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Woven Continuous Frames in Hilbert Spaces

Hafida Massit[1], **Maryam G. Alshehri**[2,*], **Mohamed Rossafi**[3]

[1] *Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco*
[2] *Department of Mathematics, Faculty of Science, University of Tabuk, P.O.Box741, Tabuk 71491, Saudi Arabia*
[3] *Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco*

[*] *Corresponding author:* mgalshehri@ut.edu.sa

Abstract. In this present paper we introduce weaving Hilbert space frames in the continuous case, we propose new approaches for manufacturing pairs of woven continuous frames, and we obtain new properties in continuous weaving frame theory related to dual frames. Also, we provide some approaches for constructing continuous weaving frames by using small perturbations. These methods not only enhance the understanding of frame theory but also open avenues for practical applications in signal processing and data representation. Future research may further explore the implications of these findings in more complex systems and their potential interdisciplinary benefits.

1. Introduction

The concept of frames in Hilbert spaces was introduced by Duffin and Schaffer [7] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [6] by Daubechies, Grossman, and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. Continuous frames defined by Ali, Antoine, and Gazeau [1]. Gabrado and Han in [9] called these frames associated with measurable spaces. For more about frames, see [2, 8, 10–24].

Recently, Bemrose et al. [3] have introduced a new concept of weaving frames in separable Hilbert space. This is motivated by a problem regarding distributed signal processing, where frames play an important role. Weaving frames has potential applications in wireless sensor networks that require distributed processing under different frames. The fundamental properties

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of weaving frames were reviewed by Casazza and Lynch in [4]. Weaving frames were further studied by Casazza, Freeman, and Lynch [5].

In this paper, we give new basic properties of weaving continuous frames related to dual frames to survey under which conditions a continuous frame with its dual constitutes woven continuous frames, and we give some approaches for constructing concrete pairs of woven continuous frames.

2. Preliminaries

Throughout this paper, we suppose $\mathcal{H}$ is a separable Hilbert space, $\mathcal{H}_m$ an $m-$ dimensional Hilbert space, I the identity operator on $\mathcal{H}$, $(\mathfrak{A}, \mu)$ be a measure space with positive measure μ.

A family of vectors $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ in a separable Hilbert $\mathcal{H}$ is called a Riesz basis if $\overline{span}\{F_\varsigma\}_{\varsigma\in\mathfrak{A}} = \mathcal{H}$ and there exist constants $0 < A_F \leq B_F < \infty$, such that

$$A_F \int_{\mathfrak{A}} |\alpha_\varsigma|^2 d\mu(\varsigma) \leq \| \int_{\mathfrak{A}} \alpha_\varsigma F_\varsigma d\mu(\varsigma)\|^2 \leq B_F \int_{\mathfrak{A}} |\alpha_\varsigma|^2 d\mu(\varsigma),\ \forall \{\alpha_\varsigma\}_\varsigma \in L^2(\mathfrak{A}).$$

The constants A_F and B_F are called Riesz basis bounds.

A family of vectors $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ in a separable Hilbert $\mathcal{H}$ is said to be a continuous frame if there exist constants $0 < A_F \leq B_F < \infty$, such that

$$A_F\|f\|^2 \leq \| \int_{\mathfrak{A}} |\langle f, F_\varsigma\rangle|^2 d\mu(\varsigma) \leq B_F\|f\|^2,\ \forall f \in \mathcal{H}, \tag{2.1}$$

then the constants A_F and B_F are called frame bounds.

The family $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is said to be a Bessel sequence whenever in 2.1, the right hand side holds. In the case of $A_F = B_F = 1$, $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ called a Parseval frame. And if $A_F = B_F$ it is called a tight frame.

Given a frame $F = \{F_\varsigma\}$, the frame operator is defined by

$$S_F f = \int_{\mathfrak{A}} \langle f, F_\varsigma\rangle F_\varsigma d\mu(\varsigma).$$

It is a bounded, invertible, and self-adjoint operator. Also, the synthesis operator $T_F : L^2(\mathfrak{A}, \mu) \to \mathcal{H}$ defined by $T_F(f) = \int_{\mathfrak{A}} f(\varsigma)F(\varsigma)d\mu(\varsigma)$. The frame operator can be written as $S_F = T_F T_F^*$ where $T_F^* : \mathcal{H} \to L^2(\mathfrak{A}, \mu)$, the adjoint of T_F, given by $T_F^*(f)(\varsigma) = \{\langle f, F(\varsigma)\rangle\}_{\varsigma\in\mathfrak{A}}$ is called the analysis operator. The family $\{S_F^{-1}F\}_{\varsigma\in\mathfrak{A}}$ is also a frame for $\mathcal{H}$, called the canonical dual frame. In general, a continuous frame $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}} \subset \mathcal{H}$ is called an alternate dual for $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ if

$$f = \int_{\mathfrak{A}} \langle f, G_\varsigma\rangle F_\varsigma d\mu(\varsigma), f \in \mathcal{H}. \tag{2.2}$$

Every dual frame is of the form $\{S_F^{-1}F_\varsigma + v_\varsigma\}_{\varsigma\in\mathfrak{A}}$, with $\{v_\varsigma\}_{\varsigma\in\mathfrak{A}}$ a Bessel sequence that satisfies

$$\int_{\mathfrak{A}} \langle f, v_\varsigma\rangle F_\varsigma = 0. \tag{2.3}$$

Definition 2.1. *A family of continuous frames $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{A},1\leq\nu\leq N}$ in Hilbert space $\mathcal{H}$ is said to be a continuous woven if there are universal constants A and B so that for every partition $\{\mathfrak{B}_\nu\}_{1\leq\nu\leq N}$ of $\mathfrak{A}$, the family $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{B}_\nu,1\leq\nu\leq N}$ is a continuous frame for $\mathcal{H}$ with bounds A and B, respectively. The family $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{B}_\nu,1\leq\nu\leq N}$ is called a continuous weaving.*

If for every partition $\{\mathfrak{B}_\nu\}_{1\leq\nu\leq N}$ of $\mathfrak{A}$, the family $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{B}_\nu,1\leq\nu\leq N}$ is a continuous frame for $\mathcal{H}$, then the family $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{A},1\leq\nu\leq N}$ is called weakly continuous woven.

Casazza, Freeman, and Lynch proved in [5] that the weaker form of weaving is equivalent to weaving using the uniform boundedness principle.

Theorem 2.1. *[5] Given two continuous frames $\{F_\varsigma\}$ and $\{G_\varsigma\}$ for $\mathcal{H}$, the following are equivalent:*

(1) *The two continuous frames are continuous woven.*
(2) *The two continuous frames are weakly continuous woven.*

Proposition 2.1. *[3] If $\{F_{\varsigma,\nu}\}_{\varsigma\in\mathfrak{A},1\leq\nu\leq N}$ is a family of Bessel sequences for $\mathcal{H}$ with a Bessel bound B_ν for all $1\leq\nu\leq N$, then every weaving is a Bessel sequence with the Bessel bound $\sum_{\nu=1}^N B_\nu$.*

Proposition 2.2. *[3] Let $F=\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous frame and T an invertible operator satisfying $\|I-T\|^2<\dfrac{A}{B}$. Then, F and T are continuous woven with the universal lower bound $(\sqrt{A}-\sqrt{B}\|I-T\|)^2$.*

Definition 2.2. *[3] If U_1 and U_2 are subspaces of $\mathcal{H}$, let*

$$d_{U_1}(U_2)=\inf\{\|f-g\|:\ f\in U_1, g\in S_{U_2}\}$$

and

$$d_{U_2}(U_1)=\inf\{\|f-g\|:\ f\in S_{U_1}, g\in U_2\},$$

where $S_{U_i}=S_{\mathcal{H}}\cup U_i$ and $S_{\mathcal{H}}$ is the unit sphere in $\mathcal{H}$. Then, the distance between U_1 and U_2 is defined as

$$d(U_1,U_2)=\min\{d_{U_1}(U_2),d_{U_2}(U_1)\}.$$

Theorem 2.2. *[3] If $F=\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ and $G=\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ are two continuous Riesz bases for $\mathcal{H}$, then the following are equivalent*

(1) *F and G are continuous woven.*
(2) *For every $K\subset\mathfrak{A}$, $d(\overline{span}\{F_\varsigma\}_{\varsigma\in K},\overline{span}\{G_\varsigma\}_{\varsigma\in K^C})>0$.*
(3) *There is a constant $t>0$ so that for every $K\subset\mathfrak{A}$,*

$$d_{F_K,G_{K^C}}:=d(\overline{span}\{F_\varsigma\}_{\varsigma\in K},\overline{span}\{G_\varsigma\}_{\varsigma\in K^C})\geq t.$$

We denote K^C the complement of K.

3. Main result

Now, we try to examining some conditions under which a continuous frame with its approximate duals is woven. This leads us to construct some concrete pairs of woven frames.

Theorem 3.1. *Suppose that $F=\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a continuous redundant frame so that*

$$\|I-S_F^{-1}\|^2<\frac{A_F}{B_F}. \tag{3.1}$$

Then, F has infinitely many dual frames $\{G_\varsigma\}_\varsigma$ for which and $\{F_\varsigma\}_\varsigma$ are continuous woven.

Proof. By Proposition 2.2 we have F and $\{S_F^{-1}F_\varsigma\}$ are woven frames for $\mathcal{H}$ with the lower bound $\mathcal{R} := (\sqrt{A} - \sqrt{B}\|I - S_F^{-1}\|)^2$. Now, let $V = \{v_\varsigma\}_\varsigma$ be a bessel sequence, which satisfies $f = \int_{\mathfrak{A}} \langle f, v_\varsigma\rangle F_\varsigma d\mu(\varsigma)$, $(f \in \mathcal{H})$ and let $\varepsilon > 0$ so that

$$\varepsilon^2 B_V + 2\varepsilon\sqrt{B_V/A_F} < \mathcal{R}. \tag{3.2}$$

Then, $G_\beta := \{S_F^{-1}F_\varsigma + \beta v_\varsigma\}_\varsigma$ is a dual frame of F, for all $0 < \beta < \varepsilon$. To show F and G_β constitute woven continuous frames for $\mathcal{H}$, by Proposition 2.1, we need to prove the existence of a lower bound.

Suppose $\mathfrak{B} \subset \mathfrak{A}$. Then

$$\begin{aligned}
&\int_{\mathfrak{B}} |\langle f, F_\varsigma\rangle|^2 d\mu(\varsigma) + \int_{\mathfrak{B}^C} |\langle f, S_F^{-1}F_\varsigma + \beta v_\varsigma\rangle|^2 d\mu(\varsigma)\\
=&\int_{\mathfrak{B}} |\langle f, F_\varsigma\rangle|^2 d\mu(\varsigma) + \int_{\mathfrak{B}^C} |\langle f, S_F^{-1}F_\varsigma\rangle + \langle f, \beta v_\varsigma\rangle|^2 d\mu(\varsigma)\\
\geq &\int_{\mathfrak{B}} |\langle f, F_\varsigma\rangle|^2 d\mu(\varsigma) + \int_{\mathfrak{B}^C} |\,|\langle f, S_F^{-1}F_\varsigma\rangle| - |\langle f, \beta v_\varsigma\rangle|\,|^2 d\mu(\varsigma)\\
\geq &\int_{\mathfrak{B}} |\langle f, F_\varsigma\rangle|^2 d\mu(\varsigma) + \int_{\mathfrak{B}^C} |\langle f, S_F^{-1}F_\varsigma\rangle d\mu(\varsigma)|^2\\
-&\int_{\mathfrak{B}^C} |\langle f, \beta v_\varsigma\rangle|^2 d\mu(\varsigma) - 2\int_{\mathfrak{B}^C} |\langle f, S_F^{-1}F_\varsigma\rangle|\,|\langle f, \beta v_\varsigma\rangle| d\mu(\varsigma)\\
\geq &(-\beta^2 B_V - 2\beta\sqrt{B_V/A_F} + \mathcal{R})\|f\|^2.
\end{aligned}$$

□

Proposition 3.1. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a redundant continuous frame for $\mathcal{H}$ and suppose there exists an operator $T \in B(\mathcal{H})$ so that*

$$\|I - T\| < 1\,, \|I - T^* S_F^{-1}\|^2 < \frac{A_F}{B_F}. \tag{3.3}$$

Then, F has infinitely many approximate dual frames such as $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$, for wich $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ and $\{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ are continuous woven.

Proof. The sequence $\{T^* S_F^{-1}F_\varsigma + v_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is an approximate dual of F, with $V = \{v_\varsigma\}_{\varsigma\in\mathfrak{A}}$ satisfies 3.1. Also by Proposition 2.2, F and $\{T^* S^{-1}F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ are continuous woven lower with the universal bound $(\sqrt{A_F} - \sqrt{B_F}\|I - T^* S_F^{-1}\|)^2$. Let $\varepsilon > 0$, such that

$$\varepsilon^2\|T_V\|^2 + 2\varepsilon\|T_V\|\,\|S_F^{-1}T\|\sqrt{B_F} < (\sqrt{A_G} - \sqrt{B_F}\|I - T^* S_F^{-1}\|)^2.$$

Then $\Phi_\beta = \{T^* S_F^{-1}F_\varsigma + \beta v_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is an approximate dual frame of F for all $0 < \beta < \varepsilon$. Using Theorem 3.1, we obtain that F and Φ_β are woven continuous frames. □

Theorem 3.2. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous frame for $\mathcal{H}$. Then, the following assertions hold:*

1. *If $G = \{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a dual frame of F and $\{F_\varsigma\}_{\varsigma\in K} \cup \{G_\varsigma\}_{\varsigma\in K^C}$ for $K \subset \mathfrak{A}$. Then, F and G are continuous woven.*
2. *If $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is continuous Riesz basis for $\mathcal{H}$. Then, F and its canonical dual are continuous woven.*

Proof. Let $f \in \mathcal{H}$ such that $f \perp \{F_\varsigma\}_{\varsigma \in K} \cup \{G_\varsigma\}_{\varsigma \in K^C}$. Then, we obtain

$$\begin{aligned}\|f\|^2 = \langle f, f\rangle &= \langle f, \int_{\mathfrak{A}} \langle f, G_\varsigma\rangle F_\varsigma d\mu(\varsigma)\rangle \\ &= \langle f, \int_K \langle f, G_\varsigma\rangle F_\varsigma d\mu(\varsigma)\rangle + \langle f, \int_{K^C} \langle f, G_\varsigma\rangle F_\varsigma d\mu(\varsigma)\rangle \\ &= \int_K \langle f, G_\varsigma\rangle\langle f, F_\varsigma\rangle d\mu(\varsigma) + \int_{K^C} \langle f, G_\varsigma\rangle\langle f, F_\varsigma\rangle d\mu(\varsigma) \\ &= 0.\end{aligned}$$

Hence $f = 0$ and consequently $\overline{span}\{\{F_\varsigma\}_{\varsigma \in K} \cup \{G_\varsigma\}_{\varsigma \in K^C}\} = \mathcal{H}$ for $K \subset \mathfrak{A}$. Then F and G are weakly continuous woven.

For (2), let $K \subset \mathfrak{A}$, $X = \int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) \in \overline{span}\{\{F_\varsigma\}_{\varsigma \in K}$ and $Y = \int_{K^C} \beta_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma) \in \overline{span}\{\{S_F^{-1} F_\varsigma\}_{\varsigma \in K^C}$ with $\|X\| = 1$ and $\|Y\| = 1$.

Then, we have

$$\begin{aligned}\|X - Y\|^2 &= \|\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma)\|^2 \\ &= \|\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma)\|^2 + \|\int_{K^C} \beta_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma)\|^2 - 2Re\langle \int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma), \int_{K^C} \beta_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma)\rangle \\ &= \|\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma)\|^2 + \|\int_{K^C} \beta_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma)\|^2 \geq 1.\end{aligned}$$

Thus, F and G are continuous woven by Theorem 2.2. □

Example 3.1. *Let $\mathcal{H} = L^2(\mathbb{N})$ and $\mathfrak{A} = (\mathbb{R}^2, \mu)$ where μ is the Lebesgue measure. Let χ_I denote the characteristic function of a set I. Let $\{\phi_i\}_{(1,2)}$ be any non-zero element in $\mathcal{H}$ such $\phi_2 = 2\phi_1$. Then, the family $\{I_x T_y \phi_1\}_{(x,y)\in\mathfrak{A}}$ and $\{I_x T_y \phi_2\}_{(x,y)\in\mathfrak{A}}$ are continuous frames for $\mathcal{H}$ with respect to μ with frame bounds A_i and B_i for $i \in \{1, 2\}$.*

For any subset K of $\mathfrak{A}$ and for $f \in \mathcal{H}$, we define the function $\psi : \mathfrak{A} \to \mathbb{C}$ by

$$\psi(x, y) = \psi_1(x, y).\chi_K(x, y) + \psi_2(x, y).\chi_{K^C}(x, y)$$

where $\psi_i(x, y) = \langle f, I_x T_y 2\phi_i\rangle$, $i \in \{1, 2\}$

We have $\{I_x T_y \phi\}_{(x,y)\in K} \cup \{I_x T_y 2\phi\}_{(x,y)\in K^C}$ is a continuous Bessel sequence with Bessel bound $\sum_{i\in\{1,2\}} B_i$.

We obtain

$$\begin{aligned}\|\psi\|^2_{L^2(\mu)} &= \int\int_K |\langle f, I_x T_y \phi\rangle|^2 dxdy + \int\int_{K^C} |\langle f, I_x T_y 2\phi\rangle|^2 dxdy \\ &\geq \int\int_K |\langle f, I_x T_y \phi\rangle|^2 dxdy + \int\int_{K^C} |\langle f, I_x T_y \phi\rangle|^2 dxdy \\ &= \int\int_{\mathfrak{A}} |\langle f, I_x T_y \phi\rangle|^2 dxdy \\ &\geq A_1\|f\|^2.\end{aligned}$$

Hence $\{I_x T_y \phi_1\}_{(x,y)\in\mathfrak{A}}$ and $\{I_x T_y \phi_2\}_{(x,y)\in\mathfrak{A}}$ are woven with universal bounds A_1 and $\sum_{i\in\{1,2\}} B_i$.

Corollary 3.1. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous frame for finite dimensional Hilbert space $\mathcal{H}$. Then, F is continuous woven with all its duals.*

Proof. Suppose that $G = \{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is an arbitrary dual continuous frame of F, then the family $\{F_\varsigma\}_{\varsigma\in K} \cup \{G_\varsigma\}_{\varsigma\in K^C}$ is a continuous frame sequence, for every $K \subset \mathfrak{A}$. Using Theorem 3.2 we have F and G are continuous woven. □

In the next theorem we show that in infinite dimension Hilbert spaces, continuous frames are continuous woven with their canonical duals.

Theorem 3.3. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a continuous frame for $\mathcal{H}$, so that the norm of its redundant elements be small enough. Then, F is continuous woven with its canonical dual.*

Proof. Without loss of generality, we can write $F = \{F_\varsigma\}_{\varsigma\in K} \cup \{F_\varsigma\}_{\varsigma\in K^C}$ where $K \subset \mathfrak{A}$ and $F = \{F_\varsigma\}_K$ is a Riesz basis for $\mathcal{H}$, then by Theorem 3.2, F and $S_F^{-1}F$ are continuous woven, then F and $S_F F$ are continuous woven with the universal lower bound A_F, and

$$\int_K \|F_\varsigma\|^2 d\mu(\varsigma) < \sqrt{\frac{A_F}{B_F}}.$$

Then, F is continuous woven with its canonical dual.

□

Theorem 3.4. *Let $F = \{F_\varsigma\}_\varsigma$ and $G = \{G_\varsigma\}_\varsigma$ be two continuous frames for $\mathcal{H}$. The followings hold:*

(1) *If $S_F^{-1} \geq I$ and $S_F S_{FK} = S_{FK} S_F$ for all $K \subset \mathfrak{A}$, then $\{F_\varsigma\}_\varsigma$ and $\{S_F^{-1}F_\varsigma\}_\varsigma$ are continuous woven.*

(2) *If F and G are two woven continuous Riesz bases and T_1 and T_2 are invertible operators so that*

$$d_{FK,GK^C} > \max\{\|T_1 - T_2\|\,\|T_1^{-1}\|, \|T_1 - T_2\|\,\|T_2^{-1}\|\}$$

where d_{FK,GK^C} is defined as in Theorem 2.2, then T_1F and T_2G are continuous woven.

Proof. For (1), we consider $F_K = \{F_\varsigma\}_{\varsigma\in K} \cup \{S_F^{-1}F_\varsigma\}_{\varsigma\in K^C}$. Then F_K is a Bessel sequence for all $K \subset \mathfrak{A}$, and

$$\begin{aligned} S_{FK}f &= \int_K \langle f, F_\varsigma\rangle F_\varsigma d\mu(\varsigma) + \int_{K^C} \langle f, S_F^{-1}F_\varsigma\rangle S_F^{-1}F_\varsigma d\mu(\varsigma) \\ &= S_{FK}f + S_F^{-1} S_{FK^C} S_F^{-1} f \\ &= S_F f - S_{FK^C} f + S_F^{-1} S_{FK^C} S_F^{-1} f \\ &= S_F f + (S_F^{-1} - I) S_{FK^C} (I + S_F^{-1}) f, \quad \forall f \in \mathcal{H}. \end{aligned}$$

Since $(S_F^{-1} - I) S_{FK^C} (I + S_F^{-1})$ is a positive operator , we obtain

$$S_{FK} \geq S_F.$$

Then, T_{FK}^* is injective and F_K is a continuous frame for all K. Hence, (1) is obtained.

For (2), let $f = \int_K \alpha_\varsigma T_1 F_\varsigma d\mu(\varsigma)$ and $g = \int_{K^C} \beta_\varsigma T_2 G_\varsigma d\mu(\varsigma)$ with $\|g\| = 1$ then

$$\begin{aligned}
\|f-g\| &= \|\int_K \alpha_\varsigma T_1 F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma T_2 G_\varsigma d\mu(\varsigma)\| \\
&= \|\int_K \alpha_\varsigma T_1 F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma T_1 G_\varsigma d\mu(\varsigma) + \int_{K^C} \beta_\varsigma T_1 G_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma T_2 G_\varsigma d\mu(\varsigma)\| \\
&\geq \|T_1(\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma))\| - \|(T_1 - T_2)\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq \|T_1^{-1}\|^{-1} \|\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \, \| \frac{\int_K \alpha_\varsigma F_{\varsigma d\mu(\varsigma)}}{\|\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\|} - \frac{\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)}{\|\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\|}\| \\
&- \|T_1 - T_2\| \, \|\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C}\|T_1\|^{-1} - \|T_1 - T_2\|) \, \|\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C}\|T_1\|^{-1} - \|T_1 - T_2\|)\|T_2\|^{-1} > 0.
\end{aligned}$$

If $\|f\| = 1$, then we obtain

$$\begin{aligned}
\|f-g\| &= \|\int_K \alpha_\varsigma T_1 F_\varsigma d\mu(\varsigma) - \int_K \alpha_\varsigma T_1 F_\varsigma d\mu(\varsigma) + \int_K \alpha_\varsigma T_2 F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma T_2 G_\varsigma d\mu(\varsigma)\| \\
&\geq \|T_2(\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma))\| - \|(T_1 - T_2)\int_K \alpha_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C}\|T_2^{-1}\|^{-1} - \|T_1 - T_2\|)\|T_1\|^{-1} > 0.
\end{aligned}$$

By considering

$$d_1 = (d_{FK,GK^C}\|T_1^{-1}\|^{-1} - \|T_1 - T_2\|)\|T_2\|^{-1}$$

and

$$d_2 = (d_{FK,GK^C}\|T_2^{-1}\|^{-1} - \|T_1 - T_2\|)\|T_2\|^{-1}$$

we obtain $d_{T_1FK,T_2GK^C} \geq \min\{d_1, d_2\} > 0$. Thus, the result is obtained. □

A consequence of the above theorem is that the canonical duals of two woven continuous frames are continuous woven.

Corollary 3.2. *Let $F = \{F_\varsigma\}_\varsigma$ and $G = \{G_\varsigma\}_\varsigma$ be two Riesz bases for $\mathcal{H}$, so that for every $K \subset \mathfrak{A}$*

$$d_{FK,GK^C} > \max\{\|S_F^{-1} - S_G^{-1}\| \, \|S_F\|, \|S_F^{-1} - S_G^{-1}\| \, \|S_G\|\}.$$

Then, $S_F^{-1}F$ and $S_G^{-1}G$ are also continuous woven.

Proof. By Theorem 3.4, we consider $f = \int_K \alpha_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma)$ and $g = \int_{K^C} \beta_\varsigma S_G^{-1} G_\varsigma d\mu(\varsigma)$ with $\|g\| = 1$ then

$$\begin{aligned}
\|f-g\| &= \| \int_K \alpha_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma S_G^{-1} G_\varsigma d\mu(\varsigma)\| \\
&= \| \int_K \alpha_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma S_F^{-1} G_\varsigma d\mu(\varsigma) + \int_{K^C} \beta_\varsigma S_F^{-1} G_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma S_G^{-1} G_\varsigma d\mu(\varsigma)\| \\
&\geq \|S_F^{-1}(\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma))\| - \|(S_F^{-1} - S_G^{-1}) \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq \|S_F\|^{-1} \| \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \| \frac{\int_K \alpha_\varsigma F_{\varsigma d\mu(\varsigma)}}{\| \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\|} - \frac{\int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)}{\| \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\|}\| \\
&- \|S_F^{-1} - S_G^{-1}\| \| \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C} \|S_F\|^{-1} - \|S_F^{-1} - S_G^{-1}\|) \| \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C} \|S_F\|^{-1} - \|S_F^{-1} - S_G^{-1}\|) \|S_G\|^{-1} > 0.
\end{aligned}$$

If $\|f\| = 1$, then we obtain

$$\begin{aligned}
\|f-g\| &= \| \int_K \alpha_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma) - \int_K \alpha_\varsigma S_F^{-1} F_\varsigma d\mu(\varsigma) + \int_K \alpha_\varsigma S_G^{-1} F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma T_2 G_\varsigma d\mu(\varsigma)\| \\
&\geq \|S_G^{-1}(\int_K \alpha_\varsigma F_\varsigma d\mu(\varsigma) - \int_{K^C} \beta_\varsigma G_\varsigma d\mu(\varsigma))\| - \|(S_F^{-1} - S_G^{-1}) \int_K \alpha_\varsigma G_\varsigma d\mu(\varsigma)\| \\
&\geq (d_{FK,GK^C} \|S_F\|^{-1} - \|S_F^{-1} - S_G^{-1}\|) \|S_F\|^{-1} > 0.
\end{aligned}$$

By considering

$$d_1 = (d_{FK,GK^C} \|S_F^{-1}\|^{-1} - \|S_F - S_G\|) \|S_G\|^{-1}$$
$$d_2 = (d_{FK,GK^C} \|S_G^{-1}\|^{-1} - \|S_F - S_G\|) \|S_G\|^{-1}$$

we obtain $d_{S_F FK, S_G GK^C} \geq \min\{d_1, d_2\} > 0$. Thus, the result is obtained. □

In the following, we obtain some invertible operators T for which F and TF are woven continuous frames and we give some conditions which continuous frames with their perturbations constitute woven continuous frames.

Let $e = \{e_i\}_{i=1}^n$ and $h = \{h_i\}_{i=1}^n$ be orthonormal bases for $\mathcal{H}_n$. Also, let $a = \{a_i\}_{i=1}^n$ and $b = \{b_i\}_{i=1}^n$ be sequences of positive constants.

An operator $T : \mathcal{H}_n \to \mathcal{H}_n$ is called admissible for (e, h, a, b), if there exists an orthonormal basis $\{\lambda_i\}_{i=1}^n$ for $\mathcal{H}_n$ satisfying

$$T^* e_i = \sum_{j=1}^n \sqrt{\frac{a_j}{b_j}} \langle e_i, h_j \rangle h_j, \ (i = 1, 2, ..., n)$$

TF is a continuous frame for $\mathcal{H}_n$ with the frame operator $S_{TF} = TS_F T^*$. By considering $\mathcal{F} = \{F_j\}_{j\in K} \cup \{TF_j\}_{j\in K^c}$, we obtain

$$S_{\mathcal{F}K} = S_{FK} + S_{TFK^C} \ for \ K \subset \{1, 2, ..., n\},$$

where S_{FK} and S_{TFK^C} are the continuous frames operators of $\{F_j\}_j$ and $\{T_{F_j}\}_j$, respectively. Then, $S_{\mathcal{F}_K}$ the continuous frame operator of $\mathcal{F}_K$, is a positive and invertible operator on $\mathcal{H}_n$ and so F and T_F are continuous woven.

Theorem 3.5. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ and $G = \{G_\varsigma\}_{\varsigma\in\mathfrak{A}}$ are two continuous frames so that for all sequences of scalars $\{\alpha_\varsigma\}_{\varsigma\in\mathfrak{A}}$, we have*

$$\|\int_{\mathfrak{A}}\alpha_\varsigma(F_\varsigma - G_\varsigma)d\mu(\varsigma)\| \le a\|\int_{\mathfrak{A}}\alpha_\varsigma F_\varsigma d\mu(\varsigma)\| + b\|\int_{\mathfrak{A}}\alpha_\varsigma G_\varsigma d\mu(\varsigma)\| + c(\int_{\mathfrak{A}}|\alpha_\varsigma|^2 d\mu(\varsigma))^{\frac{1}{2}}$$

for some positive numbers a, b, c such that

$$a\sqrt{B_F} + b\sqrt{B_G} + c < \sqrt{A_F}.$$

Then, F and G are continuous woven.

Proof. Consider T_{FK^C} and T_{GK^C} as the synthesis operators of Bessel sequences $\{F_\varsigma\}_{\varsigma\in K}$ and $\{G_\varsigma\}_{\varsigma\in K^C}$, respectively. Then, for every $K \subset \mathfrak{A}$ and $f \in \mathcal{H}$, we have

$$\begin{aligned}
&(\int_K |\langle f, F_\varsigma\rangle\rangle|^2 d\mu(\varsigma) + \int_{K^C} |\langle f, G_\varsigma\rangle|^2 d\mu(\varsigma))^{1/2}\\
&= (\int_K |\langle f, F_\varsigma\rangle\rangle|^2 d\mu(\varsigma) + \int_{K^C} |\langle f, G_\varsigma\rangle - \langle f, F_\varsigma - G_\varsigma\rangle|^2 d\mu(\varsigma))^{1/2}\\
&\ge (\int_K |\langle f, F_\varsigma\rangle\rangle|^2 d\mu(\varsigma))^{1/2} - \int_{K^C} |\langle f, F_\varsigma - G_\varsigma\rangle|^2 d\mu(\varsigma))^{1/2}\\
&\ge \sqrt{A_F}\|f\| - \|T_{FK^C}f - T_{GK^C}f\|\\
&\ge (\sqrt{A_F} - \|T_{FK^C}f - T_{GK^C})\|f\|\\
&\ge (\sqrt{A_F} - a\|T_F\| - b\|T_G\| - c)\|f\|\\
&\ge (\sqrt{A_F} - a\sqrt{B_F} - b\sqrt{B_G} - c)\|f\|.
\end{aligned}$$

By $(\sqrt{A_F} - a\sqrt{B_F} - b\sqrt{B_G} - c) > 0$, the lower bound is obtained.

Clearly $\{F_j\}_{j\in K} \cup \{G_j\}_{j\in K^C}$ is Bessel with an upper bound $B_F + B_G$. □

Applying Theorem 3.5, we will obtain the following results.

Corollary 3.3. *Let $F = \{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ is a continuous frame for $\mathcal{H}$ and $0 \neq f \in \mathcal{H}$. Also, $\{a_\varsigma\}_{\varsigma\in\mathfrak{A}}$ be a sequence of scalars so that*

$$\int_{\mathfrak{A}} |a_\varsigma|^2 d\mu(\varsigma) < b\frac{A_F}{\|f\|^2},$$

for some $b < 1$. Then, $\{F_\varsigma\}_{\varsigma\in\mathfrak{A}}$ and $\{F_\varsigma + a_\varsigma f\}_{\varsigma\in\mathfrak{A}}$ are continuous woven.

Proof. We have $\{F_\varsigma + a_\varsigma f\}_{\varsigma\in\mathfrak{A}}$ is a Bessel sequence with the upper bound

$(\sqrt{B_F} + \|\{a_\varsigma\}_{\varsigma\in\mathfrak{A}}\| \, \|f\|)^2$. And for any sequence $\{\alpha_\varsigma\}_{\varsigma\in\mathfrak{A}}$ of scalars

$$\begin{aligned}
\| \int_{\mathfrak{A}} \alpha_\varsigma (F_\varsigma + a_\varsigma f - F_\varsigma) d\mu(\varsigma)\| &= \| \int_{\mathfrak{A}} \alpha_\varsigma a_\varsigma f d\mu(\varsigma)\| \\
&\leq \int_{\mathfrak{A}} |\alpha_\varsigma| \, |a_\varsigma| \, \|f\| d\mu(\varsigma) \\
&\leq (\int_{\mathfrak{A}} |\alpha_\varsigma|^2 d\mu(\varsigma))^{1/2} (\int_{\mathfrak{A}} |a_\varsigma|^2 d\mu(\varsigma))^{1/2} \|f\| \\
&< \sqrt{bA_F} (\int_{\mathfrak{A}} |\alpha_\varsigma|^2 d\mu(\varsigma))^{1/2}.
\end{aligned}$$

The result follows by Theorem 3.5. □

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