt{h}(x) \cap \mathtt{h}(x * y) && \text{(by (3.2))} \\ &= \mathtt{h}(x * y) \cap \mathtt{h}(x). \end{aligned}$$

Hence, $\mathtt{h}$ is a hesitant fuzzy IUP-filter of X. □

Theorem 3.5. *Every hesitant fuzzy IUP-ideal of X is a hesitant fuzzy IUP-filter of X.*

Proof. Assume that $\mathtt{h}$ is a hesitant fuzzy IUP-ideal of X. Then $\mathtt{h}$ satisfies (3.2). Let $x, y \in X$. Then

$$\begin{aligned} \mathtt{h}(y) &= \mathtt{h}(0 * y) && \text{(by (IUP-1))} \\ &\supseteq \mathtt{h}(0 * (x * y)) \cap \mathtt{h}(x) && \text{(by (3.4))} \\ &= \mathtt{h}(x * y) \cap \mathtt{h}(x). && \text{(by (IUP-1))} \end{aligned}$$

Hence, $\mathtt{h}$ is a hesitant fuzzy IUP-filter of X. □

Example 3.4. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation defined by the following Cayley table:*

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	3	4	2	1
2	2	3	0	1	5	4
3	3	4	5	0	1	2
4	4	2	1	5	0	3
5	1	5	4	2	3	0

Then $X = (X, *, 0)$ is an IUP-algebra. We define a hesitant fuzzy set $\mathtt{h}$ on X as follows:

$$\mathtt{h} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \{0.3, 0.5\} & \{0.3\} & \{0.3\} & \{0.3\} & \{0.3, 0.5\} & \{0.3\} \end{pmatrix}$$

Then $\mathtt{h}$ is a hesitant fuzzy IUP-filter of X. Since $\mathtt{h}(1 * 2) = \{0.3\} \not\supseteq \{0.3, 0.5\} = \mathtt{h}(1 * (4 * 2)) \cap \mathtt{h}(4)$, we have $\mathtt{h}$ is not a hesitant fuzzy IUP-ideal of X.

Based on Theorems 3.2, 3.4, and 3.5, together with Examples 3.2–3.4, we observe a natural hierarchy among the hesitant fuzzy structures on IUP-algebras. Specifically, the class of hesitant fuzzy IUP-ideals properly generalizes that of hesitant fuzzy strong IUP-ideals. Furthermore, both hesitant fuzzy IUP-ideals and hesitant fuzzy IUP-subalgebras are shown to be generalizations of hesitant fuzzy IUP-filters, forming a layered structure of increasing generality. These relationships can be visualized through the following diagram, which captures the inclusion relations among the various hesitant fuzzy constructs.

In what follows, we turn our attention to characteristic hesitant fuzzy sets—those induced by classical subsets of IUP-algebras. We investigate the correspondence between algebraic notions such as IUP-subalgebras, filters, ideals, and strong ideals, and their representations as characteristic hesitant fuzzy sets. We also introduce and explore the concepts of prime hesitant fuzzy sets, examining how they reflect algebraic primeness in the hesitant fuzzy context.

Given a subset $S \subseteq X$, the *characteristic hesitant fuzzy set* $\mathtt{h}_S$ is the function $\mathtt{h}_S : X \to \mathcal{P}([0, 1])$ defined by

$$\mathtt{h}_S(x) = \begin{cases} [0, 1] & \text{if } x \in S, \\ \emptyset & \text{otherwise.} \end{cases}$$

By construction, $\mathtt{h}_S$ maps each element of X to either the full interval $[0, 1]$ or the empty set, and thus satisfies $\mathrm{Im}(\mathtt{h}_S) \subseteq \{\emptyset, [0, 1]\} \subseteq \mathcal{P}([0, 1])$. Therefore, $\mathtt{h}_S$ is a hesitant fuzzy set on X.

Theorem 3.6. *A nonempty subset S of X is an IUP-subalgebra of X if and only if $\mathtt{h}_S$ is a hesitant fuzzy IUP-subalgebra of X.*

Proof. Assume that S is an IUP-subalgebra of X. Let $x, y \in X$.

Case 1: $x, y \in S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = [0,1]$. Then $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = [0,1]$. Since S is an IUP-subalgebra of X, we have $x * y \in S$ and so $\mathrm{h}_S(x * y) = [0,1]$. Therefore, $\mathrm{h}_S(x * y) = [0,1] \supseteq [0,1] = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 2: $x \in S$ and $y \notin S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = \emptyset$. Thus, $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x * y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 3: $x \notin S$ and $y \in S$. Then $\mathrm{h}_S(x) = \emptyset$ and $\mathrm{h}_S(y) = [0,1]$. Thus, $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x * y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Case 4: $x \notin S$ and $y \notin S$. Then $\mathrm{h}_S(x) = \emptyset$ and $\mathrm{h}_S(y) = \emptyset$. Thus, $\mathrm{h}_S(x) \cap \mathrm{h}_S(y) = \emptyset$. Therefore, $\mathrm{h}_S(x * y) \supseteq \emptyset = \mathrm{h}_S(x) \cap \mathrm{h}_S(y)$.

Hence, h_S is a hesitant fuzzy IUP-subalgebra of X.

Conversely, assume that h_S is a hesitant fuzzy IUP-subalgebra of X. Let $x, y \in S$. Then $\mathrm{h}_S(x) = [0,1]$ and $\mathrm{h}_S(y) = [0,1]$. Then $\mathrm{h}_S(x * y) \supseteq \mathrm{h}_S(x) \cap \mathrm{h}_S(x) = [0,1]$, so $\mathrm{h}_S(x * y) = [0,1]$. Therefore, $x * y \in S$ and so S is an IUP-subalgebra of X. □

Lemma 3.3. *The constant* 0 *of* X *is in a nonempty subset* B *of* X *if and only if* $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ *for all* $x \in X$.

Proof. If $0 \in B$, then $\mathrm{h}_B(0) = [0,1]$. Thus, $\mathrm{h}_B(0) = [0,1] \supseteq \mathrm{h}_B(x)$ for all $x \in X$.

Conversely, assume that $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(x)$ for all $x \in X$. Since B is a nonempty subset of X, we have $a \in B$ for some $a \in X$. Then $\mathrm{h}_B(0) \supseteq \mathrm{h}_B(a) = [0,1]$, so $\mathrm{h}_B(0) = [0,1]$. Hence, $0 \in B$. □

Theorem 3.7. *A nonempty subset* F *of* X *is an IUP-filter of* X *if and only if* h_F *is a hesitant fuzzy IUP-filter of* X.

Proof. Assume that F is an IUP-filter of X. Since $0 \in F$, it follows from Lemma 3.3 that $\mathrm{h}_F(0) \supseteq \mathrm{h}_F(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: $x, y \in F$. Then $\mathrm{h}_F(x) = [0,1]$ and $\mathrm{h}_F(y) = [0,1]$. Therefore, $\mathrm{h}_F(y) = [0,1] \supseteq \mathrm{h}_F(x * y) = \mathrm{h}_F(x * y) \cap \mathrm{h}_F(x)$.

Case 2: $x \notin F$ and $y \in F$. Then $\mathrm{h}_F(x) = \emptyset$ and $\mathrm{h}_F(y) = [0,1]$. Thus, $\mathrm{h}_F(y) = [0,1] \supseteq \emptyset = \mathrm{h}_F(x * y) \cap \mathrm{h}_F(x)$.

Case 3: $x \in F$ and $y \notin F$. Then $\mathrm{h}_F(x) = [0,1]$ and $\mathrm{h}_F(y) = \emptyset$. Since F is an IUP-filter of X, we have $x * y \notin F$ or $x \notin F$ but $x \in F$, so $x * y \notin F$. Then $\mathrm{h}_F(x * y) = \emptyset$. Thus, $\mathrm{h}_F(y) = \emptyset \supseteq \emptyset = \mathrm{h}_F(x * y) \cap \mathrm{h}_F(x)$.

Case 4: $x \notin F$ and $y \notin F$. Then $\mathrm{h}_F(x) = \emptyset$ and $\mathrm{h}_F(y) = \emptyset$. Then $\mathrm{h}_F(y) = \emptyset$. Thus, $\mathrm{h}_F(y) = \emptyset \supseteq \emptyset = \mathrm{h}_F(x * y) \cap \mathrm{h}_F(x)$.

Hence, h_F is a hesitant fuzzy IUP-filter of X.

Conversely, assume that h_F is a hesitant fuzzy IUP-filter of X. Since $\mathrm{h}_F(0) \supseteq \mathrm{h}_F(x)$ for all $x \in X$, it follows from Lemma 3.3 that $0 \in F$. Next, let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. Then $\mathrm{h}_F(x * y) = [0,1]$ and $\mathrm{h}_F(x) = [0,1]$. Thus, $\mathrm{h}_F(y) \supseteq \mathrm{h}_F(x * y) \cap \mathrm{h}_F(x) = [0,1]$, so $\mathrm{h}_F(y) = [0,1]$. Therefore, $y \in F$ and so F is an IUP-filter of X. □

Theorem 3.8. *A nonempty subset* B *of* X *is an IUP-ideal of* X *if and only if* h_B *is a hesitant fuzzy IUP-ideal of* X.

Proof. Assume that B is an IUP-ideal of X. Since $0 \in B$, it follows from Lemma 3.3 that $\mathtt{h}_B(0) \supseteq \mathtt{h}_B(x)$ for all $x \in X$. Next, let $x, y, z \in X$.

Case 1: $x * (y * z) \in B$ and $y \in B$. Then $\mathtt{h}_B(x * (y * z)) = [0,1]$ and $\mathtt{h}_B(y) = [0,1]$. Thus, $\mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y) = [0,1]$. Since $x*(y*z) \in B$ and $y \in B$, we have $x*z \in B$ and so $\mathtt{h}_B(x*z) = [0,1]$. Therefore, $\mathtt{h}_B(x*z) = [0,1] \supseteq [0,1] = \mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y)$.

Case 2: $x * (y * z) \in B$ and $y \notin B$. Then $\mathtt{h}_B(x * (y * z)) = [0,1]$ and $\mathtt{h}_B(y) = \emptyset$. Thus, $\mathtt{h}_B(x * (y * z)) \cap \mathtt{h}_B(y) = \emptyset$. Therefore, $\mathtt{h}_B(x*z) \supseteq \emptyset = \mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y)$.

Case 3: $x * (y * z) \notin B$ and $y \in B$. Then $\mathtt{h}_B(x * (y * z)) = \emptyset$ and $\mathtt{h}_B(y) = [0,1]$. Thus, $\mathtt{h}_B(x * (y * z)) \cap \mathtt{h}_B(y) = \emptyset$. Therefore, $\mathtt{h}_B(x*z) \supseteq \emptyset = \mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y)$.

Case 4: $x * (y * z) \notin B$ and $y \notin B$. Then $\mathtt{h}_B(x * (y * z)) = \emptyset$ and $\mathtt{h}_B(y) = \emptyset$. Thus, $\mathtt{h}_B(x * (y * z)) \cap \mathtt{h}_B(y) = \emptyset$. Therefore, $\mathtt{h}_B(x*z) \supseteq \emptyset = \mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y)$.

Hence, $\mathtt{h}_B$ is a hesitant fuzzy IUP-ideal of X.

Conversely, assume that $\mathtt{h}_B$ is a hesitant fuzzy IUP-ideal of X. Since $\mathtt{h}_B(0) \supseteq \mathtt{h}_B(x)$ for all $x \in X$, it follows from Lemma 3.3 that $0 \in B$. Next, let $x, y, z \in X$ be such that $x * (y * z) \in B$ and $y \in B$. Then $\mathtt{h}_B(x * (y * z)) = [0,1]$ and $\mathtt{h}_B(y) = [0,1]$. Thus, $\mathtt{h}_B(x*z) \supseteq \mathtt{h}_B(x*(y*z)) \cap \mathtt{h}_B(y) = [0,1]$, so $\mathtt{h}_B(x*z) = [0,1]$. Therefore, $x * z \in B$ and so B is an IUP-ideal of X. □

Theorem 3.9. *A nonempty subset C of X is a strong IUP-ideal of X if and only if $\mathtt{h}_C$ is a hesitant fuzzy strong IUP-ideal of X.*

Proof. It is a direct consequence of Theorem 3.1. □

Example 3.5. *[9] Let $\mathbb{R}^*$ be the set of all nonzero real numbers. Then $(\mathbb{R}^*, *, 1)$ is an IUP-algebra, where $*$ is the binary operation on $\mathbb{R}^*$ defined by $x * y = \dfrac{y}{x}$ for all $x, y \in \mathbb{R}^*$. Let $S = \{x \in \mathbb{R}^* \mid x \geq 1\}$. Then S is an IUP-ideal and an IUP-filter of $\mathbb{R}^*$ but it is not an IUP-subalgebra of $\mathbb{R}^*$. From Theorems 3.7, 3.8, and 3.9, we have $\mathtt{h}_S$ is a hesitant fuzzy IUP-ideal and a hesitant fuzzy IUP-filter of $\mathbb{R}^*$ but it is not a hesitant fuzzy IUP-subalgebra of $\mathbb{R}^*$.*

Definition 3.6. *A hesitant fuzzy set $\mathtt{h}$ on X is called a prime hesitant fuzzy set on X if it satisfies the following property:*

$$(\forall x, y \in X)(\mathtt{h}(x * y) \subseteq \mathtt{h}(x) \cup \mathtt{h}(y)). \tag{3.6}$$

Definition 3.7. *A hesitant fuzzy IUP-subalgebra (resp., hesitant fuzzy IUP-filter, hesitant fuzzy IUP-ideal, hesitant fuzzy strong IUP-ideal) $\mathtt{h}$ of X is called a prime hesitant fuzzy IUP-subalgebra (resp., hesitant fuzzy IUP-filter, hesitant fuzzy IUP-ideal, hesitant fuzzy strong IUP-ideal) if $\mathtt{h}$ is a prime hesitant fuzzy set on X.*

Theorem 3.10. *Let $\mathtt{h}$ be a hesitant fuzzy set on X. Then the following statements are equivalent:*

- (i) *$\mathtt{h}$ is a prime hesitant fuzzy IUP-subalgebra (resp., hesitant fuzzy IUP-filter, hesitant fuzzy IUP-ideal, hesitant fuzzy strong IUP-ideal) of X,*
- (ii) *$\mathtt{h}$ is a constant hesitant fuzzy set on X, and*
- (iii) *$\mathtt{h}$ is a hesitant fuzzy strong IUP-ideal of X.*

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathtt{h}$ is a prime hesitant fuzzy IUP-subalgebra (resp., hesitant fuzzy IUP-filter, hesitant fuzzy IUP-ideal, hesitant fuzzy strong IUP-ideal) of X. Then $\mathtt{h}(0) \supseteq \mathtt{h}(x)$ for all $x \in X$. By (IUP-2), we have $\mathtt{h}(0) = \mathtt{h}(x * x) \subseteq \mathtt{h}(x) \cup \mathtt{h}(x) = \mathtt{h}(x)$ for all $x \in X$ and so $\mathtt{h}(x) = \mathtt{h}(0)$ for all $x \in X$. Hence, $\mathtt{h}$ is a constant hesitant fuzzy set on X.

$(ii) \Rightarrow (i)$ Clearly, every constant hesitant fuzzy set on X is a prime hesitant fuzzy IUP-subalgebra (resp., hesitant fuzzy IUP-filter, hesitant fuzzy IUP-ideal, hesitant fuzzy strong IUP-ideal) of X.

$(ii) \Leftrightarrow (iii)$ It is straightforward by Theorem 3.1.

□

Theorem 3.11. *A nonempty subset S of X is a prime subset of X if and only if $\mathtt{h}_S$ is a prime hesitant fuzzy set on X.*

Proof. Assume that S is a prime subset of X. Let $x, y \in X$.

Case 1: $x * y \in S$. Then $\mathtt{h}_S(x * y) = [0, 1]$. Since S is a prime subset of X, we have $x \in S$ or $y \in S$. Then $\mathtt{h}_S(x) = [0, 1]$ or $\mathtt{h}_S(y) = [0, 1]$, so $\mathtt{h}_S(x) \cup \mathtt{h}_S(y) = [0, 1]$. Therefore, $\mathtt{h}_S(x * y) = [0, 1] \subseteq [0, 1] = \mathtt{h}_S(x) \cup \mathtt{h}_S(y)$.

Case 2: $x * y \notin S$. Then $\mathtt{h}_S(x * y) = \emptyset \subseteq \mathtt{h}_S(x) \cup \mathtt{h}_S(y)$.

Therefore, $\mathtt{h}_S$ is a prime hesitant fuzzy set on X.

Conversely, assume that $\mathtt{h}_S$ is a prime hesitant fuzzy set on X. Let $x, y \in X$ be such that $x * y \in S$. Then $\mathtt{h}_S(x * y) = [0, 1]$, so $[0, 1] = \mathtt{h}_S(x * y) \subseteq \mathtt{h}_S(x) \cup \mathtt{h}_S(y)$. Thus, $\mathtt{h}_S(x) \cup \mathtt{h}_S(y) = [0, 1]$, so $\mathtt{h}_S(x) = [0, 1]$ or $\mathtt{h}_S(y) = [0, 1]$. Hence, $x \in S$ or $y \in S$ and so S is a prime subset of X. □

4. Conclusion

In this paper, we introduced hesitant fuzzy substructures on IUP-algebras, including hesitant fuzzy subalgebras, filters, ideals, and strong ideals. We established generalization relationships among them and showed that strong ideals correspond exactly to constant hesitant fuzzy sets. Characteristic representations and primeness conditions were also analyzed, supported by illustrative examples.

Future research may focus on level-set characterizations of these structures. In particular, the study of α-level sets of hesitant fuzzy subsets can provide graded insights into their algebraic behavior and support applications in uncertainty modeling and decision-making processes.

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