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Fixed Point Results for Multivalued Graphic Contractions in $\mathcal{F}$-Metric Spaces

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Abstract. The present paper is devoted to the introduction and development of the notions of multivalued graphic contractions and multivalued *GF*-contractions in the setting of $\mathcal{F}$-metric spaces. By extending the idea of contractions to multivalued mappings associated with an underlying graph structure, we aim to enrich the existing theory of fixed point results in generalized metric frameworks. The main contribution of this study is the establishment of new fixed point theorems for these classes of mappings in $\mathcal{F}$-metric spaces, which provide a natural extension of classical fixed point principles. Furthermore, in order to demonstrate the validity and applicability of our theoretical findings, we construct a non-trivial illustrative example that highlights how the proposed conditions can be effectively utilized. These results not only advance the fixed point theory in abstract metric settings but also open potential avenues for applications in mathematical analysis and applied sciences.

1. Introduction

In the theory of fixed points, Banach contraction principle [1] is pioneer theorem which has plenty of extensions in different directions (see. [2-4]). Jachymski [5] generalized this theorem for single valued mappings in the situation of complete metric spaces (CMSs) equipped with the graph. Wardowski et al. [6] gave a fashionable assortment of contraction by employing a precise function is said to be *F*-contraction and furnished some examples to show the boldness of such generalizations. Wardowski et al. [6] proved a fixed point result by utilizing the concept of *F*-contraction and generalized conventional Banach contraction principle (BCP). Vetro [7] proved fixed point results for Hardy-Rogers type *F*-contractions and applied their results to multistage decision processes.

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In other direction, the supreme part in fixed point theory is the underlying space. The idea of MS was laid by the French mathematician Maurice Fréchet [8] in 1905, laying the foundation for subsequent advancements in the field. Over the past several decades, numerous researchers have explored generalizations of MSs by modifying the traditional triangle inequality. Notable examples include b-MSs by Czerwik [9], rectangular MSs by Branciari [10], and θ-MSs by Khojasteh et al. [11]. Building upon these developments, Jleli et al. [12] proposed a comprehensive extension known as $\mathcal{F}$-MSs. Al-Mezel [13] established some fixed point theorems for generalized ($\alpha\beta$-ψ)-contractions in the background of $\mathcal{F}$-MSs. Later on, Hussain et al. [14] enlarged the concept of $\mathcal{F}$-MSs by establishing some new fixed point results and solving a differential equation as application. For further exploration of this topic, we recommend consulting the bibliography, specifically references [15-22].

This paper introduces the novel concept of multivalued graphic contractions and multivalued GF-contractions within the foundation of $\mathcal{F}$-MSs and establishes corresponding fixed point theorems. To illustrate the applicability of our results, we offer an illustrative example.

2. Preliminaries

The Banach Contraction Principle asserts that any self-mapping $\mathfrak{J}$ defined on a CMS $(\mathcal{O}, d)$ satisfying the condition

$$d(\mathfrak{J}\mathfrak{w}, \mathfrak{J}\mathfrak{h}) \leq \lambda d(\mathfrak{w}, \mathfrak{h})$$

for all $\mathfrak{w}, \mathfrak{h} \in \mathcal{O}$, where $\lambda \in [0,1)$ has a unique fixed point.

We will present graph-theoretic concepts based on the work of Jachymski [5]. Let $(\mathcal{O}, d)$ be a MS and let Δ denotes the diagonal of $\mathcal{O} \times \mathcal{O}$. Consider a directed graph G composed of a vertex set $V(G)$ identical to $\mathcal{O}$ and an edge set $E(G)$ encompassing all loops $(\Delta \subseteq E(G))$. Importantly, G contains no multiple edges.

Jachymski [5] introduced the following definition of G-contraction:

Definition 2.1. *([5]) Let $(\mathcal{O}, d)$ be a MS and $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$. A mapping $\mathfrak{J}$ is termed a Banach graphic contraction if*

(a) $\forall\ \mathfrak{w}, \mathfrak{h} \in \mathcal{O}$ with $(\mathfrak{w}, \mathfrak{h}) \in E(G)$, we have $(\mathfrak{J}(\mathfrak{w}), \mathfrak{J}(\mathfrak{h})) \in E(G)$,

(b) there exists $\lambda \in (0,1)$ such that, $\forall\ \mathfrak{w}, \mathfrak{h} \in \mathcal{O}$ with $(\mathfrak{w}, \mathfrak{h}) \in E(G)$, we have

$$d(\mathfrak{J}(\mathfrak{w}), \mathfrak{J}(\mathfrak{h})) \leq \lambda d(\mathfrak{w}, \mathfrak{h}). \tag{2.1}$$

G^{-1} is the converse graph of G that is the edge set of G^{-1} is established by reversing the direction of edges of G, that is

$$E\left(G^{-1}\right) = \{(\mathfrak{w}, \mathfrak{h}) \in \mathcal{O} \times \mathcal{O} : (\mathfrak{h}, \mathfrak{w}) \in E\,(G)\}.$$

Given two vertices, $\mathfrak{w}$ and $\mathfrak{h}$ in a graph G, a path connecting $\mathfrak{w}$ to $\mathfrak{h}$ of length N (a natural number) is a sequence $\{\mathfrak{w}_i\}_{i=0}^{N}$ of $N+1$ vertices such that $\mathfrak{w}_0 = \mathfrak{w}$, $\mathfrak{w}_N = \mathfrak{h}$ and $(\mathfrak{w}_{i-1}, \mathfrak{w}_i) \in E(G), \forall\ i = 1, \cdots, N$. A graph G is connected if any two distinct vertices within the graph can be joined by a path. If

$\widetilde{G} = \big(\mathcal{O}, E(\widetilde{G})\big)$ represents symmetric graph established by putting the vertices of both G and G^{-1} that is

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

then the graph G is said to be weakly connected if $\widetilde{G}$ is connected.

Let $\Omega = \{G : G \text{ is a directed graph with } V(G) = \mathcal{O} \text{ and } \Delta \subseteq E(G)\}$. If $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$, then we represent set of all fixed points of $\mathfrak{J}$ by $\mathcal{O}_{\mathfrak{J}}$ and let $\mathcal{O}_{\mathfrak{J}} := \{\mathfrak{w} \in \mathcal{O} : (\mathfrak{w}, \mathfrak{J}(\mathfrak{w})) \in E(G)\}$.

In 2008, Jachymski [5] gave the following property which is also required in the proof of our result.

(P) for $\{\mathfrak{w}_n\} \subseteq \mathcal{O}$, if $\mathfrak{w}_n \to \mathfrak{w}$ as $n \to \infty$ and $(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \in E(G)$, then there exists a subsequence $\{\mathfrak{w}_{n_k}\}$ such that $(\mathfrak{w}_{n_k}, \mathfrak{w}) \in E(G)$, $\forall\, n \in \mathbb{N}$.

Definition 2.2. *([15]) Consider the MS $(\mathcal{O}, d)$ and a mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$.The mapping $\mathfrak{J}$ is termed a Picard operator if it possesses a unique fixed point denoted by $\mathfrak{w}^*$ and $\mathfrak{J}^n\mathfrak{w} \to \mathfrak{w}^*$, as $n \to \infty$, for all $\mathfrak{w} \in \mathcal{O}$.*

Definition 2.3. *([5]) Given a MS $(\mathcal{O}, d)$ and a mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$. Then $\mathfrak{J}$ is classified as a weakly Picard operator if for any $\mathfrak{w} \in \mathcal{O}$, $\lim_{n\to\infty} \mathfrak{J}^n\mathfrak{w}$ exists and its limit is a fixed point of $\mathfrak{J}$.*

Wardowski [6] introduced the following new notion of F-contraction in 2012.

Definition 2.4. *Let $(\mathcal{O}, d)$ be a metric space and $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$. Then $\mathfrak{J}$ is said to be a F-contraction if there exists $\lambda > 0$ such that for $\mathfrak{w}, \mathfrak{h} \in \mathcal{O}$;*

$$d(\mathfrak{J}\mathfrak{w}, \mathfrak{J}\mathfrak{h}) > 0 \Longrightarrow \lambda + F\Big(d(\mathfrak{J}\mathfrak{w}, \mathfrak{J}\mathfrak{h})\Big) \leq F\Big(d(\mathfrak{w}, \mathfrak{h})\Big)$$

where, $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying

($\mathcal{F}_1$) $0 < \mathfrak{w}_1 < \mathfrak{w}_2 \Rightarrow \xi(\mathfrak{w}_1) \leq \xi(\mathfrak{w}_2)$.
($\mathcal{F}_2$) $\forall \{\mathfrak{w}_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \mathfrak{w}_n = 0 \iff \lim_{n\to\infty} F(\mathfrak{w}_n) = -\infty$.
($\mathcal{F}_3$) $\exists\, 0 < r < 1$ such that $\lim_{\mathfrak{w}\to 0^+} \mathfrak{w}^r F(\mathfrak{w}) = 0$.

We represnts F, the set of the functions $\mathfrak{J} : \mathbb{R}^+ \to \mathbb{R}$ satisfying ($\mathcal{F}_1$)-($\mathcal{F}_3$).

Jleli et al. [12] initiated a novel extension of MSs, termed $\mathcal{F}$-MSs, by considering a specific set of functions ξ mapping the positive real numbers to the real numbers satisfying only ($\mathcal{F}_1$) and ($\mathcal{F}_2$).

Definition 2.5. *([12])Let $\mathcal{O}$ be nonempty set, and let $d : \mathcal{O} \times \mathcal{O} \to [0, +\infty)$. Let (ξ, α) be an element of the Cartesian product $\mathcal{F}$ and the non-negative real numbers such that*

(D$_1$) $(\mathfrak{w}, \mathfrak{h}) \in \mathcal{O} \times \mathcal{O}$, $d(\mathfrak{w}, \mathfrak{h}) = 0$ iff $\mathfrak{w} = \mathfrak{h}$,
(D$_2$) $d(\mathfrak{w}, \mathfrak{h}) = d(\mathfrak{h}, \mathfrak{w})$, for all $\mathfrak{w}, \mathfrak{h} \in \mathcal{O}$.
(D$_3$) for all $(\mathfrak{w}, \mathfrak{h}) \in \mathcal{O} \times \mathcal{O}$, and $(\mathfrak{w}_i)_{i=1}^N \subset \mathcal{O}$, with $(\mathfrak{w}_1, \mathfrak{w}_N) = (\mathfrak{w}, \mathfrak{h})$, we have

$$d(\mathfrak{w}, \mathfrak{h}) > 0 \Rightarrow \xi(d(\mathfrak{w}, \mathfrak{h})) \leq \xi(\sum_{i=1}^{N-1} d(\mathfrak{w}_i, \mathfrak{w}_{i+1})) + \alpha.$$

$\forall N \geq 2$. Consequently, the pair $(\mathcal{O}, d)$ is classified as an $\mathcal{F}$-MS.

Example 2.1. *([12]) Let O be the set of natural numbers. Then $d : O \times O \to [0, +\infty)$ defined by*

$$d(\mathfrak{w}, \mathfrak{h}) = \begin{cases} (\mathfrak{w} - \mathfrak{h})^2 & \text{if } (\mathfrak{w}, \mathfrak{h}) \in [0,3] \times [0,3] \\ |\mathfrak{w} - \mathfrak{h}| & \text{if } (\mathfrak{w}, \mathfrak{h}) \notin [0,3] \times [0,3] \end{cases}$$

with $\xi(t) = \ln(t)$ and $\alpha = \ln(3)$ is an $\mathcal{F}$-metric.

Definition 2.6. *([12]) Let (O, d) be a $\mathcal{F}$-MS.*

(i) A sequence $\{\mathfrak{w}_n\}$ in O is termed $\mathcal{F}$-convergent to $\mathfrak{w} \in O$ if $\{\mathfrak{w}_n\}$ is convergent to $\mathfrak{w}$ if it converges to $\mathfrak{w}$ under the $\mathcal{F}$-metric d.

(ii) A sequence $\{\mathfrak{w}_n\}$ is considered $\mathcal{F}$-Cauchy, if

$$\lim_{n,m \to \infty} d(\mathfrak{w}_n, \mathfrak{w}_m) = 0.$$

(iii) (O, d) is said to be $\mathcal{F}$-complete, if each $\mathcal{F}$-Cauchy sequence in O is $\mathcal{F}$-convergent to a certain point in O.

Lemma 2.1. *([2]) If $A, B \in CB(O)$ and $a \in A$ then for each positive number γ there exists a number $b \in B$ such that*

$$d(a, b) \leq H_{\mathcal{F}}(A, B) + \gamma.$$

Lemma 2.2. *([2]) Let $\{A_n\}$ be a sequence of sets in $CB(O)$ and $\lim_{n\to\infty} H_{\mathcal{F}}(A_n, A) = 0$ for $A \in CB(O)$. Moreover, if $\mathfrak{w}_n \in A_n$ and*

$$\lim_{n \to \infty} d(\mathfrak{w}_n, \mathfrak{w}) = 0,$$

then $\mathfrak{w} \in A$.

3. Main Results

In the whole section, we suppose that O is a $\mathcal{F}$-MS with a $\mathcal{F}$-metric d and $G = \{G : G$ is a directed graph with $V(G) = O$ and $\Delta \subseteq E(G)\}$. We proceed to define the concept of a multivalued G-contraction as follows.

Definition 3.1. *Let (O, d) be a $\mathcal{F}$-MS equipped with a graph G. A mapping $\mathfrak{J} : O \to CB(O)$ is termed a multivalued graphic contraction (multivalued G-contraction) if there exists some $\lambda \in (0, 1)$ such that*

$$H_{\mathcal{F}}(\mathfrak{J}\mathfrak{w}, \mathfrak{J}\mathfrak{h}) \leq \lambda d(\mathfrak{w}, \mathfrak{h}), \text{ for all } \mathfrak{w}, \mathfrak{h} \in O \text{ with } (\mathfrak{w}, \mathfrak{h}) \in E(G), \tag{3.1}$$

and if $u \in \mathfrak{J}\mathfrak{w}$ and $v \in \mathfrak{J}\mathfrak{h}$ are such that

$$d(u, v) \leq \lambda d(\mathfrak{w}, \mathfrak{h}) + \gamma, \text{ for each } \gamma > 0, \tag{3.2}$$

then $(u, v) \in E(G)$.

Proposition 3.1. *Let (O, d) be a $\mathcal{F}$-MS and $\mathfrak{J} : O \to CB(O)$ be a multivalued G contraction. Then $\mathfrak{J} : O \to CB(O)$ is multivalued graphic contraction for both G^{-1} and $\widetilde{G}$, it means (3.1) and (3.2) holds for G^{-1} and $\widetilde{G}$.*

Proof. As $\mathcal{F}$-metric is symmetric, so $\mathfrak{J} : O \to CB(O)$ is also multivalued graphic contraction for both G^{-1} and $\widetilde{G}$. □

Theorem 3.1. *Let (O, d) be a $\mathcal{F}$-complete $\mathcal{F}$-MS and suppose that the triple (O, d, G) has the Property P. Suppose that the mapping $\mathfrak{J} : O \to CB(O)$ is multivalued G-contraction and the set*

$$O_{\mathfrak{J}} = \{\mathfrak{w} \in O \colon (\mathfrak{w}, u) \in E(G), \text{ for some } u \in \mathfrak{J}(\mathfrak{w})\}$$

is nonempty. Then the following statements hold:

(i) for any $\mathfrak{w} \in O_{\mathfrak{J}}$, $\mathfrak{J}|_{[\mathfrak{w}]_{\widetilde{G}}}$ has a fixed point,
(ii) if G is weakly connected and $O_{\mathfrak{J}} \neq \emptyset$, then $\mathfrak{J}$ has a fixed point in O,
(iii) if $O' = \cup\left\{[\mathfrak{w}]_{\widetilde{G}} : \mathfrak{w} \in O_{\mathfrak{J}}\right\}$, then the restriction $\mathfrak{J}|_{\mathfrak{w}'}$ has a fixed point,
(iv) if $\mathfrak{J} \subseteq E(G)$, consequently, $\mathfrak{J}$ has a fixed point.
(v) $Fix\mathfrak{J} \neq \emptyset \iff O_{\mathfrak{J}} \neq \emptyset$.

Proof. (i). Let $\mathfrak{w}_0 \in O_{\mathfrak{J}}$, then there exists a point $\mathfrak{w}_1 \in \mathfrak{J}(\mathfrak{w}_0)$ such that $(\mathfrak{w}_0, \mathfrak{w}_1) \in E(G)$. Since the mapping $\mathfrak{J} : O \to CB(O)$ is multivalued G-contraction, so we have

$$H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_0), \mathfrak{J}(\mathfrak{w}_1)) \leq \lambda d(\mathfrak{w}_0, \mathfrak{w}_1).$$

By Lemma (2.1), there exists a point $\mathfrak{w}_2 \in \mathfrak{J}(\mathfrak{w}_1)$ such that

$$d(\mathfrak{w}_1, \mathfrak{w}_2) \leq H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_0), \mathfrak{J}(\mathfrak{w}_1)) + \lambda \leq \lambda d(\mathfrak{w}_0, \mathfrak{w}_1) + \lambda. \tag{3.3}$$

Given that the mapping $\mathfrak{J}$ is a multivalued G-contraction, so $(\mathfrak{w}_1, \mathfrak{w}_2) \in E(G)$. Hence by (3.1), we have

$$H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_1), \mathfrak{J}(\mathfrak{w}_2)) \leq \lambda d(\mathfrak{w}_1, \mathfrak{w}_2).$$

Again by Lemma (2.1), there exists a point $\mathfrak{w}_3 \in \mathfrak{J}(\mathfrak{w}_2)$ such that

$$d(\mathfrak{w}_2, \mathfrak{w}_3) \leq H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_1), \mathfrak{J}(\mathfrak{w}_2)) + \lambda^2. \tag{3.4}$$

Using inequality (3.3) in (3.4), we have

$$d(\mathfrak{w}_2, \mathfrak{w}_3) \leq \lambda^2 d(\mathfrak{w}_0, \mathfrak{w}_1) + 2\lambda^2. \tag{3.5}$$

Proceeding in this manner, we get $\mathfrak{w}_{n+1} \in \mathfrak{J}(\mathfrak{w}_n)$ such that $(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \in E(G)$ and

$$d(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \leq \lambda^n d(\mathfrak{w}_0, \mathfrak{w}_1) + n\lambda^n \tag{3.6}$$

which yields

$$\sum_{i=1}^{m-1} d(\mathfrak{w}_i, \mathfrak{w}_{i+1}) \leq \frac{\lambda^n}{1-\lambda} d(\mathfrak{w}_0, \mathfrak{w}_1) + \frac{n\lambda^n}{1-\lambda}$$

for $m > n$. Since

$$\lim_{n\to\infty} \left(\frac{\lambda^n}{1-\lambda} d(\mathfrak{w}_0, \mathfrak{w}_1) + \frac{n\lambda^n}{1-\lambda} \right) = 0$$

there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} d(\mathfrak{w}_0, \mathfrak{w}_1) + \frac{n\lambda^n}{1-\lambda} < \delta$$

for $n \geq N$. Now let $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied and $\epsilon > 0$ be fixed. From $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } \xi(t) < \xi(\epsilon) - \alpha. \tag{3.7}$$

Now since

$$0 < \frac{\lambda^n}{1-\lambda} d(\mathfrak{w}_0, \mathfrak{w}_1) + \frac{n\lambda^n}{1-\lambda} < \delta,$$

for $n \geq N$. Hence, by (3.7) and $(\mathcal{F}_1)$, we get

$$\xi\left(\sum_{i=1}^{m-1} d(\mathfrak{w}_i, \mathfrak{w}_{i+1})\right) \leq \xi\left(\frac{\lambda^n}{1-\lambda} d(\mathfrak{w}_0, \mathfrak{w}_1) + \frac{n\lambda^n}{1-\lambda}\right) < \xi(\epsilon) - a \tag{3.8}$$

for $m > n \geq N$. Using (D_3) and (3.8), we get

$$d(\mathfrak{w}_n, \mathfrak{w}_m) > 0, \text{ for } m > n \geq N$$

implies

$$\xi\left(d(\mathfrak{w}_n, \mathfrak{w}_m)\right) \leq \xi\left(\sum_{i=1}^{m-1} d(\mathfrak{w}_i, \mathfrak{w}_{i+1})\right) + a < \xi(\epsilon)$$

which, from $(\mathcal{F}_1)$, gives that

$$d(\mathfrak{w}_n, \mathfrak{w}_m) < \epsilon$$

for $m > n \geq N$. This proves that $\{\mathfrak{w}_n\}$ is Cauchy Sequence. Since $(\mathcal{O}, d)$ is an $\mathcal{F}$-complete $\mathcal{F}$-MS, so $\{\mathfrak{w}_n\}$ converges to some point $\mathfrak{w}^*$ in $\mathcal{O}$. We aim to establish that $\mathfrak{w}^*$ is a fixed point under the action of $\mathfrak{J}$. Given Property P and the multivalued G-contractiveness of $\mathfrak{J}$, it follows that

$$H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_n), \mathfrak{J}(\mathfrak{w}^*)) \leq \lambda d(\mathfrak{w}_n, \mathfrak{w}^*).$$

Since $\mathfrak{w}_{n+1} \in \mathfrak{J}(\mathfrak{w}_n)$ and $\mathfrak{w}_n \to \mathfrak{w}^*$, then by Lemma 2.2, we get that $\mathfrak{w}^* \in \mathfrak{J}(\mathfrak{w}^*)$. Next, as $(\mathfrak{w}_n, \mathfrak{w}^*) \in E(G)$, for $n \in \mathbb{N}$, we conclude that $(\mathfrak{w}_0, \mathfrak{w}_1, \mathfrak{w}_2, ..., \mathfrak{w}_n, \mathfrak{w}^*)$ is a path in G and so $\mathfrak{w}^* \in [\mathfrak{w}_0]_{\widetilde{G}}$. □

(ii) Since $\mathcal{O}_{\mathfrak{J}} \neq \emptyset$, there exists a point $\mathfrak{w}_0 \in \mathcal{O}_{\mathfrak{J}}$ and since G is weakly connected, then $[\mathfrak{w}_0]_{\widetilde{G}} = \mathcal{O}$ and by the conclusion of (i), mapping $\mathfrak{J}$ has a fixed point.

(iii) Condition (iii) is direct consequence of (i) and (ii).

(iv) let $\mathfrak{J} \subseteq E(G)$, implies that all $\mathfrak{w} \in \mathcal{O}$ are such that there exists some $u \in \mathfrak{J}(\mathfrak{w})$ with $(\mathfrak{w}, u) \in E(G)$, so $\mathcal{O}_{\mathfrak{J}} = \mathcal{O}$, and consequently by (ii) and (iii), $\mathfrak{J}$ has a fixed point.

(v) let $Fix\mathfrak{J} \neq \emptyset$, this implies that there exists a point $\mathfrak{w} \in Fix\mathfrak{J}$ such that $\mathfrak{w} \in \mathfrak{J}(\mathfrak{w})$. Then $\Delta \subseteq E(G)$, therefore $(\mathfrak{w}, \mathfrak{w}) \in E(G)$ which

implies that $\mathfrak{w} \in \mathcal{O}_{\mathfrak{J}}$. So $\mathcal{O}_{\mathfrak{J}} \neq \emptyset$. Conversely if $\mathcal{O}_{\mathfrak{J}} \neq \emptyset$, then it follows from (ii) and (iii) that $Fix\mathfrak{J} \neq \emptyset$.

A straightforward consequence of Theorem 3.1 is the following result.

Corollary 3.1. *Let $(\mathcal{O}, d)$ be a $\mathcal{F}$-complete $\mathcal{F}$-MS and suppose that the triple $(\mathcal{O}, d, G)$ has the Property P. If G is weakly connected then every multivalued G-contraction $\mathfrak{J} : \mathcal{O} \to CB(\mathcal{O})$ such that $(\mathfrak{w}_0, \mathfrak{w}_1) \in E(G)$ for some $\mathfrak{w}_1 \in \mathfrak{J}(\mathfrak{w}_0)$ has a fixed point.*

Remark 3.1. *Assuming G satisfies $E(G) = \mathcal{O} \times \mathcal{O}$, G is necessarily connected. This, combined with Theorem 3.1, yields Nadler's theorem in the setting of $\mathcal{F}$-MSs.*

Remark 3.2. *If G satisfies $E(G) = \mathcal{O} \times \mathcal{O}$, G is necessarily connected and furthermore, if $\mathfrak{J}$ is a single-valued function, then we obtain the main result of Jleli et al. [12] which is Banach contraction theorem in the framework of $\mathcal{F}$-MSs.*

4. Fixed Point Results for GF-Contractions

Let G be a directed graph on an $\mathcal{F}$-MS $(\mathcal{O}, d)$ and $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$. Define $\mathfrak{J}_G = \{(\mathfrak{w}, \mathfrak{h}) \in E(G) : H_{\mathcal{F}}(\mathfrak{J}\mathfrak{w}, \mathfrak{J}\mathfrak{h}) > 0\}$.

Definition 4.1. *([25]) Let $(\mathcal{O}, d)$ be a $\mathcal{F}$-MS equipped with a graph G and $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$ be a multivalued mapping. $\mathfrak{J}$ has the weakly graph-preserving property, whenever for each $a \in \mathcal{O}$ and $b \in \mathfrak{J}(a)$ with $(a, b) \in E(G)$ implies $(b, c) \in E(G)$ for all $c \in \mathfrak{J}(b)$.*

Definition 4.2. *Let $(\mathcal{O}, d)$ be a $\mathcal{F}$-MS equipped with a graph G. A mapping $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$ is termed a multivalued GF- contraction if there exists a function $F \in F$ and a constant $\lambda > 0$ such that*

$$\lambda + F\left(H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}), \mathfrak{J}(\mathfrak{h}))\right) \leq F\left(d(\mathfrak{w}, \mathfrak{h})\right), \tag{4.1}$$

for all $\mathfrak{w}, \mathfrak{h} \in \mathcal{O}$ with $(\mathfrak{w}, \mathfrak{h}) \in \mathfrak{J}_G$.

Theorem 4.1. *() Let $(\mathcal{O}, d)$ be a $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a directed graph G and $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$ is a multivalued GF- contraction and satisfy weakly graph-preserving property . If the set*

$$\mathcal{O}_{\mathfrak{J}} = \{\mathfrak{w} \in \mathcal{O} \colon (\mathfrak{w}, \mathfrak{h}) \in E(G), \text{ for some } \mathfrak{h} \in \mathfrak{J}(\mathfrak{w})\}$$

is nonempty. If $\mathfrak{J}$ is continuous or if $\{\mathfrak{w}_n\}$ is a sequence in $\mathcal{O}$ converging to $\mathfrak{w} \in \mathcal{O}$ as $n \to \infty$ and $(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \in E(G)$ for all n, there is a subsequence $\{\mathfrak{w}_{n_k}\}$ of $\{\mathfrak{w}_n\}$ in $\mathcal{O}$ such that $(\mathfrak{w}_{n_k}, \mathfrak{w}) \in E(G)$ for all k, then $\mathfrak{J}$ has a fixed point.

Proof. Let $\mathfrak{w}_0 \in \mathcal{O}_{\mathfrak{J}}$, then there exists a point $\mathfrak{w}_1 \in \mathfrak{J}(\mathfrak{w}_0)$ such that $(\mathfrak{w}_0, \mathfrak{w}_1) \in E(G)$. Now if $\mathfrak{w}_1 \in \mathfrak{J}(\mathfrak{w}_1)$, then $\mathfrak{w}_1$ is fixed point and we have nothing to prove. So we assume that $\mathfrak{w}_1 \notin \mathfrak{J}(\mathfrak{w}_1)$, so $d(\mathfrak{w}_1, \mathfrak{J}(\mathfrak{w}_1)) > 0$. Now, since

$$0 < d(\mathfrak{w}_1, \mathfrak{J}(\mathfrak{w}_1)) \leq H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_0), \mathfrak{J}(\mathfrak{w}_1)). \tag{4.2}$$

Thus $(\mathfrak{w}_0, \mathfrak{w}_1) \in \mathfrak{J}_G$. Now by $(\mathcal{F}_1)$, (4.1) and (4.2), we have

$$F\left(d(\mathfrak{w}_1, \mathfrak{J}(\mathfrak{w}_1))\right) \leq F\left(H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_0), \mathfrak{J}(\mathfrak{w}_1))\right) \leq F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - \lambda. \tag{4.3}$$

Due to the compactness of $\mathfrak{J}(\mathfrak{w}_1)$, there exists $\mathfrak{w}_2 \in \mathfrak{J}(\mathfrak{w}_1)$ such that

$$d(\mathfrak{w}_1, \mathfrak{w}_2) = d(\mathfrak{w}_1, \mathfrak{J}(\mathfrak{w}_1)). \tag{4.4}$$

Hence by (4.3) and (4.4), we have

$$F\left(d(\mathfrak{w}_1, \mathfrak{w}_2)\right) \leq F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - \lambda. \tag{4.5}$$

Now since $(\mathfrak{w}_0, \mathfrak{w}_1) \in E(G)$, $\mathfrak{w}_1 \in \mathfrak{J}(\mathfrak{w}_0)$ and $\mathfrak{w}_2 \in \mathfrak{J}(\mathfrak{w}_1)$ by the weakly graph-preserving property, one writes

$$(\mathfrak{w}_1, \mathfrak{w}_2) \in E(G).$$

Now if $\mathfrak{w}_2 \in \mathfrak{J}(\mathfrak{w}_2)$, then $\mathfrak{w}_2$ is fixed point of $\mathfrak{J}$. So we assume that $\mathfrak{w}_2 \notin \mathfrak{J}(\mathfrak{w}_2)$, then

$$0 < d(\mathfrak{w}_2, \mathfrak{J}(\mathfrak{w}_2)) \leq H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_1), \mathfrak{J}(\mathfrak{w}_2)). \tag{4.6}$$

Thus $(\mathfrak{w}_1, \mathfrak{w}_2) \in \mathfrak{J}_G$. Now by $(\mathcal{F}_1)$, (4.1) and (4.6), we have

$$F\left(d(\mathfrak{w}_2, \mathfrak{J}(\mathfrak{w}_2))\right) \leq F\left(H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}_1), \mathfrak{J}(\mathfrak{w}_2))\right) \leq F\left(d(\mathfrak{w}_1, \mathfrak{w}_2)\right) - \lambda. \tag{4.7}$$

Again, the compactness of $\mathfrak{J}(\mathfrak{w}_2)$ implies that there exists a point $\mathfrak{w}_3 \in \mathfrak{J}(\mathfrak{w}_2)$ such that

$$d(\mathfrak{w}_2, \mathfrak{w}_3) = d(\mathfrak{w}_2, \mathfrak{J}(\mathfrak{w}_2)), \tag{4.8}$$

Hence by (4.7) and (4.8), we have

$$\begin{aligned} F\left(d(\mathfrak{w}_2, \mathfrak{w}_3)\right) &\leq F\left(d(\mathfrak{w}_1, \mathfrak{w}_2)\right) - \lambda \\ &\leq F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - 2\lambda. \end{aligned}$$

In this way, we can construct a sequence $\{\mathfrak{w}_n\}$ in $\mathcal{O}$ such that $\mathfrak{w}_{n+1} \in \mathfrak{J}(\mathfrak{w}_n)$, $(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \in E(G)$ and

$$\begin{aligned} F\left(d(\mathfrak{w}_n, \mathfrak{w}_{n+1})\right) &\leq F\left(d(\mathfrak{w}_{n-1}, \mathfrak{w}_n)\right) - \lambda \\ &\leq F\left(d(\mathfrak{w}_{n-2}, \mathfrak{w}_{n-1})\right) - 2\lambda \\ &\leq \ldots \leq F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - n\lambda. \end{aligned} \tag{4.9}$$

Taking $n \to \infty$ in above inequality (4.9), we have

$$\lim_{n\to\infty} F\left(d(\mathfrak{w}_n, \mathfrak{w}_{n+1})\right) \leq \lim_{n\to\infty} F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - n\lambda = -\infty. \tag{4.10}$$

Hence by $(\mathcal{F}_2)$, we get

$$\lim_{n\to\infty} d(\mathfrak{w}_n, \mathfrak{w}_{n+1}) = 0.$$

From the condition $(\mathcal{F}_3)$, there exists $0 < r < 1$ such that

$$\lim_{n\to\infty} [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r F\left(d(\mathfrak{w}_n, \mathfrak{w}_{n+1})\right) = 0. \tag{4.11}$$

From (4.9) and (4.11), we have

$$\begin{aligned} & [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r F\left(d(\mathfrak{w}_n, \mathfrak{w}_{n+1})\right) - [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) \\ \leq\ & [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r [F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) - n\lambda] - [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r F\left(d(\mathfrak{w}_0, \mathfrak{w}_1)\right) \\ \leq\ & -n\lambda [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r \leq 0. \end{aligned}$$

Taking $n \to \infty$, we have

$$\lim_{n\to\infty} n\lambda [d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r = 0.$$

So there exists n_1 (a positive integer) such that

$$n[d(\mathfrak{w}_n, \mathfrak{w}_{n+1})]^r < 1$$

for all $n \geq n_1$, or

$$d(\mathfrak{w}_n, \mathfrak{w}_{n+1}) < \frac{1}{n^{\frac{1}{r}}} \tag{4.12}$$

$$\sum_{i=1}^{n_1} d(\mathfrak{w}_i, \mathfrak{w}_{i+1}) \leq \sum_{i=1}^{n_1} \frac{1}{i^{\frac{1}{r}}} \tag{4.13}$$

ffor all $n \geq n_1$. Now let $(\xi, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied and $\epsilon > 0$ be fixed. From $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } \xi(t) < \xi(\epsilon) - \alpha. \tag{4.14}$$

Now since

$$0 < \sum_{i=1}^{n_1} \frac{1}{i^{\frac{1}{r}}} < \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}} < \delta, \tag{4.15}$$

for $n > n_1$. Hence, by $(\mathcal{F}_1)$, (4.13), (4.14) and (4.15), we get

$$\xi\left(\sum_{i=1}^{m} d(\mathfrak{w}_i, \mathfrak{w}_{i+1})\right) \leq \xi\left(\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{r}}}\right) < \xi\left(\epsilon\right) - a \tag{4.16}$$

for $m > n \geq n_1$. Using (D_3) and (3.8), we get

$$d(\mathfrak{w}_n, \mathfrak{w}_m) > 0,\ m > n > n_1 \Longrightarrow \xi\left(d(\mathfrak{w}_n, \mathfrak{w}_m)\right) \leq \xi\left(\sum_{i=1}^{m} d(\mathfrak{w}_i, \mathfrak{w}_{i+1})\right) + a < \xi\left(\epsilon\right)$$

which, from $(\mathcal{F}_1)$, gives that

$$d(\mathfrak{w}_n, \mathfrak{w}_m) < \epsilon$$

Thus $d(\mathfrak{w}_n, \mathfrak{w}_m) \to 0$ as $n \to \infty$ and hence $\{\mathfrak{w}_n\}$ is a Cauchy sequence. □

Case 1. If $\mathfrak{I}$ is continuous.

Since $\mathfrak{w}_{n+1} \in \mathfrak{I}(\mathfrak{w}_n)$, so we have

$$\lim_{n\to\infty} d(\mathfrak{w}_{n+1}, \mathfrak{I}(\mathfrak{w})) \leq \lim_{n\to\infty} H_{\mathcal{F}}(\mathfrak{I}(\mathfrak{w}_n), \mathfrak{I}(\mathfrak{w})) = 0.$$

This shows that $\mathfrak{w} \in \mathfrak{I}(\mathfrak{w})$, that is, $\mathfrak{w}$ is a fixed point of $\mathfrak{I}$ in $\mathcal{O}$.

Case 2. If $\{\mathfrak{w}_n\}$ is a sequence in $\mathcal{O}$ converging to $\mathfrak{w} \in \mathcal{O}$ as $n \to \infty$ and $(\mathfrak{w}_n, \mathfrak{w}_{n+1}) \in E(G)$ for all n, there is a subsequence $\{\mathfrak{w}_{n_k}\}$ of $\{\mathfrak{w}_n\}$ in $\mathcal{O}$ such that $(\mathfrak{w}_{n_k}, \mathfrak{w}) \in E(G)$ for all k. By the inequality (4.1), we have

$$\begin{aligned} F\Big(d(\mathfrak{w}_{n_k+1}, \mathfrak{I}(\mathfrak{w}))\Big) &= F\left(H_{\mathcal{F}}(\mathfrak{I}(\mathfrak{w}_{n_k}), \mathfrak{I}(\mathfrak{w}))\right) \leq F\left(d(\mathfrak{w}_{n_k}, \mathfrak{w})\right) - \lambda \\ &< F\left(d(\mathfrak{w}_{n_k}, \mathfrak{w})\right) \end{aligned}$$

which implies due to $(\mathcal{F}_1)$ that

$$d(\mathfrak{w}_{n_k+1}, \mathfrak{I}(\mathfrak{w})) \leq d(\mathfrak{w}_{n_k}, \mathfrak{w}). \tag{4.17}$$

Taking the limit as $k \to \infty$, we have $d(\mathfrak{w}, \mathfrak{I}(\mathfrak{w})) = 0$. Since $\mathfrak{I}(\mathfrak{w})$ is closed, so $\mathfrak{w} \in \mathfrak{I}(\mathfrak{w})$.

Example 4.1. *Let $\mathcal{O} = \{0,1,2,3,4,5,6,7,8\}$ and $d : \mathcal{O}\times\mathcal{O} \to [0,+\infty)$ be defined by*

$$d(\mathfrak{w},\mathfrak{h}) = \begin{cases} 0, \text{ if } \mathfrak{w} = \mathfrak{h}, \\ e^{\mathfrak{w}+\mathfrak{h}}, \text{ if } \mathfrak{w} \neq \mathfrak{h}. \end{cases}$$

Let G be a directed graph with $V(G) = \mathcal{O}$ and

$$E(G) = \left\{ \begin{array}{c} (0,0),(0,1),(0,4),(0,5),(1,0),(1,1),(1,2),(1,3),(2,2), \\ (2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5),(6,6), \\ (6,7),(7,1),(7,7),(8,7),(8,8). \end{array} \right\}$$

Then G is a directed graph and $(\mathcal{O},d)$ is $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a directed graph. Define $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$ by

$$\mathfrak{J}(\mathfrak{w}) = \begin{cases} \{2,3\} \text{ if } \mathfrak{w} \in \{1,2,3\} \\ \{4,5\}, \text{ if } \mathfrak{w} \in \{0,4,5\} \\ \{1\}, \quad \text{if } \mathfrak{w} = 7 \\ \{7\}, \text{ if } \mathfrak{w} \in \{6,8\}. \end{cases}$$

Let $F(t) = \ln t$, for $t > 0$ and $\lambda = \ln(1.23)$. Given that the premises of Theorem 4.1 are met , we can conclude that $\mathfrak{J}$ has fixed points $\{2,3,4,5\}$.

Corollary 4.1. *([7]) Let $(\mathcal{O},d)$ be a $\mathcal{F}$-complete $\mathcal{F}$-MS and $\mathfrak{J} : \mathcal{O} \to K(\mathcal{O})$ be a continuous multivalued mapping. If there exists a function $F \in F$ and a constant $\lambda > 0$ such that*

$$\lambda + F\left(H_{\mathcal{F}}(\mathfrak{J}(\mathfrak{w}),\mathfrak{J}(\mathfrak{h}))\right) \leq F\left(d(\mathfrak{w},\mathfrak{h})\right),$$

for all $\mathfrak{w},\mathfrak{h} \in \mathcal{O}$, then $\mathfrak{J}$ has a fixed point.

Proof. Take $E(G) = \mathcal{O}\times\mathcal{O}$ in Theorem 4.1. □

Corollary 4.2. *([19]) Let $(\mathcal{O},d)$ be a $\mathcal{F}$-complete $\mathcal{F}$-MS equipped with a directed graph G and $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ be a self mapping. f there exists a function $F \in F$ and a constant $\lambda > 0$ such that*

$$\lambda + F\left(d(\mathfrak{J}(\mathfrak{w}),\mathfrak{J}(\mathfrak{h}))\right) \leq F\left(d(\mathfrak{w},\mathfrak{h})\right),$$

for all $(\mathfrak{w},\mathfrak{h}) \in E(G)$ with $d(\mathfrak{J}\mathfrak{w},\mathfrak{J}\mathfrak{h}) > 0$. If $\mathfrak{J}$ is continuous or if $\{\mathfrak{w}_n\}$ is a sequence in $\mathcal{O}$ converging to $\mathfrak{w} \in \mathcal{O}$ as $n \to \infty$ and $(\mathfrak{w}_n,\mathfrak{w}_{n+1}) \in E(G)$ for all n, there is a subsequence $\{\mathfrak{w}_{n_k}\}$ of $\{\mathfrak{w}_n\}$ in $\mathcal{O}$ such that $(\mathfrak{w}_{n_k},\mathfrak{w}) \in E(G)$ for all k, then $\mathfrak{J}$ has a fixed point.

Proof. Take $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1. □

Corollary 4.3. *([20]) Let $(\mathcal{O},d)$ be a $\mathcal{F}$-complete $\mathcal{F}$-MS and $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ be a continuous self mapping. If there exists a function $F \in F$ and a constant $\lambda > 0$ such that*

$$\lambda + F\left(d(\mathfrak{J}(\mathfrak{w}),\mathfrak{J}(\mathfrak{h}))\right) \leq F\left(d(\mathfrak{w},\mathfrak{h})\right),$$

for all $\mathfrak{w},\mathfrak{h} \in \mathcal{O}$, then $\mathfrak{J}$ has a fixed point.

Proof. Take $E(G) = \mathcal{O}\times\mathcal{O}$ and the self mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1. □

Remark 4.1. *If we take $\xi(t) = \ln t$ and $\alpha = 0$ in Definition (2.5), then the notion of $\mathcal{F}$-MS reduces to classical MS. Moreover if we take $E(G) = \mathcal{O}\times\mathcal{O}$ in Theorem 4.1, then we obtain the main result of Vetro et al. [7].*

Remark 4.2. *If we take $\xi(t) = \ln t$ and $\alpha = 0$ in Definition (2.5), then the notion of $\mathcal{F}$-MS reduces to classical MS. Moreover if we take the mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1, we get the leading result of Batra et al. [21].*

Remark 4.3. *If we take $\xi(t) = \ln t$ and $\alpha = 0$ in Definition (2.5), then the notion of $\mathcal{F}$-MS reduces to classical MS. Moreover if we take $E(G) = \mathcal{O}\times\mathcal{O}$ and the mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1, then we obtain the main result of Wardowski et al. [6].*

Remark 4.4. *If we take $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1, then we get the prime result of Faraji et al. [22].*

Remark 4.5. *If we take $\xi(t) = \ln t$ and $\alpha \geq 1$ in Definition (2.5), then the notion of $\mathcal{F}$-MS reduces to b-MS. Moreover if we take $E(G) = \mathcal{O}\times\mathcal{O}$ and the mapping $\mathfrak{J} : \mathcal{O} \to \mathcal{O}$ in Theorem 4.1, then we obtain the main result of Cosentino et al. [23].*

Conclusion

In this study, we successfully introduced the novel concept of multivalued graphic contractions and multivalued *GF*-contractions within the framework of $\mathcal{F}$-MSs. By establishing new fixed point theorems in this newly defined metric space, we have expanded the existing body of knowledge in this area. The provided example serves as concrete evidence of the validity and applicability of our obtained results. These findings contribute significantly to the ongoing research in fixed point theory and its related fields.

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