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Approximated Solution Based on the Frame Bounds in Hilbert C^*-Modules

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Abstract. Given $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$, and given a frame in $\mathcal{H}$. We introduce several iterative methods for solving the operator equation:

$$Lu = f \tag{*}$$

Where L is a bounded, invertible, and symmetric operator on $\mathcal{H}$. We present algorithms that utilize frame bounds, the Chebyshev method, and the conjugate gradient method to provide approximate solutions to the problem. Additionally, we analyze the convergence and optimality of these methods.

1. Introduction and Preliminaries

Frames for Hilbert spaces were introduced by Duffin and Schaefer [3] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [6] for signal processing. In fact, in 1946, Gabor showed that any function $f \in L^2(\mathbb{R})$ can be reconstructed via a Gabor system $\{g(x-ka)e^{2\pi imbx} : k, m \in \mathbb{Z}\}$ where g is a continuous compact support function. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmann, and Meyer [2] in 1986, where they developed the class of tight frames for signal reconstruction and they showed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ that are very similar to the expansions using orthonormal bases. After this innovative work, the theory of frames began to be widely studied.

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While orthonormal bases have been widely used for many applications, it is the redundancy that makes frames useful in applications.

Formally, a frame in a separable Hilbert space $\mathcal{H}$ is a sequence $\{f_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ called frame bounds such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \ \forall x \in \mathcal{H}.$$

It is remarkable that the above inequalities imply the existence of a dual frame $\{\tilde{f_i}\}_{i\in I}$, such that the following reconstruction formula holds for every $x \in \mathcal{H}$: $\sum_{i\in I}\langle x, \tilde{f_i}\rangle f_i$. In particular, any orthonormal basis for $\mathcal{H}$ is a frame. However, in general, a frame need not be a basis, and, in fact, most useful frames are overcomplete. The redundancy that frames carry is what makes them very useful in many applications.

Today, frame theory is an exciting, dynamic, and fast-paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory, wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks, and more. The last decades have seen tremendous activity in the development of frame theory, and many generalizations of frames have come into existence. For more detailed information, readers are recommended to consult [1,4,7–11,13,15–30].

In the context of Hilbert C^*-modules, the iterative method typically refers to the process of constructing solutions to certain problems (e.g., equations or inequalities) by iterating through successive approximations, usually with the aim of converging to a desired solution. A Hilbert C^*-module is a generalization of a Hilbert space, where the scalars are elements of a C^*-algebra (instead of the field of complex numbers) and the inner product satisfies certain properties related to the algebraic structure of the C^*-algebra. These modules play an important role in various areas of functional analysis, particularly in operator theory and the theory of C^*-algebra.

In this work we present an algorithm to approximate the solution of the operator equation $Lx = y$ where $L : \mathcal{H} \longrightarrow \mathcal{H}$ is a boundedly invertible and self-adjoint operator on a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$.

Definition 1.1. *[12] Let $\mathbb{A}$ be a C^*-algebra and H be a left $\mathbb{A}$-module. We assume that the linear operations of $\mathbb{A}$ and H are compatible, i.e., $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in \mathbb{C}$, $a \in \mathbb{A}$, and $x \in H$. Recall that H is a pre-Hilbert $\mathbb{A}$-module if there exists a sesquilinear mapping $\langle\cdot,\cdot\rangle : H\times H \to \mathbb{A}$ with the properties*

(1) $\langle x, x\rangle \geq 0$; *if* $\langle x, x\rangle = 0$, *then* $x = 0$ *for every* $x \in H$.
(2) $\langle x, y\rangle = \langle y, x\rangle^*$ *for every* $x, y \in H$.
(3) $\langle ax, y\rangle = a\langle x, y\rangle$ *for every* $a \in \mathbb{A}$, $x, y \in H$.
(4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *for every* $x, y, z \in H$.

The map $H \to \mathbb{R}^+, x \mapsto \|x\|_H = \|\langle x, x\rangle_H\|_A^{\frac{1}{2}}$ is a norm on H and the following properties hold:

(i) $\|ax\|_H \leq \|x\|_H\|a\|_A$ *for all* $x \in H$, $a \in \mathbb{A}$,

(ii) $\langle x, y\rangle_H\langle y, x\rangle_H \leq \|y\|_H\langle x, x\rangle_H$ *for all* $x, y \in H$,

(iii) $\|\langle x, y\rangle_H\|_{\mathbb{A}} \leq \|x\|_H\|y\|_H$ *for all* $x, y \in H$.

If H is complete under the norm $\|\cdot\|_H$, *then H is called a Hilbert A-module or Hilbert* C^**-module over A.*

Lemma 1.1. *[14]. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If* $T \in End^*_{\mathcal{A}}(\mathcal{H})$, *then*

$$\langle Tx, Tx\rangle \leq \|T\|^2\langle x, x\rangle, \forall x \in \mathcal{H}.$$

Theorem 1.1. *[12] (Spectral Mapping). Let a be a normal element of a unital* C^**-algebra A, and* $f \in C(\sigma(a))$. *Then*

$$\sigma(f(a)) = f(\sigma(a))$$

Definition 1.2. *Let a be a element of a unital* C^**-algebra A. We define the spectral radius by:*

$$r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|.$$

Theorem 1.2. *[12] If a is a self-adjoint element of a* C^**-algebra A, then* $r(a) = \|a\|$.

Theorem 1.3. *Let a be a element of a unital* C^**-algebra A.*
Suppose T is a bounded self-adjoint linear operator and $f \in C(\sigma(a))$. *Then*

$$\|f(T)\| = \max_{t\in\sigma(T)} \left|f(t)\right|.$$

Proof. The proof is similar to the one in the case of Hilbert. □

Definition 1.3. *[5] A set of vectors* $\{x_n\}_{n\in\Gamma}$ *is a frame of a Hilbert space* C^*- *modules H if there exist two constants* $A > 0$ *and* $B \geq A > 0$ *such that*

$$\forall x \in H, \quad A\langle x, x\rangle \leq \sum_{n\in\Gamma}\langle x, x_n\rangle\langle x_n, x\rangle \leq B\langle x, x\rangle.$$

The frame is said to be tight if A and B are equal.

Since a frame $\{x_n\}_{k\geq1}$ is a Bessel sequence, the operator

$$\begin{array}{cccc} T & : & \ell^2(\mathbb{A}) & \rightarrow & H \\ & & \{c_k\}_{k\geq1} & \mapsto & \sum_{k\geq1} c_k x_k \end{array}$$

is bounded, where

$$\ell^2(\mathbb{A}) = \left\{\{q_i\}_{i\in I} \in \mathbb{A} : \sum_{i\in I} q_i q_i^* \textit{ converge in } \mathbb{A}\right\}$$

T is called the synthesis operator or the pre-frame operator.

The adjoint operator is given by

$$\begin{array}{cccc} T^* & : & H & \rightarrow & \ell^2(\mathbb{A}) \\ & & x & \mapsto & \{\langle x, x_k\rangle\}_{k\geq1} \end{array}$$

The operator T^* is called the analysis operator.

By composing T and T^*, we obtain the frame operator

$$\begin{array}{rcl} S \; : \; H & \rightarrow & H \\ x & \mapsto & TT^*x = \sum_{k\geq 1}\langle x, x_k\rangle x_k \end{array}$$

2. An approximated solution based on the frame bounds

Throughout the paper, we consider $\mathcal{A}$ an unital C^*-algebra.

2.1. **Frame Algorithm.** We know that if $\{x_k\}_{k\geq 1}$ is a frame for H, and S is its frame operator, then for every element $x \in H$:

$$x = \sum_{i\geq 1}\langle x, S^{-1}x_i\rangle x_i.$$

However, for this formula to be useful, we must invert the frame operator, which can be complicated if the dimension of H is large. To address this, an algorithm is used to obtain approximations of x. A classical algorithm for this is known as the *frame algorithm*.

Lemma 2.1. *(Frame Algorithme) Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module, and a family $\{f_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$ with bounds A et B, we define de sequence $\{y_i\}_{i\in I}$ for all $x \in \mathcal{H}$:*

$$y_0 = 0,\ y_k = y_{k-1} + \frac{2}{A+B}S(x - y_{k-1})\ k \geq 1$$

then y_k converge to x with error estimation: $\|x - y_k\| \leq (\frac{B-A}{B+A})^k\ \|x\|$.

Proof. Let $\{f_k\}_{k\in I}^m$ a frame for $\mathcal{H}$ with bounds A et B, then for all $x \in H$, and $x \in H$:

we have:

$$y_k = y_{k-1} + \frac{2}{A+B}S(x - y_{k-1})$$

then

$$\begin{aligned} x - y_k &= x - y_{k-1} - \frac{2}{A+B}S\,(x - y_{k-1}) \\ &= \left(I - \frac{2}{A+B}S\right)(x - y_{k-1}) \end{aligned}$$

then

$$x - y_k = \left(I - \frac{2}{A+B}S\right)^k (x - y_0)$$

$$x - y_k = \left(I - \frac{2}{A+B}S\right)^k (x)$$

For the frame operator S, we have as well:

$$\left\langle \left(I - \frac{2}{A+B}S\right)x, x\right\rangle = \langle x, x\rangle - \frac{2}{A+B}\sum_{k\in I}\langle x, x_k\rangle\langle x_k, x\rangle$$

then,

$$\langle x, x\rangle - \frac{2B}{A+B}\langle x, x\rangle \leq \left\langle \left(I - \frac{2}{A+B}S\right)x, x\right\rangle \leq \langle x, x\rangle - \frac{2A}{A+B}\langle x, x\rangle.$$

$$-\frac{B-A}{A+B}\langle x,x\rangle \leq \left\langle \left(I-\frac{2}{A+B}S\right)x,x\right\rangle \leq \frac{B-A}{A+B}\langle x,x\rangle.$$

then:

$$0 \leq \left\| \left\langle \left(I-\frac{2}{A+B}S\right)x,x\right\rangle \right\| \leq \frac{B-A}{A+B}\left\|\langle x,x\rangle\right\|.$$

So

$$\left\| I-\frac{2}{A+B}S \right\| \leq \frac{B-A}{B+A}$$

then

$$\begin{aligned}\left\|x-y_k\right\| &= \left\| \left(I-\frac{2}{A+B}S\right)^k x \right\| \\ &\leq \left\| I-\frac{2}{A+B}S \right\|^k \|x\| \leq \left(\frac{B-A}{B+A}\right)^k \|x\|.\end{aligned}$$

So we observe that $\{y_k\}_{k\geq 1}$ converge to x. □

2.2. **Transformation of Frames via a boundedly invertible and self adjoint operator.**

Lemma 2.2. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module, $L:\mathcal{H}\to\mathcal{H}$ is a boundedly invertible and self adjoint operator, and a family $\mathcal{F}=\{f_i\}_{i\in I}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$ with bounds A et B.*
Then the sequence $\mathcal{G}=\{g_i\}_{i\in I}=\{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds $\frac{A}{\|L^{-1}\|}^2$ and $B\|L\|^2$. Its operator frame is denoted by S' where $S'=LSL$.

Proof. $\mathcal{F}=\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds A et B, and $\forall x\in\mathcal{H},\ Lx\in\mathcal{H}$, then:

$$A\langle Lx,Lx\rangle \leq \sum_{i\in I}\langle Lx,f_i\rangle\langle f_i,Lx\rangle \leq B\langle Lx,Lx\rangle$$

Since $L\in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ then

$$A\langle Lx,Lx\rangle \leq \sum_{i\in I}\langle x,Lf_i\rangle\langle Lf_i,x\rangle \leq B\langle Lx,Lx\rangle$$

and

$$\langle Lx,Lx\rangle \leq \|L\|^2\langle x,x\rangle$$

L is invertible then:

$$\langle L^{-1}Lx,L^{-1}Lx\rangle \leq \|L^{-1}\|^2\langle Lx,Lx\rangle$$

So:

$$\langle x,x\rangle \leq \|L^{-1}\|^2\langle Lx,Lx\rangle$$

and

$$\frac{1}{\|L^{-1}\|^2}\langle x,x\rangle \leq \langle Lx,Lx\rangle$$

In conclusion:

$$\frac{A}{\|L^{-1}\|^2}\langle Lx,Lx\rangle \leq \sum_{i\in I}\langle x,Lf_i\rangle\langle Lf_i,x\rangle \leq B\|L\|^2\langle Lx,Lx\rangle$$

However:

$$\begin{aligned} S'x &= \sum_{i\in I} \langle x, g_i\rangle g_i \\ &= \sum_{i\in I} \langle x, Lf_i\rangle Lf_i \\ &= L\sum_{i\in I} \langle Lx, f_i\rangle f_i \\ &= LSLx \end{aligned}$$

□

Theorem 2.1. *Let $\mathcal{H}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module, a family $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with operator frame S, and $\{g_i\}_{i\in I} = \{Lf_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with operator frame S' and bounds A et B, we define de sequence $\{y_i\}_{i\in I}$ for all $x \in \mathcal{H}$:*

$$y_0 = 0,\ y_k = y_{k-1} + \frac{2}{A+B} LS(y - Ly_{k-1})\ k \geq 1$$

then y_k converge to x with error estimation: $\|x - y_k\| \leq (\frac{B-A}{B+A})^k \|x\|$.

Proof. Let $\{f_k\}_{k\in I}$ a frame for $\mathcal{H}$ with bounds A et B, then $\forall x \in H$, and $x \in H$:

Since $Lx = y$ then:

$$y_k = y_{k-1} + \frac{2}{A+B} LS(Lx - Ly_{k-1})$$

then

$$y_k = y_{k-1} + \frac{2}{A+B} LSL(x - y_{k-1})$$

So:

$$\begin{aligned} x - y_k &= x - y_{k-1} - \frac{2}{A+B} LSL\,(x - y_{k-1}) \\ &= \left(I - \frac{2}{A+B} LSL\right)(x - y_{k-1}) \end{aligned}$$

then

$$x - y_k = \left(I - \frac{2}{A+B} LSL\right)^k (x - y_0)$$

$$x - y_k = \left(I - \frac{2}{A+B} LSL\right)^k (x)$$

Other wise: S, we have as well:

$$\left\langle \left(I - \frac{2}{A+B} LSL\right) x, x \right\rangle = \langle x, x\rangle - \frac{2}{A+B} \sum_{k\in I} \langle x, f_k\rangle \langle f_k, x\rangle$$

then,

$$\langle x, x\rangle - \frac{2B}{A+B}\langle x, x\rangle \leq \left\langle \left(I - \frac{2}{A+B} LSL\right) x, x \right\rangle \leq \langle x, x\rangle - \frac{2A}{A+B}\langle x, x\rangle.$$

$$-\frac{B-A}{A+B}\langle x, x\rangle \leq \left\langle \left(I - \frac{2}{A+B} LSL\right) x, x \right\rangle \leq \frac{B-A}{A+B}\langle x, x\rangle.$$

then:

$$0 \leq \left\| \left\langle \left(I - \frac{2}{A+B} LSL\right) x, x \right\rangle \right\| \leq \frac{B-A}{A+B} \|\langle x, x\rangle\|.$$

So

$$\left\|I - \frac{2}{A+B}LSL\right\| \leq \frac{B-A}{B+A}$$

then

$$\begin{aligned}\left\|x - y_k\right\| &= \left\|\left(I - \frac{2}{A+B}LSL\right)^k x\right\| \\ &\leq \left\|I - \frac{2}{A+B}LSL\right\|^k \|x\| \leq \left(\frac{B-A}{B+A}\right)^k \|x\|.\end{aligned}$$

So we observe that $\{y_k\}_{k\geq 1}$ converge to x. □

If the upper bound of the frame is significantly larger than the lower bound, convergence may be slow. However, when the frame is close to a tight frame, faster convergence can be achieved.

Remark 2.1. *If $\{f_k\}_{k\in I}$ is a tight frame (i.e $A = B$), then $x = \frac{1}{A}LSy$*

Proof. if $A = B$ then $(S')^{-1} = \frac{1}{A}I$, we have:

$$\begin{aligned}& Lx = y \\ \Longleftrightarrow\ & LSLx = LSy \\ \Longleftrightarrow\ & S'x = LSy \\ \Longleftrightarrow\ & x = (S')^{-1}LSy \\ \Longleftrightarrow\ & x = \frac{1}{A}LSy\end{aligned}$$

□

2.3. **Chebyshev Acceleration Method in Hilbert $\mathcal{A}$-Modules.** In this section, we extend the iterative method based on frames to the setting where $\mathcal{H}$ is a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$. Our goal is to solve the operator equation

$$Lu = f,$$

where $L : \mathcal{H} \to \mathcal{H}$ is a bounded adjointable operator, subject to given boundary conditions.

2.3.1. *Richardson Iteration on Hilbert $\mathcal{A}$-Modules.* The equation can be reformulated as

$$u = (I - L)u + f.$$

Starting with an initial guess $u_0 \in \mathcal{H}$, define the iteration

$$u_{k+1} = (I - L)u_k + f, \quad k \geq 0.$$

Since $Lu = f$, it follows that

$$u_{k+1} - u = (I - L)(u_k - u).$$

Applying the module norm $\|\cdot\|_{\mathcal{H}}$ and the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$ yields

$$\|u_{k+1} - u\|_{\mathcal{H}} \leq \|I - L\|_{\mathcal{L}(\mathcal{H})}\|u_k - u\|_{\mathcal{H}}.$$

Therefore, the sequence (u_k) converges to u if

$$\|I - L\|_{\mathcal{L}(\mathcal{H})} < 1.$$

The non-stationary Richardson iteration in this context is

$$u_{k+1} = u_k + a_k(f - Lu_k), \quad k \geq 0,$$

where $a_k \in \mathbb{R}^+$ are parameters to be chosen. Denote the residual $r_k := f - Lu_k$. Then,

$$r_k = Q_k(L)r_0, \quad u - u_k = Q_k(L)(u - u_0),$$

where

$$Q_k(x) := \prod_{i=0}^{k-1}(1 - a_i x), \quad Q_k(0) = 1,$$

with the operator functional calculus applied in $\mathcal{L}(\mathcal{H})$.

2.3.2. *Frame-Based Scheme and Frame Bounds.* Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{H}$ be a frame with associated frame operator $S : \mathcal{H} \to \mathcal{H}$, which is bounded, positive and invertible in $\mathcal{L}(\mathcal{H})$. Assume the family $(L\psi_\lambda)_{\lambda\in\Lambda}$ is also a frame for $\mathcal{H}$ with frame bounds $A, B > 0$, i.e., for all $v \in \mathcal{H}$,

$$A\langle v, v\rangle_{\mathcal{A}} \leq \sum_{\lambda} \langle v, L\psi_\lambda\rangle_{\mathcal{A}} \langle L\psi_\lambda, v\rangle_{\mathcal{A}} \leq B\langle v, v\rangle_{\mathcal{A}}.$$

Define the iterative sequence by

$$u_k = u_{k-1} + \frac{2}{A+B} LS(f - Lu_{k-1}), \quad k \geq 1.$$

Theorem 2.2. *With the above notations, the operator*

$$R := I - \frac{2}{A+B} LSL$$

satisfies

$$\|R\|_{\mathcal{L}(\mathcal{H})} \leq \frac{B-A}{B+A}.$$

Consequently, for any initial $u_0 \in \mathcal{H}$, the sequence (u_k) converges to the solution u of $Lu = f$.

Proof. For any $v \in \mathcal{H}$, using the $\mathcal{A}$-valued inner product and frame inequalities, we have

$$\begin{aligned}
\left\langle \left(I - \frac{2}{A+B} LSL\right) v, v \right\rangle_{\mathcal{A}} &= \langle v, v\rangle_{\mathcal{A}} - \frac{2}{A+B} \langle LSLv, v\rangle_{\mathcal{A}} \\
&= \langle v, v\rangle_{\mathcal{A}} - \frac{2}{A+B} \sum_{\lambda} \langle v, L\psi_\lambda\rangle_{\mathcal{A}} \langle L\psi_\lambda, v\rangle_{\mathcal{A}} \\
&\leq \left(\frac{B-A}{B+A}\right) \langle v, v\rangle_{\mathcal{A}}.
\end{aligned}$$

By symmetry, a corresponding lower bound holds, proving the norm estimate. □

2.3.3. *Chebyshev Polynomial Acceleration.* Since LSL is a positive operator in $\mathcal{L}(\mathcal{H})$, the spectral theorem extends naturally in this framework. The spectral radius of R is contained in $[-\alpha_0, \alpha_0]$ with $\alpha_0 := \frac{B-A}{B+A}$.

To accelerate convergence, we define weighted combinations

$$h_n = \sum_{k=1}^{n} a_{n,k} u_k, \quad \text{with} \quad \sum_{k=1}^{n} a_{n,k} = 1,$$

and the corresponding polynomial

$$Q_n(x) = \sum_{k=1}^{n} a_{n,k} x^k.$$

The Chebyshev polynomials $(C_n)_{n\geq 0}$ satisfy the recurrence

$$C_0(x) = 1, \quad C_1(x) = x, \quad C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x),$$

and exhibit the minimax property, minimizing

$$\min_{Q_n(1)=1} \max_{|x|\leq\alpha_0} |Q_n(x)|.$$

Setting

$$P_n(x) := \frac{C_n\left(\frac{x}{\alpha_0}\right)}{C_n\left(\frac{1}{\alpha_0}\right)},$$

the error satisfies

$$\|u - h_n\|_{\mathcal{H}} \leq \frac{1}{C_n\left(\frac{1}{\alpha_0}\right)} \|u - u_0\|_{\mathcal{H}}.$$

2.3.4. *Algorithm.* Using the above, the accelerated Chebyshev iteration in $\mathcal{H}$ proceeds as follows:

Algorithm 2.1 Chebyshev Frame Method in Hilbert $\mathcal{A}$-Modules

1: **Input:** L, f, ϵ, A, B, m
2: Set $\alpha_0 = \frac{B-A}{B+A}$, $\quad \sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$
3: Initialize $h_0 := 0$, $h_1 := \frac{2}{A+B} LSf$, $\beta_1 := 2$, $n := 1$
4: **while** $\frac{2\sigma^n}{1+\sigma^{2n}} \frac{\|f\|_{\mathcal{H}}}{m} > \epsilon$ **do**
5: $\quad n := n + 1$
6: $\quad \beta_n := \left(1 - \frac{\alpha_0^2}{4}\beta_{n-1}\right)^{-1}$
7: $\quad h_n := \beta_n \left(h_{n-1} - h_{n-2} + \frac{2}{A+B} LS(f - Lh_{n-1})\right) + h_{n-2}$
8: **end while**
9: **Output:** $u_c := h_{n-1}$

The output satisfies the error bound

$$\|u - u_c\|_{\mathcal{H}} \leq \epsilon.$$

2.4. **Conjugate gradient method:** In this section, we demonstrate how to solve an operator equation with prescribed boundary conditions using frames and the conjugate gradient method. Additionally, we introduce a stopping criterion based on the frame bounds.

For the Chebyshev method to be effective, it is necessary to know an interval $[a, b]$that contains the spectrum of$L$. If this interval is too broad, the process becomes less efficient. A key benefit of the conjugate gradient method is that it does not require any prior knowledge about the spectrum's location. Also in contrast to Chebyshev method, the conjugate gradient method is adaptive. The hidden polynomials Q_n depend nonlinearly on u and arise from a minimization problem.

Suppose that S is the frame operator of the frame $(\psi_\lambda)_{\lambda\in\Lambda}$. Let a mapping:

$$\langle , \rangle_{LSL} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A} \tag{2.1}$$

$$(x, y) \longmapsto \langle x, y\rangle_{LSL} = \langle x, LSLy\rangle \tag{2.2}$$

For all $x,\ y\ et\ z \in \mathcal{H}$ and $a \in \mathcal{A}$. Since LSL is Positive definite we have:

(1) $\langle\ x, x\rangle_{LSL} = \langle\ x, LSLx\rangle = \langle\ LSLx, x\rangle \geqslant 0.$

(2)

$$\langle x, x\rangle_{LSL} = 0 \Leftrightarrow \langle x, LSLx\rangle = 0 \Leftrightarrow \langle SLx, Lx\rangle = 0 \Leftrightarrow Lx = 0 \Leftrightarrow x = 0.$$

(3) $\langle \alpha x + z, y\rangle_{LSL} = \langle \alpha x + z, LSLy\rangle = \alpha\langle x, LSLy\rangle + \langle z, LSLy\rangle = \alpha\langle x, y\rangle_{LSL} + \langle z, y\rangle_{LSL}.$

(4) $\langle x, y\rangle^*_{LSL} = \langle x, LSLy\rangle^* = \langle LSLy, x\rangle = \langle y, LSLx\rangle = \langle y, x\rangle_{LSL}.$

So $\langle , \rangle_{LSL}$ is a inner product for $\mathcal{H}$.
Then we can define the $LSL-$norm for the space $\mathcal{H}$ by

$$|||f||| = \|\langle f, f\rangle_{LSL}\|^{\frac{1}{2}} = \|\langle f, LSLf\rangle\|^{\frac{1}{2}}, \quad \forall f \in H$$

corresponding to the inner product $\langle , \rangle_{LSL}$

In this case if u is the solution of the equation (*) then:

$$|||u|||^2 = \|\langle u, LSLu\rangle\| = \|\langle Lu, SLu\rangle\| = \|\langle f, Sf\rangle\| \leq B\|f\|^2.$$

Then

$$|||u||| \leq \sqrt{B}\|f\|. \tag{2.3}$$

Then, f can be reconstructed iteratively from the frame coefficients (f, e,) by the following algorithm: Put $h_0 = 0, \quad v_{-1} = 0, \quad r_0 = v_0 = LSf, \quad n = 0$

$$\lambda_n = \langle r_n, v_n\rangle \langle v_n, LSLv_n\rangle^{-1} \tag{2.4}$$

$$h_{n+1} = h_n + \lambda_n v_n \tag{2.5}$$

$$r_{n+1} = r_n - \lambda_n LSLv_n \tag{2.6}$$

$$v_{n+1} = LSLv_n - \langle LSLv_n, LSLv_n\rangle \langle v_n, LSLv_n\rangle^{-1} v_n - \langle LSLv_n, LSLv_{n-1}\rangle \langle v_{n-1}, LSLv_{n-1}\rangle^{-1} v_{n-1}.$$

Theorem 2.3. *The following error estimate holds in the LSL-norm:*

$$|||u - h_n|||\leq \frac{2\sigma^n}{1+\sigma^{2n}}|||u||| = \frac{2\sigma^n}{1+\sigma^{2n}}\sqrt{B}\|f\| \text{ where } \sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}.$$

Proof. Considering the problem (*), define $v_{-1} = 0, v_0 = LSf$ and

$$v_{n+1} = LSLv_n - \langle LSLv_n, LSLv_n\rangle \langle v_n, LSLv_n\rangle^{-1} v_n - \langle LSLv_n, LSLv_{n-1}\rangle \langle v_{n-1}, LSLv_{n-1}\rangle^{-1} v_{n-1}$$

Assume that u is the solution of the problem (*), then the following lemma holds.

Lemma 2.3. *Let* $H_n = \operatorname{span}\left\{(LSL)^j u : j = 1,2,3,\ldots,n\right\}$*, then*

$$\{v_0, v_1, \ldots, v_{n-1}\} \subseteq H_n$$

Proof. you can prove this lemma by induction.
For $n = 0$:

$$\begin{aligned} v_1 =& LSLv_0 - \langle LSLv_0, LSLv_0\rangle \langle v_0, LSLv_0\rangle^{-1} v_0 - \langle LSLv_0, LSLv_{-1}\rangle \langle v_{-1}, LSLv_{-1}\rangle^{-1} v_{-1} \\ =& (LSL)^2 u - \langle LSLv_0, LSLv_0\rangle \langle v_0, LSLv_0\rangle^{-1} LSLu \end{aligned}$$

So true for $n = 1$.
Now assume that, it is true for $n \in \mathbb{N}$ (ie: $\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathcal{H}_n$)
For $n + 1$ we have: $v_n =$
$LSLv_{n-1} - \langle LSLv_{n-1}, LSLv_{n-1}\rangle \langle v_{n-1}, LSLv_{n-1}\rangle^{-1} v_{n-1} - \langle LSLv_{n-1}, LSLv_{n-2}\rangle \langle v_{n-2}, LSLv_{n-2}\rangle^{-1} v_{n-2}$
Alors $v_n \subset \mathcal{H}_{n+1}$. as we desired. □

Lemma 2.4. *The system* $\{v_0, v_1, \ldots, v_{n-1}\}$ *forms an orthogonal basis for* H_n *with respect to the LSL-inner product.*

Proof. Since $\dim H_n \leq n$ and span $\{v_0, v_1, \ldots, v_{n-1}\}$ is a subspace of H_n, then it is enough to show that $\{v_0, v_1, \ldots, v_{n-1}\}$ is an orthogonal system. By induction:
For n=2, since $v_{-1} = 0$:

$$v_1 = LSLv_0 - \langle LSLv_0, LSLv_0\rangle \langle v_0, LSLv_0\rangle^{-1} v_0,$$

and so,

$$\langle v_1, v_0\rangle_{LSL} = \langle v_1, LSLv_0\rangle = \langle LSLv_0, LSLv_0\rangle - \langle LSLv_0, LSLv_0\rangle \langle v_0, LSLv_0\rangle^{-1} \langle v_0, LSLv_0\rangle = 0.$$

So $\{v_0, v_1\}$ is an LSL orthogonal basis for $\mathcal{H}_2$.
Now assume that, it is true for $n \in \mathbb{N}$ (ie: $\{v_0, v_1, \ldots, v_{n-1}\}$is an LSL orthogonal basis for$\mathcal{H}_n$)
For $n + 1$ by putting

$$\alpha = \langle LSLv_{n-1}, LSLv_{n-1}\rangle \langle v_{n-1}, LSLv_{n-1}\rangle^{-1}$$

and

$$\beta = \langle LSLv_{n-1}, LSLv_{n-2}\rangle \langle v_{n-2}, LSLv_{n-2}\rangle^{-1}$$

we have:

$$\alpha \langle v_{n-1}, LSLv_{n-1}\rangle = \langle LSLv_{n-1}, LSLv_{n-1}\rangle$$

and

$$\beta \langle v_{n-2}, LSLv_{n-2} \rangle = \langle LSLv_{n-1}, LSLv_{n-2} \rangle .$$

So

$$\begin{aligned}\langle v_n, LSLv_{n-1} \rangle &= \langle LSLv_{n-1} - \alpha v_{n-1} - \beta v_{n-2}, LSLv_{n-1} \rangle \\ &= \langle LSLv_{n-1}, LSLv_{n-1} \rangle - \alpha \langle v_{n-1}, LSLv_{n-1} \rangle - \beta \langle v_{n-2}, LSLv_{n-1} \rangle \\ &= \langle LSLv_{n-1}, LSLv_{n-1} \rangle - \langle LSLv_{n-1}, LSLv_{n-1} \rangle = 0.\end{aligned}$$

The same for $\langle v_n, LSLv_{n-2} \rangle$:

$$\begin{aligned}\langle v_n, LSLv_{n-2} \rangle &= \langle LSLv_{n-1} - \alpha v_{n-1} - \beta v_{n-2}, LSLv_{n-2} \rangle \\ &= \langle LSLv_{n-1}, LSLv_{n-2} \rangle - \alpha \langle v_{n-1}, LSLv_{n-2} \rangle - \beta \langle v_{n-2}, LSLv_{n-2} \rangle \\ &= \langle LSLv_{n-1}, LSLv_{n-2} \rangle - \langle LSLv_{n-1}, LSLv_{n-2} \rangle = 0.\end{aligned}$$

for $j < n-1$, we observe that $LSLv_j \in LSL\left(H_{n-1}\right) \subset H_n$. Since $\{v_0, \ldots, v_{n-1}\}$ is a basis for H_n, hence, there exist $c_1, c_2, \ldots, c_{i-1}$ that:

$$LSLv_j = \sum_{i=0}^{n-1} c_i v_i$$

$$\begin{aligned}\left\langle v_n, LSLv_j \right\rangle = \left\langle LSLv_{n-1} - \alpha v_{n-1} - \beta v_{n-2}, LSLv_j \right\rangle = \left\langle LSLv_n, LSLv_j \right\rangle - \alpha \left\langle v_n, LSLv_j \right\rangle \\ = \sum_{i=0}^{n-1} c_i \langle v_n, LSLv_i \rangle = 0,\end{aligned}$$

for every $j < n-1$. □

Theorem 2.4. *The approximated solution h_n is the orthogonal projection of the solution u of the problem * onto H_n. That is*

$$\|u - h_n\|_{LSL} \leq \|u - g\|_{LSL} \quad \forall g \in H_n$$

Proof. By 2.5 we have $h_n = \sum_{j=0}^{n-1} \lambda_j v_j \in H_n$.
Then, by Lemma (2.4), $\langle h_n, v_n \rangle_{LSL} = \left\langle \sum_{j=0}^{n-1} \lambda_j v_j, v_n \right\rangle_{LSL} = 0$
Also by (2.6) we have: $r_n = r_0 - LSL\left(\sum_{j=0}^{n-1} \lambda_j v_j\right) = LSLu - LSLh_n = LSL\left(u - h_n\right)$ and by (2.4) we have:

$$\lambda_n = \langle r_n, v_n \rangle \langle v_n, LSLv_n \rangle^{-1} = \langle u - h_n, v_n \rangle_{LSL} \langle v_n, v_n \rangle_{LSL}^{-1}$$

So:

$$\begin{aligned}\langle u - h_n, h_n \rangle_{LSL} &= \left\langle u - \sum_{j=0}^{n-1} \lambda_j v_j, \sum_{j=0}^{n-1} \lambda_j v_j \right\rangle_{LSL} \\ &= \sum_{j=0}^{n-1} \left(\left\langle u, v_j \right\rangle_{LSL} - \lambda_j \left\langle v_j, v_j \right\rangle_{LSL} \right) \lambda_j^* \\ &= \sum_{j=0}^{n-1} \left(\left\langle u, v_j \right\rangle_{LSL} - \left\langle u - h_j, v_j \right\rangle_{LSL} \left\langle v_j, v_j \right\rangle_{LSL}^{-1} \left\langle v_j, v_j \right\rangle_{LSL} \right) \lambda_j^*\end{aligned}$$

$$= \sum_{j=0}^{n-1} \left\langle h_j, v_j \right\rangle_{LSL} \lambda_j^{*}$$

$$= \sum_{j=0}^{n-1} \left\langle \sum_{i=0}^{j-1} \lambda_i v_i, v_j \right\rangle_{LSL} \lambda_j^{*} = 0.$$

Since $h_n =\in H_n$ and $\langle u - h_n, h_n \rangle_{LSL} = 0$,so we get our result. □

Now let's return to our theorem (2.3).

We have $h_n \in H_n = \text{span}\left\{(LSL)^j u : j = 1, 2, 3, \ldots, n\right\}$ The definition of h_n implies that $h_n = q_{n-1}(LSL)LSLu$, where $q_{n-1}(x)$ is a polynomial of degree $n-1$. Then

$$u - h_n = (I - q_{n-1}(LSL)LSL)\, u = \Phi_n(I - LSL)u$$

where $\Phi_n(x) = 1 - (1-x)q_{n-1}(1-x)$ is a polynomial of degree n with $\Phi_n(1) = 1$. Thus

$$|||u - h_n||| = |||\Phi_n(I - LSL)u||| \leq |||P_n(I - LSL)u|||$$

for all polynomials P_n of degree n with $P_n(1) = 1$.
Since

$$\forall x \in \mathcal{H},\ |||x||| = \|\langle x, LSLx\rangle\||^{\frac{1}{2}} = \|\langle (LSL)^{\frac{1}{2}} x, (LSL)^{\frac{1}{2}} x\rangle\||^{\frac{1}{2}} = \|(LSL)^{\frac{1}{2}} x\|$$

So

$$\|P_n(I - LSL)u||| = \left\|(LSL)^{\frac{1}{2}} P_n(I - LSL)u\right\|$$

Hence

$$\begin{aligned}
|||u - h_n||| \leq |||P_n(I - LSL)u||| &= \left\|(LSL)^{\frac{1}{2}} P_n(I - LSL)(LSL)^{\frac{-1}{2}} (LSL)^{\frac{1}{2}} u\right\| \\
&\leq \left\|(LSL)^{\frac{1}{2}} P_n(I - LSL)(LSL)^{\frac{-1}{2}}\right\| \left\|(LSL)^{\frac{1}{2}} u\right\| \\
&= \left\|P_n(I - LSL)\right\|\ |||u||| \\
&\leq \max_{-\frac{B-A}{A+B} \leq x \leq \frac{B-A}{A+B}} |P_n(x)|\ |||u|||\ \text{(by theorem(1.3))}. \\
&= \frac{1}{C_n\left(\frac{1}{\frac{B-A}{B+A}}\right)} |||u|||
\end{aligned}$$

Therefore, by (2.3) we conclude that

$$\|u - h_n\|_{LSL} \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \|u\|_{LSL} = \frac{2\sigma^n}{1 + \sigma^{2n}} \sqrt{B} \|f\|$$

where $\sigma = \frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}$.

□

3. NUMERICAL EXAMPLES

Example 3.1. ***:Frame-Algorithme:***

Let $\mathbb{H}$ be a Hilbert space of dimension 3, and the matrix of our operator L is:

$$\mathcal{M}_L = \begin{pmatrix} 2 & 4 & 7 \\ 1 & 3 & 8 \\ 5 & 9 & 2 \end{pmatrix}$$

Let the frame $\mathcal{F} = \{f_1, f_2, f_3, f_4, f_5\} = \{2e_1 + 3e_2, 2e_1 + 4e_2 + 6e_3, 2e_1 + 3e_2 + e_3, 3e_1 + 4e_3, 5e_3\}$.

Therefore, the respective matrices of S *and* S'=LSL *are:*

$$\mathcal{M}_S = \begin{pmatrix} 21 & 20 & 26 \\ 20 & 34 & 27 \\ 26 & 27 & 78 \end{pmatrix}$$

$$\mathcal{M}_{S'} = \begin{pmatrix} 4503 & 8665 & 6460 \\ 4571 & 8749 & 6189 \\ 3779 & 7489 & 7097 \end{pmatrix}$$

The frame bounds of S' *are* $A = 0.29956005504190586$ *and* $B = 19143.563287821737$. *If we apply our frame algorithm to the vector* $x = (3, 4, 2)$ *with an error* $e = 10^{-6}$, *after 495242 iterations we obtain the vector* $x' = (2.99999953.999999361.99999946)$.

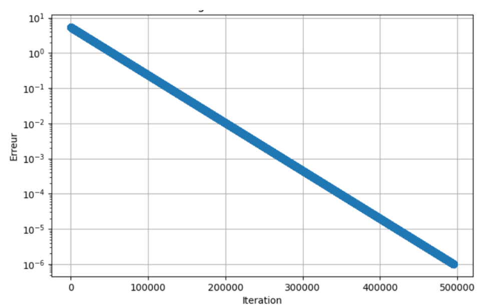

Figure1 : **Convergence of Algorithme Frame**

Example 3.2. *:***ChebychevMethod** :

We want to solve the operator equation $Lu = f$, *where L is the operator already mentioned in the previous experiment,* $f = (4, 7, 9))$ *and we take the same Frame* $\mathcal{F}$. *Then, using the Chebyshev method and an error tolerance of 0.0000001, we obtain after 2199 iterations an approximation* $u_e = (-9.99999938\ 6.63636333 - 0.3636363)$.

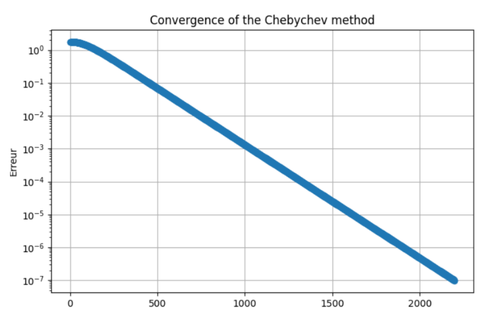

Figure2 : **Convergence of the chebychev mathod**

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