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Essential Norm of Composition Operators on Harmonic Zygmund Spaces and Their Derivative Spaces

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Abstract. Let ψ represent the analytic self-mapping within the unit disk $\mathbb{D}$. We define the composition operator C_ψ as $C_\psi f = f \circ \psi$ for every f belonging to the space of harmonic functions $\mathcal{H}(\mathbb{D})$. The essential norm of composition operators within specific harmonic mapping spaces is investigated in this research. Explicitly, we outline the essential norm of composition operators on the harmonic Zygmund spaces $\mathcal{Z}^{\mathcal{H}}$ and the derivative of harmonic Zygmund spaces $\mathcal{V}^{\mathcal{H}}$. Notably, these results extend and build upon results that were established previously for the analytic settings.

1. Introduction

Given Ω to be a simply connected region in the complex plane. A harmonic mapping is a complex function h defined on Ω satisfying the Laplace equation such that:

$$\Delta h := 4\frac{\partial^2 h}{\partial\omega\partial\overline{\omega}} \equiv 0.$$

A harmonic mapping h always admits a representation in the form $f + \overline{g}$, where f and g are analytic functions. This representation achieves uniqueness when a fixed point ω_0 within Ω is specified, and g is chosen such that $g(\omega_0) = 0$. Let's denote $\mathbb{D}$ as the open unit disk within $\mathbb{C}$ and Aut($\mathbb{D}$) as the group of disk automorphisms. The class of analytic functions on $\mathbb{D}$ is represented by $H(\mathbb{D})$, while $\mathcal{H}(\mathbb{D})$ symbolizes the class of harmonic mappings on $\mathbb{D}$. For the scope of our study, we will focus on the harmonic mappings with the domain $\mathbb{D}$ and will use $\omega_0 = 0$ as the base point. Thus, the typical representation of a mapping $h \in \mathcal{H}(\mathbb{D})$ is $h = f + \overline{g}$, where $f, g \in H(\mathbb{D})$ and $g(0) = 0$.

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The Bloch space $\mathcal{B}$ is characterized as the Banach space consisting of the functions $f \in H(\mathbb{D})$ such that

$$\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|f'(\omega)| < \infty.$$

The norm of $f \in \mathcal{B}$ is defined as

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|f'(\omega)|.$$

The point-evaluation estimate, given by

$$|f(\omega)| \leq \log\frac{e}{1-|\omega|^2}\|f\|_{\mathcal{B}},$$

is a well-established property of functions f in $\mathcal{B}$.

The closed subspace $\mathcal{B}_0$ of $\mathcal{B}$, comprising of the functions f satisfying

$$\lim_{|\omega|\to 1}(1-|\omega|^2)|f'(\omega)| = 0,$$

is commonly referred as the little Bloch space. In a study by [5], it was demonstrated that $\mathcal{B}_0$ is the closure in $\mathcal{B}$ of the polynomial functions, thereby establishing its separability.

In complex function theory, extensive research has been conducted on the classical Zygmund space $\mathcal{Z}$. This space defined as the set of analytic functions f on $\mathbb{D}$ with extensions to the unit circle obtained through means of the radial limits. Formally, a function $f \in \mathcal{Z}$ if

$$\|f\|_* := \sup\frac{|f(e^{i(\theta+\gamma)}) + f(e^{i(\theta-\gamma)}) - 2f(e^{i\theta})|}{\gamma} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $\gamma > 0$.

Suppose $C(\overline{\mathbb{D}})$ denotes the space of continuous complex-valued functions on the closed unit disk $\overline{\mathbb{D}}$. It is well-established that the Zygmund space $\mathcal{Z}$ is subset of the disk algebra $H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$.

According to Theorem 5.3 in [11], an analytic function f on $\mathbb{D}$ is in the Zygmund space $\mathcal{Z}$ if and only if

$$\|f\|_{s\mathcal{Z}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|f''(\omega)| < \infty.$$

Moreover, $\|f\|_{s\mathcal{Z}} \asymp \|f\|_*$.

The Zygmund space $\mathcal{Z}$ is a Banach space with the respect of the norm

$$\|f\|_{\mathcal{Z}} := |f(0)| + |f'(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|f''(\omega)| < \infty.$$

Moreover, the functions in the $\mathcal{Z}$ satisfy the point-evaluation estimate

$$|f(\omega)| \leq \|f\|_{\mathcal{Z}}$$

The closed subspace $\mathcal{Z}_0$ of $\mathcal{Z}$, consisting of the functions f satisfying the condition

$$\lim_{|\omega|\to 1}(1-|\omega|^2)|f''(\omega)| = 0.$$

The space $\mathcal{V}$ consists of analytic functions f on unit disk $\mathbb{D}$ such that their first derivative belong to $\mathcal{Z}$, and they satisfy

$$\|f\|_{s\mathcal{V}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2|f'''(\omega)| < \infty.$$

The space $\mathcal{V}$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{V}} := |f(0)| + |f'(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|f'''(\omega)| < \infty.$$

It is well known that the space $\mathcal{V}$ is properly contained in $\mathcal{Z}$. Moreover, for $f \in \mathcal{V}$

$$\|f\|_{\mathcal{B}} \leq \|f\|_{\mathcal{Z}} \leq \|f\|_{\mathcal{V}}.$$

For more details see [10], where the authors studied weighted composition operators on iterated weighted type Banach space of analytic functions.

Colonna, in reference [9], introduced the concept of the harmonic Bloch spaces $\mathcal{B}^{\mathcal{H}}$ as the set of harmonic mappings on $\mathbb{D}$ that act as Lipschitz functions when considered as maps between the hyperbolic disk and $\mathbb{C}$ equipped with the Euclidean metric. An additional characteristic of such harmonic mappings h is the condition

$$\|h\|_{s\mathcal{B}^{\mathcal{H}}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[|h_\omega(\omega)| + |h_{\overline{\omega}}(\omega)|\Big] < \infty,$$

where h_ω and $h_{\overline{\omega}}$ are the first complex partial derivatives of h.

Subsequently, the authors, as detailed in [1] studied the harmonic Bloch spaces. In particular, it was shown that the $B^{\mathcal{H}}$ is a Banach space under the norm

$$\|h\|_{\mathcal{B}^{\mathcal{H}}} := |h(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big(|h_\omega(\omega)| + |h_{\overline{\omega}}(\omega)|\Big).$$

The space $\mathcal{B}^{\mathcal{H}}$ can be regraded as the collection of harmonic mappings $h \in \mathcal{H}(\mathbb{D})$ such that $h_\omega + h_{\overline{\omega}}$ lies in the harmonic growth space $\mathcal{A}^{\mathcal{H}}$. The latter is defined as the set of $h \in \mathcal{H}(\mathbb{D})$ such that

$$\|h\|_{\mathcal{A}^{\mathcal{H}}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|h(\omega)| < \infty.$$

The functions in $\mathcal{B}^{\mathcal{H}}$ satisfy the growth condition:

$$|h(\omega)| \leq \log\frac{e}{1-|\omega|^2}\|h\|_{\mathcal{B}^{\mathcal{H}}}. \tag{1.1}$$

The harmonic Zygmund space $\mathcal{Z}^{\mathcal{H}}$ comprises all harmonic mappings $h \in C(\overline{\mathbb{D}})$ such that

$$\|h\| := \sup\frac{|h(e^{i(\theta+\gamma)}) + h(e^{i(\theta-\gamma)}) - 2h(e^{i\theta})|}{\gamma} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $\gamma > 0$. According to Theorem 3.4 of [2], $h \in \mathcal{Z}^{\mathcal{H}}$ if and only if the following holds

$$\|h\|_{s\mathcal{Z}^{\mathcal{H}}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big(|h_{\omega\omega}(\omega)| + |h_{\overline{\omega\omega}}(\omega)|\Big) < \infty,$$

where $h_{\omega\omega}$, $h_{\overline{\omega\omega}}$ represent the second complex partial derivatives of h.

$\mathcal{Z}^{\mathcal{H}}$ is a Banach space under the given norm

$$\|h\|_{\mathcal{Z}^{\mathcal{H}}} = |h(0)| + |h_\omega(0)| + |h_{\overline{\omega}}(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big(|h_{\omega\omega}(\omega)| + |h_{\overline{\omega\omega}}(\omega)|\Big).$$

The functions in $\mathcal{Z}^{\mathcal{H}}$ satisfy

$$|h(\omega)| \leq \|h\|_{\mathcal{Z}^{\mathcal{H}}}. \tag{1.2}$$

The little harmonic Zygmund space $\mathcal{Z}_0^{\mathcal{H}}$ is the collection of all harmonic mapping h such that

$$\lim_{|\omega|\to 1}(1-|\omega|^2)\Big(|h_{\omega\omega}(\omega)| + |h_{\overline{\omega\omega}}(\omega)|\Big) = 0.$$

Expanding on the concept of the first derivative Zygmund space $\mathcal{V}$, [4] the first author introudced the space $\mathcal{V}^{\mathcal{H}}$ as set of all harmonic mappings h such that

$$\|h\|_{s\mathcal{V}^{\mathcal{H}}} := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\left[\left|\frac{\partial^3}{\partial\omega^3}h(\omega)\right| + \left|\frac{\partial^3}{\partial\overline{\omega}^3}h(\omega)\right|\right] < \infty.$$

Furthermore,

$$\begin{aligned}\|h\|_{\mathcal{V}^{\mathcal{H}}} &:= |h(0)| + |h_\omega(0)| + |h_{\overline{\omega}}(0)| + |h_{\omega\omega}(0)| + |h_{\overline{\omega\omega}}(0)| + \|h\|_{s\mathcal{V}^{\mathcal{H}}} \\ &= |f(0)| + \|f_\omega + f_{\overline{\omega}}\|_{\mathcal{Z}^{\mathcal{H}}}.\end{aligned}$$

The space $\mathcal{V}^{\mathcal{H}}$ forms a Banach space with respect to the norm described above.

In [4] the first author proves that the space $\mathcal{V}^{\mathcal{H}}$ is contained in $C(\overline{\mathbb{D}})$. Moreover, for $h \in \mathcal{V}^{\mathcal{H}}$,

$$\|h\|_\infty \leq 2\log 2\|h\|_{\mathcal{V}^{\mathcal{H}}}. \tag{1.3}$$

The composition operator C_ψ induced by an analytic or a conjugate analytic self-map ψ of $\mathbb{D}$, is defined as

$$C_\psi h = h\circ\psi,$$

representing a linear transformation over $\mathbb{C}$ that acts on the class of harmonic mappings on $\mathbb{D}$.

Extensive research has been conducted on operator theory in spaces of analytic functions defined within the unit disk, resulting in numerous research papers across various settings. Nevertheless, the exploration of similar research in the context of harmonic settings remains relatively limited.

In recent years, there has been a growing interest in the study of harmonic mappings. Notably, work presented in [1] has been complemented by efforts to characterize Bloch-type spaces for harmonic mappings. In [3], the same authors have undertaken a comprehensive examination of the compactness and boundedness of mappings in C_ψ into weighted Banach spaces of harmonic mappings.

For a more in-depth understanding of the field of harmonic mappings, we encourage further exploration of additional references. Colonna studied the Bloch constant of bounded harmonic mappings in [9], and [6] for Harmonic Function Theory. In [14], the authors characterized the Bloch spaces and Besov spaces of pluriharmonic mappings. A characterization of the harmonic Bloch space and the harmonic Besov spaces by an oscillation has been studied in [18]. The authors in [8] discussed the harmonic Bloch spaces in the unit ball of $\mathbb{C}^n$. The Q-Type Spaces of Harmonic Mappings and Harmonic Bergman spaces was studied in [15] and [16], respectively. Finally, in [13] the author studied the harmonic mapping in the plane and [7] discussed Landau's theorem and Marden constant for harmonic ν-Bloch mappings.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\|\cdot\|_{\mathcal{X}\to\mathcal{Y}}$ denotes the operator norm. The essential norm of a bounded linear operator $\mathcal{S}:\mathcal{X}\to\mathcal{Y}$ is its distance to the set of compact operators L mapping $\mathcal{X}$ to $\mathcal{Y}$, that is

$$\|\mathcal{S}\|_{e,\mathcal{X}\to\mathcal{Y}} = \inf\{\|\mathcal{S}-L\|_{\mathcal{X}\to\mathcal{Y}} \ : \text{L is compact operator}\}.$$

If $\mathcal{X}=\mathcal{Y}$, we denote the essential norm of bounded linear operator by $\|\mathcal{S}\|_{e,\mathcal{X}}$.

It is well known that $\|\mathcal{S}\|_{e,\mathcal{X}\to\mathcal{Y}} = 0$ if and only if $\mathcal{S}:\mathcal{X}\to\mathcal{Y}$ is compact.

One of the primary aims in this work is to provide estimates the essential norm of a composition operator acting on $\mathcal{Z}^{\mathcal{H}}$ and $\mathcal{V}^{\mathcal{H}}$.

In this work we shall use the notation $A \preceq B$ to mean that for some $c>0$, $A \leq c\,B$, whereas $A \asymp B$ means $A \preceq B$ and $B \preceq A$.

2. Essential norm on harmonic Zygmund space $\mathcal{Z}^{\mathcal{H}}$

In this section, we focus on discussing the essential norm of composition operators on $\mathcal{Z}^{\mathcal{H}}$. We begin with a useful lemmas to prove the main result of this section.

The following lemma is an extension of Lemma 3.7 in [17], and the proof is straightforward.

Lemma 2.1. ([17], Lemma 3.7) *Let X, Y be Banach spaces whose elements are harmonic functions on $\mathbb{D}$, and $T: X\to Y$ a bounded linear operator. Assume*

(i) *the point evaluation functionals on X are continuous;*
(ii) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*
(iii) *$T: X\to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n\to 0$ uniformly on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Lemma 2.2. *If $\{h_k\}$ is a sequence in $\mathcal{Z}_0^{\mathcal{H}}$ converging uniformly to 0 on compact subsets of $\mathbb{D}$, then $\{h_k\}$ converges to 0 weakly.*

Proof. Let $\{h_k\}$ be as in the statement. Let Λ be a bounded linear functional on $\mathcal{Z}^{\mathcal{H}}$. We wish to show that $\{\Lambda h_k\}$ converges to 0. Since $h_k\in\mathcal{Z}_0^{\mathcal{H}}$ for each k, given $\varepsilon>0$ there exists $\delta>0$ such that for $\delta<|\omega|<1$

$$(1-|\omega|^2)|(h_k)_{\omega\omega}(\omega)| + |(h_k)_{\overline{\omega\omega}}(\omega)| < \varepsilon.$$

Then

$$\begin{aligned}|\Lambda h_k| &\le \|\Lambda\|\|h_k\|_{\mathcal{Z}^{\mathcal{H}}}\\ &= \|\Lambda\|\left[|h_k(0)| + |(h_k)_\omega(0)| + |(h_k)_{\overline{\omega}}(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\left[|(h_k)_{\omega\omega}(\omega)| + |(h_k)_{\overline{\omega\omega}}(\omega)|\right]\right]\\ &\le \|\Lambda\|\Bigg[|h_k(0)| + |(h_k)_\omega(0)| + |(h_k)_{\overline{\omega}}(0)| + \sup_{|\omega|\le\delta}(1-|\omega|^2)\left[|(h_k)_{\omega\omega}(\omega)| + |(h_k)_{\overline{\omega\omega}}(\omega)|\right]\\ &\qquad + \sup_{\delta<|\omega|<1}(1-|\omega|^2)\left[|(h_k)_{\omega\omega}(\omega)| + |(h_k)_{\overline{\omega\omega}}(\omega)|\right]\Bigg]\\ &< \|\Lambda\|\left[|h_k(0)| + |(h_k)_\omega(0)| + |(h_k)_{\overline{\omega}}(0)| + \sup_{|\omega|\le\delta}(1-|\omega|^2)\left[|(h_k)_{\omega\omega}(\omega)| + |(h_k)_{\overline{\omega\omega}}(\omega)| + \varepsilon\right]\right].\end{aligned}$$

Since $\{h_k\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, it follows that $\lim_{k\to\infty}|\Lambda f_k| \le \|\Lambda\|\varepsilon$, hence $\Lambda h_k \to 0$ as $k\to\infty$. □

Lemma 2.3. *For $0<\rho<1$, T_ρ be the linear operator mapping a harmonic function h in $\mathcal{Z}^{\mathcal{H}}$ to its dilation $h_\rho(\omega) = h(\rho\omega)$, ω in $\mathbb{D}$. Then the following statements hold.*
(a) *For $\rho\in(0,1)$, T_ρ is bounded operator on $\mathcal{Z}^{\mathcal{H}}$. Moreover*

$$\|T_\rho\|_{\mathcal{Z}^{\mathcal{H}}} = 1.$$

(b)*Let $\delta\in(0,1)$ and $\epsilon>0$, then there exists $\rho\in(0,1)$ so that*

$$\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le 1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)(\omega)| < \epsilon,\quad \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_\omega(\omega)| + |((I-T_\rho)h)_{\overline{\omega}}(\omega)| < \epsilon,$$

$$\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega}(\omega)| + |((I-T_\rho)h)_{\overline{\omega\omega}}(\omega)| < \epsilon.$$

(c) *T_ρ is compact on $\mathcal{Z}^{\mathcal{H}}$.*

Proof. (a) Let $\rho\in(0,1)$ and h in $\mathcal{Z}^{\mathcal{H}}$. Then

$$\begin{aligned}\|T_\rho h\|_{\mathcal{Z}^{\mathcal{H}}} &= |h(0)| + \rho|h_\omega(0)| + \rho|h_{\overline{\omega}}(0)| + \rho^2\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\left[|h_{\omega\omega}(\rho\omega)| + |h_{\overline{\omega\omega}}(\rho\omega)|\right]\\ &= |h(0)| + \rho|h_\omega(0)| + \rho|h_{\overline{\omega}}(0)| + \sup_{|z|<\rho}(\rho^2-|z|^2)\left[|h_{\omega\omega}(z)| + |h_{\overline{\omega\omega}}(z)|\right]\\ &\le |h(0)| + |h_\omega(0)| + |h_{\overline{\omega}}(0)| + \sup_{|z|<1}(1-|z|^2)\left[|h_{\omega\omega}(z)| + |h_{\overline{\omega\omega}}(z)|\right]\\ &= \|h\|_{\mathcal{Z}^{\mathcal{H}}}.\end{aligned} \tag{2.1}$$

Thus T_ρ is bounded and observing that

$$1 = \|T_\rho 1\|_{\mathcal{Z}^{\mathcal{H}}} \le \|T_\rho\|_{\mathcal{Z}^{\mathcal{H}}\to\mathcal{Z}^{\mathcal{H}}}\|1\|_{\mathcal{Z}^{\mathcal{H}}} = \|T_\rho\|_{\mathcal{Z}^{\mathcal{H}}\to\mathcal{Z}^{\mathcal{H}}}, \tag{2.2}$$

combining (2.1) and (2.2), we obtain $\|T_\rho\|_{\mathcal{Z}^{\mathcal{H}}\to\mathcal{Z}^{\mathcal{H}}} = 1$.

(b) Let $h \in \mathcal{Z}^{\mathcal{H}}$ with $\|h\|_{\mathcal{Z}^{\mathcal{H}}} \leq 1$. Let $\delta \in (0,1)$ and assume that $\{\rho_n\}$ is a sequence in $(0,1)$ converging to 1 as $n \to \infty$. By the continuity of h on the closed unit disk, for each $\omega \in \mathbb{D}$,

$$\lim_{n\to\infty}((I - T_{\rho_n})h)(\omega) = \lim_{n\to\infty}(h(\omega) - h(\rho_n \omega)) = 0,$$

and by linearity and part (a)

$$\|(I - T_{\rho_n})h\|_{\mathcal{Z}^{\mathcal{H}}} \leq \|h\|_{\mathcal{Z}^{\mathcal{H}}} + \|T_{\rho_n}h\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2\|h\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2.$$

Thus $\frac{1}{2}(I - T_{\rho_n})h$ is in unit ball of $\mathcal{Z}^{\mathcal{H}}$. By (1.2), $\mathcal{Z}^{\mathcal{H}}$ satisfies the hypothesis (ii) in Lemma 2.1, therefore the sequence $\{(I - T_{\rho_n})h\}$ has subsequence $\{(I - T_{\rho_{n_j}})h\}$ converging uniformly to 0 on every compact subset of $\mathbb{D}$. Since every sequence in $\mathcal{Z}^{\mathcal{H}}$ converging uniformly on compact subsets of $\mathbb{D}$ converges uniformly on $\overline{\mathbb{D}}$, the subsequence $\{(I - T_{\rho_{n_j}})h\}$ converges uniformly to 0 on $\overline{\mathbb{D}}$. Therefore for every $\varepsilon > 0$ there are $\rho \in (0,1)$ such that

$$\sup_{\omega\in\mathbb{D}} |((I - T_\rho)h)(\omega)| < \varepsilon.$$

By Montel's theorem, the functions $((I - T_{\rho_n})h)_\omega$, $((I - T_{\rho_n})h)_{\overline{\omega}}$, $((I - T_{\rho_n})h)_{\omega\omega}$, $((I - T_{\rho_n})h)_{\overline{\omega\omega}}$ converge uniformly to 0 on every compact subset of $\mathbb{D}$. Thus

$$\sup_{|\omega|\leq\delta}((I - T_{\rho_n})h)_\omega| + |((I - T_{\rho_n})h)_{\overline{\omega}}| < \varepsilon, \quad \sup_{|\omega|\leq\delta}((I - T_{\rho_n})h)_{\omega\omega}| + |((I - T_{\rho_n})h)_{\overline{\omega\omega}}| < \varepsilon.$$

The conclusion follows after taking the supremum over all functions in the unit ball of $\mathcal{Z}^{\mathcal{H}}$.

(c) Fix $0 < \rho < 1$. To show that T_ρ is compact on $\mathcal{Z}^{\mathcal{H}}$, by Lemma 2.1, it suffices to show that

$$\lim_{k\to\infty} \|T_\rho f_k\|_{\mathcal{Z}^{\mathcal{H}}} = 0$$

for each bounded sequence $\{h_k\}$ on $\mathcal{Z}^{\mathcal{H}}$ converging uniformly on every compact subset of $\mathbb{D}$.

Let $\{h_k\}$ bounded sequence on $\mathcal{Z}^{\mathcal{H}}$ converges uniformly to 0 on every compact subset of $\mathbb{D}$ and $h_k = f_k + \overline{g}_k$ with $g_k(0) = 0$. It is clear that $f_k, g_k \in \mathcal{Z}$. Then we have

$$\begin{aligned}
\|T_\rho h_k\|_{s\mathcal{Z}^{\mathcal{H}}} &= \sup_{\omega\in\mathbb{D}}((1 - |z|^2)|(T_\rho f_k)''(\omega)| + |(T_\rho g_k)''(\omega))| \\
&= \rho^2 \sup_{\omega\in\mathbb{D}}(1 - |\omega|^2)|f_k''(\rho\omega)| + |g_k''(\rho\omega)| \\
&= \sup_{|z|<\rho}(\rho^2 - |z|^2)|f_k''(z)| + |g_k''(z)| \\
&\leq \sup_{|z|<\rho}|f_k''(z)| + |g_k''(z)|.
\end{aligned}$$

Since $\{f_k\}$ and $\{g_k\}$ converge uniformly to zero on the disk with radius ρ, then $\|T_\rho h_k\|_{\delta\mathcal{Z}^{\mathcal{H}}} \to 0$ as $k \to \infty$. Since $\{0\}$ is compact, $(T_\rho h_k)(0)$, and $((T_\rho h_k)_\omega + (T_\rho h_k)_{\overline{\omega}})(0)$ converge to 0, so $\|T_\rho h_k\|_{\mathcal{Z}^{\mathcal{H}}} \to 0$ as $k \to \infty$. Thus T_ρ is compact on $\mathcal{Z}^{\mathcal{H}}$. $\square$

Before we characterize the main theorem for this section, we shall recall the result in [3] that the composition operator C_ψ is bounded on $\mathcal{Z}^{\mathcal{H}}$ if and only if

$$\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2},\quad \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[\frac{|\psi'(\omega)|^2}{(1-|\psi(\omega)|^2}\Big]$$

are finite.

We are now ready to prove the main result for this section.

Theorem 2.1. *Assume ψ be a self map on $\mathbb{D}$ such that C_ψ is bounded on $\mathcal{Z}^{\mathcal{H}}$. Then*

$$\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}}\asymp\max\left\{\limsup_{|\psi(\omega)|\to1}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2},\ \limsup_{|\psi(\omega)|\to1}(1-|\omega|^2)\Big[\frac{|\psi'(\omega)^2|}{1-|\psi(\omega)|^2}\Big]\right\}.$$

Proof. Set $N_1:=\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}$, $N_2:=\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[\frac{|\psi'(\omega)|^2}{(1-|\psi(\omega)|^2}\Big]$.

Let $\delta\in(0,1)$ and $\epsilon>0$. So, by Lemma 2.3, there is $\rho\in(0,1)$ so that

$$\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)(\omega)|<\epsilon,\quad \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_\omega(\omega)|+|((I-T_\rho)h)_{\overline{\omega}}(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega}(\omega)|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\omega)|<\epsilon.$$

It follows that

$$\begin{aligned}&|(C_\psi(I-T_\rho)h)(0)|+|(C_\psi(I-T_\rho)h)_\omega(0)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega}}(0)|\\&\le|((I-T_\rho)h)(\psi(0))|+|\psi'(0)|\Big[((I-T_\rho)h)_\omega(\psi(0)|+|((I-T_\rho)h)_{\overline{\omega}}(\psi(0))|\Big]\\&\le(1+|\psi'(0)|)\epsilon.\end{aligned}$$

So

$$A_1:=\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\Big[|(C_\psi(I-T_\rho)h)(0)|+|(C_\psi(I-T_\rho)h)_\omega(0)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega}}(0)|\le\Big(1+|\psi'(0)|\Big)\epsilon,$$

$$A_2:=\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\sup_{|\omega|\le\delta}(1-|\omega|^2)|\psi''(\omega)|\Big[|((I-T_\rho)h)_\omega(\psi(\omega))|+|((I-T_\rho)h)_{\overline{\omega}}(\psi(\omega))|\Big]\le N_1\epsilon,$$

$$A_3:=\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\sup_{|\omega|\le\delta}(1-|\omega|^2)|\psi'(\omega)|^2\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\psi(\omega))|\Big]\le N_2\epsilon.$$

Since T_ρ is compact on $\mathcal{Z}^{\mathcal{H}}$ by part (c) in Lemma 2.3, we have

$$\begin{aligned}&\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}}\le\|C_\psi(I-T_\rho)\|_{\mathcal{Z}^{\mathcal{H}}\to\mathcal{Z}^{\mathcal{H}}}\\&=\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\|C_\psi(I-T_\rho)h\|_{\mathcal{Z}^{\mathcal{H}}}\\&=\sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\le1}\Big(|(C_\psi(I-T_\rho)h)(0)|+|(C_\psi(I-T_\rho)h)_\omega(0)|+|(C_\psi((I-T_\rho)h)_{\overline{\omega}}(0)|\\&\quad+\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[|(C_\psi(I-T_\rho)h)_{\omega\omega}(\omega)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega\omega}}(\omega)|\Big]\Big)\end{aligned}$$

$$\begin{aligned}
&\leq A_1 + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[|(C_\psi(I-T_\rho)h)_{\omega\omega}(\omega)| + |(C_\psi(I-T_\rho)h)_{\overline{\omega}\overline{\omega}}(\omega)|\Big]\\
&\leq A_1 \quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi''(\omega)|\Big[|((I-T_\rho)h)_{\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}}(\psi(\omega))|\Big]\\
&\quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'(\omega)|^2\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}\overline{\omega}}(\psi(\omega))|\Big]\\
&\leq A_1 + A_2 + A_3\\
&\quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}\frac{|((I-T_\rho)h)_{\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}}(\psi(\omega))|}{\log\frac{e}{1-|\psi(\omega)|^2}}\\
&\quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)\Big[\frac{|\psi'(\omega)^2|}{1-|\psi(\omega)|^2}\Big](1-|\psi(\omega)|^2)\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}\overline{\omega}}(\psi(\omega))|\Big]\\
&\leq \Big(1+|\psi'(0)|\Big)\epsilon + N1\epsilon + N_2\epsilon\\
&\quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}\frac{|((I-T_\rho)h)_{\omega}(\psi(\omega))| + |(I-T_\rho)h)_{\overline{\omega}}(\psi(z))|}{\log\frac{e}{1-|\psi(\omega)|^2}}\\
&\quad + \sup_{\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1}\sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^2|}{1-|\psi(\omega)|^2}(1-|\psi(\omega)|^2)\Big[|((I-T_\rho)h)_{\omega\omega}(\omega)| + |((I-T_\rho)h)_{\overline{\omega}\overline{\omega}}(\omega)|\Big].
\end{aligned} \tag{2.3}$$

Let $h\in\mathcal{Z}^{\mathcal{H}}$ such that $\|h\|_{\mathcal{Z}^{\mathcal{H}}}\leq 1$. Then, by part (a) of Lemma 2.3,

$$\|(I-T_\rho)h\|_{\mathcal{Z}_{\mathcal{H}}} \leq \|h\|_{\mathcal{Z}_{\mathcal{H}}} + \|T_\rho h\|_{\mathcal{Z}^{\mathcal{H}}} \leq \|h\|_{\mathcal{Z}^{\mathcal{H}}} + \|h\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2.$$

Therefore

$$(1-|\psi(\omega)|^2)|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}\overline{\omega}}(\psi(\omega))| \leq \|(I-T_\rho)h\|_{\mathcal{Z}^{\mathcal{H}}} \leq 2. \tag{2.4}$$

Let $\omega\in\mathbb{D}$, and since the function $((I-T_\rho)h)_\omega + ((I-T_\rho)h)_{\overline{\omega}}$ in $\mathcal{B}^{\mathcal{H}}$, we have by (1.1)

$$\begin{aligned}
|((I-T_\rho)h)_{\omega}(\omega)| + |((I-T_\rho)h)_{\overline{\omega}}(\omega)| &\leq \log\frac{e}{1-|\psi(\omega)|^2}\|((I-T_\rho)h)_{\omega} + ((I-T_\rho)h)_{\overline{\omega}}\|_{\mathcal{B}^{\mathcal{H}}}\\
&\leq 2\Big[\log\frac{e}{1-|\psi(\omega)|^2}\Big].
\end{aligned} \tag{2.5}$$

Thus, by using (2.5), and (2.4) in (2.3), we have

$$\begin{aligned}
\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} &\leq \Big(1+|\psi'(0)| + M_1 + M_2 + M_3\Big)\epsilon\\
&\quad + 2\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|\psi'')(\omega)|\log\frac{e}{1-|\psi(\omega)|^2} + 2\sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^2|}{1-|\psi(\omega)|^2},
\end{aligned}$$

where ϵ is arbitrary. Let $\delta\to 1$, we obtain

$$\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} \leq 2\lim_{s\to 1}\Big(\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2} + \sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^2|}{1-|\psi(\omega)|^2}\Big) \tag{2.6}$$

which proves the upper bound.

For $b \in \mathbb{C}$ such that $|b| > 1/2$, consider the analytic one-parameter family $\{f_b\}$ defined on $\mathbb{D}$ by

$$f_b(\omega) = \frac{(1-|b|^2)\Big(\log(1-\overline{b}\omega) + \frac{1-|b|^2}{1-\overline{b}\omega}\Big)}{\overline{b}^2}. \tag{2.7}$$

Staightforward calculation shows that

$$\begin{aligned}
\|f_b\|_{s\mathcal{Z}^{\mathcal{H}}} &= \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)(1-|b|^2)\Big|-\frac{1}{(1-\overline{b}\omega)^2} + \frac{(1-|b|^2)2}{(1-\overline{b}\omega)^3}\Big| \\
&\le 4\sup_{\omega\in\mathbb{D}}(1-|\omega|)(1-|b|)\frac{|-(1-\overline{b}\omega)+(1-|b|^2)2|}{|1-\overline{b}\omega|^3} \\
&\le 4\sup_{\omega\in\mathbb{D}}\Big|\frac{1-\overline{b}\omega+\overline{b}2(\omega-b)}{1-\overline{b}\omega}\Big| \\
&\le 12.
\end{aligned}$$

Therefore $f_b \in \mathcal{Z}^{\mathcal{H}}$. Moreover $\sup_{|b|>1/2}\|f_b\|_{\mathcal{Z}^{\mathcal{H}}} < \infty$.

Now, let's consider $\{z_n\}_{n\in\mathbb{N}}$ as a sequence in the unit disk and $|\psi(z_n)| \to 1$ as $n \to \infty$. For $\omega \in \mathbb{D}$, define $f_n(\omega) = f_{\psi(z_n)}(\omega)$ as in (2.7), and let $K = \sup_{n\in\mathbb{N}}\|f_n\|_{\mathcal{Z}^{\mathcal{H}}}$. Observing that f_n is bounded on $\mathcal{Z}^{\mathcal{H}}$ and converges uniformly to 0 on $\overline{\mathbb{D}}$. Moreover, we note that $\lim_{|\omega|\to 1}(1-|\omega|^2)|f_n''(\omega)| = 0$. So, $f_n \in \mathcal{Z}_0^{\mathcal{H}}$. Thus, by Lemma 2.2, f_n converges weakly to 0 in $\mathcal{Z}^{\mathcal{H}}$. Let T be a compact operator on $\mathcal{Z}^{\mathcal{H}}$. Then according to Lemma 2.1, we have

$$\lim_{n\to\infty}\|Tf_n\|_{\mathcal{Z}^{\mathcal{H}}} = 0.$$

Thus

$$\begin{aligned}
K\|C_\psi - T\|_{\mathcal{Z}^{\mathcal{H}}} &\ge \limsup_{n\to\infty}\|(C_\psi - T)f_n\|_{\mathcal{Z}^{\mathcal{H}}} \\
&\ge \limsup_{n\to\infty}\|C_\psi f_n\|_{\mathcal{Z}^{\mathcal{H}}} - \limsup_{n\to\infty}\|Tf_n\|_{\mathcal{Z}^{\mathcal{H}}} \\
&= \limsup_{n\to\infty}\|C_\psi f_n\|_{\mathcal{Z}^{\mathcal{H}}}.
\end{aligned}$$

Hence

$$K\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} \ge \limsup_{n\to\infty}\|C_\psi f_n\|_{\mathcal{Z}^{\mathcal{H}}}. \tag{2.8}$$

Therefore

$$K\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} \ge \limsup_{n\to\infty}(1-|z_n|^2)|\psi'(z_n)^2)||f_n''(\psi(z_n))| - \limsup_{n\to\infty}\Big[(1-|z_n|^2)|\psi''(z_n)||f_n'(\psi(z_n))|\Big].$$

Note that $f_n'(\psi(z_n)) = 0$ and $|f_n''(\psi(z_n))| = \frac{1}{1-|\psi(z_n)|^2}$. Thus

$$\limsup_{n\to\infty} 1-|z_n|^2\Big[\frac{|\psi'(z_n)^2|}{1-|\psi(z_n)|^2}\Big] \le K\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}}. \tag{2.9}$$

Again, let $b \in \mathbb{C}$ such that $|b| > 1/2$. Define an analytic one-parameter family $\{g_b\}$ on $\mathbb{D}$ by

$$g_b(\omega) = \log \frac{e}{1-|b|^2}\Big)^{-1} \frac{-(1-\bar{b}\omega)}{\bar{b}}\Big(5 - 4\log(1-\bar{b}\omega) + \log^2(1-\bar{b}\omega)\Big)$$

We have

$$\begin{aligned} \|g_b\|_{s\mathcal{Z}^{\mathcal{H}}} &= \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big(\log\frac{e}{1-|b|^2}\Big)^{-1}\frac{2|b|}{|1-\bar{b}\omega|}\Big|\log\frac{e}{1-\bar{b}\omega}\Big| \\ &\leq 4\Big(\log\frac{e}{1-|b|^2}\Big)^{-1}\Big(\log\frac{e}{1-|b|}+\pi\Big) \\ &= 4\left(1+\frac{\pi+\log 2}{\log\frac{e}{1-|b|^2}}\right) \\ &\leq 4(1+\log 2+\pi). \end{aligned}$$

Therefore $g_b \in \mathcal{Z}^{\mathcal{H}}$. Moreover $\sup\limits_{|b|>1/2} \|g_b\|_{\mathcal{Z}^{\mathcal{H}}} < \infty$. Assume $\{z_n\}_{n\in\mathbb{N}}$ be in the unit disk and $|\psi(z_n)| \to 1$ as $n \to \infty$. Let $n \in \mathbb{N}$, $z \in \mathbb{D}$, and assume $g_n = g_{\psi(z_n)}$ as in (2.10), and let $L = \sup\limits_{n\in\mathbb{N}} \|g_n\|_{\mathcal{Z}^{\mathcal{H}}}$.

We know that the sequence g_n is a bounded on $\mathcal{Z}^{\mathcal{H}}$ and converges to 0 uniformly on $\overline{\mathbb{D}}$. Moreover $\lim_{|\omega|\to 1}(1-|\omega|^2)|g_n''(\omega)| = 0$. Therefore $g_n \in \mathcal{Z}_0^{\mathcal{H}}$. Thus, by Lemma 2.2, g_n converges weakly to 0 in $\mathcal{Z}^{\mathcal{H}}$. Let T be a compact operator on $\mathcal{Z}^{\mathcal{H}}$. Therefore

$$\lim_{n\to\infty} \|Tg_n\|_{\mathcal{Z}^{\mathcal{H}}} = 0.$$

It follows that

$$\begin{aligned} L\|C_\psi - T\|_{\mathcal{Z}^{\mathcal{H}}\to\mathcal{Z}^{\mathcal{H}}} &\geq \limsup_{k\to\infty} \|(C_\psi - T)g_n\|_{\mathcal{Z}^{\mathcal{H}}} \\ &\geq \limsup_{n\to\infty}\|C_\psi g_n\|_{\mathcal{Z}^{\mathcal{H}}} - \limsup_{n\to\infty}\|Tg_n\|_{\mathcal{Z}^{\mathcal{H}}} \\ &\geq \limsup_{n\to\infty}\|C_\psi g_n\|_{\mathcal{Z}^{\mathcal{H}}}. \end{aligned}$$

Hence

$$L\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} \geq \limsup_{n\to\infty}\|C_\psi g_n\|_{\mathcal{Z}^{\mathcal{H}}}. \tag{2.10}$$

Therefore

$$L\|C_{\theta,\psi}\|_{e,\mathcal{Z}^{\mathcal{H}}} \geq \limsup_{n\to\infty}(1-|z_n|^2)|\psi''(z_n)||g_n'(\psi(z_n))| - \limsup_{n\to\infty}\Big[(1-|z_n|^2)|\psi'(z_n)^2||g_n''(\psi(z_n))|\Big].$$

Then after simple calculations, we get

$$\begin{aligned} \limsup_{n\to\infty} 3(1-|z_n|^2)|\psi'')(z_n)|\log\frac{e}{1-|\psi(z_n)|^2} &\leq L\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} + \limsup_{n\to\infty}\frac{2(1-|z_n|^2)|\psi'(z_n)^2|}{1-|\psi(z_n)|^2} \\ &\leq \|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}}. \end{aligned} \tag{2.11}$$

As consequence, by (2.6), (2.9) and (2.11)

$$\|C_\psi\|_{e,\mathcal{Z}^{\mathcal{H}}} \asymp \max\left\{\limsup_{|\psi(\omega)|\to 1}(1-|\omega|^2)|\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}, \limsup_{|\psi(\omega)|\to 1}(1-|\omega|^2)\Big[\frac{|\psi'(\omega)^2|}{1-|\psi(\omega)|^2}\Big]\right\}$$

which completes our proof. □

3. Essential norm on harmonic space, $\mathcal{V}^{\mathcal{H}}$

In this section, we move our attention to characterize the essential norm of the composition operators acting on the space $\mathcal{V}^{\mathcal{H}}$.

This section begins with a result in [4], the compostion operator C_ψ is bounded on $\mathcal{V}^{\mathcal{H}}$ if and only if $\psi \in \mathcal{V}^{\mathcal{H}}$ and

$$\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'(\omega)\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}<\infty.$$

In order to characterize the main result, we recall the following lemmas.

Lemma 3.1. *Assume the sequence $\{h_k\} \in \mathcal{V}_0^{\mathcal{H}}$ and converges uniformly to 0 on compact subsets of $\mathbb{D}$, then $\{h_k\}$ converges to 0 weakly.*

The proof is similar to that used in the proof of Lemma 2.2

Lemma 3.2. *Let T_ρ be the linear operator mapping a harmonic function h in $\mathcal{V}^{\mathcal{H}}$ to its dilation $h_\rho(\omega) = h(\rho\omega)$, ω in $\mathbb{D}$ and $0<\rho<1$. Then the following statements hold.*
(a) *For each $\rho\in(0,1)$, T_ρ is bounded operator on $\mathcal{V}^{\mathcal{H}}$. Moreover*

$$\|T_\rho\|_{\mathcal{V}^{\mathcal{H}}}=1.$$

(b) *For each $\delta\in(0,1)$ and each $\epsilon>0$, there exists $\rho\in(0,1)$ such that*

$$\sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}}\le 1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}}\le 1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)_\omega(\omega)|+|((I-T_\rho)h)_{\overline{\omega}}(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega}(\omega)|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega\omega}(\omega)|+|((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\omega)|<\epsilon.$$

(c) *T_ρ is compact on $\mathcal{V}^{\mathcal{H}}$.*

Proof. The argument is similar to that in the proof of Lemma 2.3. □

We are now ready to present the main theorem of this section.

Theorem 3.1. *Let ψ be a self map on $\mathbb{D}$ such that C_ψ is bounded on $\mathcal{V}^{\mathcal{H}}$. Then*

$$\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}}\asymp\max\left\{\limsup_{|\psi(\omega)|\to1}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\log\frac{e}{1-|\psi(\omega)|^2},\ \limsup_{|\psi(\omega)|\to1}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2}\right\}.$$

Proof. Let $M_1 := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'''(\omega)|$, $M_2 := \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'(\omega)\psi''(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}$,
$M_3 := \sup_{\omega\in\mathbb{D}}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{(1-|\psi(\omega)|^2}$.
Let $\delta\in(0,1)$ and $\epsilon>0$. So, by Lemma 3.2, there is $\rho\in(0,1)$ such that

$$\sup_{\|h\|_{\mathcal{VH}}\le 1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{VH}}\le 1}\sup_{\omega\in\mathbb{D}}|((I-T_\rho)h)_\omega(\omega)|+|((I-T_\rho)h)_{\overline{\omega}}(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{VH}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega}(\omega)|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\omega)|<\epsilon,$$

$$\sup_{\|h\|_{\mathcal{VH}}\le 1}\sup_{|\omega|\le\delta}|((I-T_\rho)h)_{\omega\omega\omega}(\omega)|+|((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\omega)|<\epsilon.$$

It follows that

$$\begin{aligned}
&|(C_\psi(I-T_\rho)h)(0)|+|(C_\psi(I-T_\rho)h)_\omega(0)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega}}(0)|+|(C_\psi(I-T_\rho)h)_{\omega\omega}(0)|\\
&\quad+|(C_\psi(I-T_\rho)h)_{\overline{\omega\omega}}(0)|\\
&\le|((I-T_\rho)h)(\psi(0))|+|\psi'(0)|\Big[((I-T_\rho)h)_\omega(\psi(0))|+|((I-T_\rho)h)_{\overline{\omega}}(\psi(0))|\Big]\\
&\quad+|\psi''(0)|\Big[|((I-T_\rho)h)_\omega(\psi(0))|+|((I-T_\rho)h)_{\overline{\omega}}(\psi(0))|\Big]\\
&\quad+|\psi'(0)^2|\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(0))|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\psi(0))|\Big]\\
&\le(1+|\psi'(0)|+|\psi''(0)|)+|\psi'(0)^2|)\epsilon.
\end{aligned}$$

So

$$\begin{aligned}
B_1:&=\sup_{\|h\|_{\mathcal{VH}_{\le1}}}\Big[|(C_\psi(I-T_\rho)h)(0)|+|(C_\psi(I-T_\rho)h)_\omega(0)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega}}(0)|\\
&\qquad+|(C_\psi(I-T_\rho)h)_{\omega\omega}(0)|+|(C_\psi(I-T_\rho)h)_{\overline{\omega\omega}}(0)|\Big]\\
&\le\Big(1+|\psi'(0)|+|\psi''(0)|+|\psi'(0)^2|\Big)\epsilon,\\
B_2:&=\sup_{\|h\|_{\mathcal{VH}_{\le1}}}\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'''(\omega)|\Big[|((I-T_\rho)h)_\omega(\psi(\omega))|+|((I-T_\rho)h)_{\overline{\omega}}(\psi(\omega))|\Big]\\
&\le M_1\epsilon,\\
B_3:&=\sup_{\|h\|_{\mathcal{VH}_{\le1}}}\sup_{|\omega|\le\delta}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))|+|((I-T_\rho)h)_{\overline{\omega\omega}}(\psi(\omega))|\Big]\\
&\le M_2\epsilon,\\
B_4:&=\sup_{\|h\|_{\mathcal{VH}_{\le1}}}\sup_{|\omega|<\delta}(1-|\omega|^2)|\psi'(\omega)^3|\Big[|((I-T_\rho)h)_{\omega\omega\omega}(\psi(\omega))|+|((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\psi(\omega))|\Big]\\
&\le M_3\epsilon.
\end{aligned}$$

Since T_ρ is compact on $\mathcal{V}^{\mathcal{H}}$ by part (c) in Lemma 3.2, we have

$$
\begin{aligned}
&\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \le \|C_\psi(I-T_\rho)\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \sup_{\|h\|_{V\le 1}} \|C_\psi(I-T_\rho)h\|_{\mathcal{V}^{\mathcal{H}}}\\
&= \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \Bigg(|(C_\psi(I-T_\rho)h)(0)| + |(C_\psi(I-T_\rho)h)_\omega(0)| + |(C_\psi(I-T_\rho)h)_{\overline{\omega}}(0)|\\
&\qquad + |(C_\psi(I-T_\rho)h)_{\omega\omega}(0)| + |(C_\psi(I-T_\rho)h)_{\overline{\omega\omega}}(0)|\\
&\qquad + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[|(C_\psi(I-T_\rho)h)_{\omega\omega\omega}(\omega)| + |(C_\psi(I-T_\rho)h)_{\overline{\omega\omega\omega}}(\omega)|\Big]\Bigg)\\
&\le B_1 + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\Big[|(C_\psi(I-T_\rho)h)_{\omega\omega\omega}(\omega)| + |(C_\psi(I-T_\rho)h)_{\overline{\omega\omega\omega}}(\omega)|\Big]\\
&\le B_1 \quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'''(\omega)|\Big[|((I-T_\rho)h)_\omega(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega}}(\psi(\omega))|\Big]\\
&\quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\Big[|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega\omega}}(\psi(\omega))|\Big]\\
&\quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|((I-T_\rho)h)_{\omega\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\psi(\omega))|\\
&\le B_1 + B_2 + B_3 + B_4\\
&\qquad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}\frac{|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega\omega}}(\psi(\omega))|}{\log\frac{e}{1-|\psi(\omega)|^2}}\\
&\quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2}(1-|\psi(\omega)|^2)|\psi'(\omega)|^3(|((I-T_\rho)h)_{\omega\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\psi(\omega))|\\
&\le \Big(1+|\psi'(0)|+|\psi''(0)|+|\psi'(0)^2|\Big)\epsilon + M_1\epsilon + M_2\epsilon + M_3\epsilon\\
&\quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}\frac{|((I-T_\rho)h)_{\omega\omega}(\psi(\omega))| + |(I-T_\rho)h)_{\overline{\omega\omega}}(\psi(\omega))|}{\log\frac{e}{1-|\psi(\omega)|^2}}\\
&\quad + \sup_{\|h\|_{\mathcal{V}^{\mathcal{H}}\le 1}} \sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2}(1-|\psi(\omega)|^2)|((I-T_\rho)h)_{\omega\omega\omega}(\omega)| + |((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\omega)|
\end{aligned}
\tag{3.1}
$$

Let $h \in \mathcal{V}^{\mathcal{H}}$ such that $\|h\|_{\mathcal{V}^{\mathcal{H}}} \le 1$. Then, by part (a) of Lemma 3.2,

$$\|(I-T_\rho)h\|_{\mathcal{V}^{\mathcal{H}}} \le \|h\|_{\mathcal{V}^{\mathcal{H}}} + \|T_\rho h\|_{\mathcal{V}^{\mathcal{H}}} \le \|h\|_{\mathcal{V}^{\mathcal{H}}} + \|h\|_{\mathcal{V}^{\mathcal{H}}} \le 2.$$

Therefore

$$(1-|\psi(\omega)|^2)|((I-T_\rho)h)_{\omega\omega\omega}(\psi(\omega))| + |((I-T_\rho)h)_{\overline{\omega\omega\omega}}(\psi(\omega))| \le \|(I-T_\rho)h\|_{\mathcal{V}^{\mathcal{H}}} \le 2. \tag{3.2}$$

Since the function $(I-T_\rho)h)_{\omega\omega} + (I-T_\rho)h)_{\overline{\omega\omega}}$ in $\mathcal{B}_H$, for $\omega \in \mathbb{D}$, we obtain

$$|((I-T_\rho)h)_{\omega\omega}(\omega)| + |((I-T_\rho)h)_{\overline{\omega\omega}}(\omega)| \le \log\frac{e}{1-|\psi(\omega)|^2}\|((I-T_\rho)h)_{\omega\omega} + ((I-T_\rho)h)_{\overline{\omega\omega}}\|_{\mathcal{B}_H}$$

$$\begin{aligned}&\leq \log \frac{e}{1-|\psi(\omega)|^2}\|(I-T_\rho)h\|_{\mathcal{V}^{\mathcal{H}}}\\ &\leq 2\log \frac{e}{1-|\psi(\omega)|^2}. \end{aligned} \tag{3.3}$$

Thus, by using (3.3), and (3.2) in (3.1), we have

$$\begin{aligned}\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \leq &\Big(1+|\psi'(0)|+|\psi''(0)|+|\psi'(0)^2|+M_1+M_2+M_3\Big)\epsilon\\ &+2\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\log\frac{e}{1-|\psi(\omega)|^2}+2\sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2}.\end{aligned}$$

Since ϵ is arbitrary, and by letting $\delta\to 1$, we obtain

$$\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \leq 2\lim_{\delta\to 1}\Big(\sup_{|\psi(\omega)|>\delta}(1-|\omega|^2)|(\psi'\psi'')(\omega)|\log\frac{e}{(1-|\psi(\omega)|^2)}+\sup_{|\psi(\omega)|>\delta}\frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2}\Big). \tag{3.4}$$

which proves the upper bound.

To prove the lowerr bound, fix $b\in\mathbb{D}$ such that $1/2<|b|<1$ and define the analytic one parameter family of test function F_b as follows. For $\omega\in\mathbb{D}$, let

$$F_b(\omega)=\frac{(1-|b|^2)^2}{\overline{b}^3}\Big(2\log(1-\overline{b}\omega)+\frac{1-|b|^2}{1-\overline{b}\omega}\Big). \tag{3.5}$$

Noting f_b is analytic on $\mathbb{D}$, and a direct calculations show that

$$\begin{aligned}F_b'(\omega)&=\frac{(1-|b|^2)^2}{\overline{b}^2}\Big(-\frac{2}{1-\overline{b}\omega}+\frac{1-|b|^2}{(1-\overline{b}\omega)^2}\Big),\\ F_b''(\omega)&=\frac{2(1-|b|^2)^2}{\overline{b}}\Big(-\frac{1}{(1-\overline{b}\omega)^2}+\frac{1-|b|^2}{(1-\overline{b}\omega)^3}\Big),\\ F_b'''(\omega)&=2(1-|b|^2)^2\Big(-\frac{2}{(1-\overline{b}\omega)^3}+\frac{3(1-|b|^2)}{(1-\overline{b}\omega)^4}\Big).\end{aligned}$$

Then,

$$\begin{aligned}(1-|\omega|^2)|F_b'''(\omega)|&=2(1-|\omega|^2)(1-|b|^2)^2\Big|-\frac{2}{1-\overline{b}\omega)^3}+\frac{3(1-|b|^2)}{(1-\overline{b}\omega)^4}\Big|\\ &\leq 2(1-|\omega|^2)(1-|b|^2)^2\Big[\frac{2}{|1-\overline{b}\omega|^3}+\frac{3(1-|b|^2)}{|1-\overline{b}\omega|^4}\Big]\\ &\leq 2(1-|\omega|^2)(1-|b|^2)^2\Big[\frac{2}{(1-|\omega|)(1-|b|)^2}+\frac{3(1-|b|^2)}{(1-|\omega|)(1-|b|)^3}\Big]\\ &\leq 16(2+6)=128.\end{aligned}$$

Therefore,

$$\begin{aligned}\|F_b\|_{\mathcal{V}^{\mathcal{H}}}&=|F_b(0)|+|F_b'(0)|+|F_b''(0)|+\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|F_b'''(\omega)|\\ &=\frac{(1-|b|^2)^3}{|b|^3}+\frac{(1+|b|^2)(1-|b|^2)^2}{|b|^2}+2|b|(1-|b|^2)^2+\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|F'''(\omega)|\end{aligned}$$

$$\leq 9 + 128$$
$$= 137.$$

Thus $F_b \in \mathcal{V}^{\mathcal{H}}$. Moreover, $\sup_{\frac{1}{2}<|b|<1} \|F_a\|_{\mathcal{V}^{\mathcal{H}}} \leq 137$.

Let's consider a sequence $\{z_n\}_{n\in\mathbb{N}}$ within the unit disk such that $|\psi(z_n)| \to 1$ as $n \to \infty$. For $\omega \in \mathbb{D}$, let $F_n := F_{\psi(z_n)}$ be defined as in (3.5), and let $Q = \sup_{n\in\mathbb{N}} \|F_n\|_{\mathcal{V}^{\mathcal{H}}}$. It is obsered that F_n is bounded on $\mathcal{V}^{\mathcal{H}}$ and uniformly converges to 0 on $\overline{\mathbb{D}}$. Moreover, we note that $\lim_{|\omega|\to 1}(1-|\omega|^2)|F_n'''(\omega)| = 0$. So, $F_n \in \mathcal{V}_0^{\mathcal{H}}$. Therefore, by Lemma 3.1, F_n converges weakly to 0 in $\mathcal{V}^{\mathcal{H}}$. Let T be a compact operator on $\mathcal{V}^{\mathcal{H}}$. Then, according to Lemma 2.1

$$\lim_{n\to\infty} \|TF_n\|_{\mathcal{V}^{\mathcal{H}}} = 0.$$

Thus

$$\begin{aligned} Q\|C_\psi - T\|_{\mathcal{V}^{\mathcal{H}}\to\mathcal{V}^{\mathcal{H}}} &\geq \limsup_{n\to\infty} \|(C_\psi - T)F_n\|_{\mathcal{V}^{\mathcal{H}}} \\ &\geq \limsup_{n\to\infty} \|C_\psi F_n\|_{\mathcal{V}^{\mathcal{H}}} - \limsup_{n\to\infty} \|TF_n\|_{\mathcal{V}^{\mathcal{H}}} \\ &= \limsup_{n\to\infty} \|C_\psi F_n\|_{\mathcal{V}^{\mathcal{H}}}. \end{aligned}$$

Hence

$$Q\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \geq \limsup_{n\to\infty} \|C_\psi F_n\|_{\mathcal{V}^{\mathcal{H}}}. \tag{3.6}$$

Therefore

$$\begin{aligned} Q\|C_\psi\|_{e,V} \geq& \limsup_{n\to\infty}(1-|z_n|^2)|\psi'(z_n)^3)\|F_n'''(\psi(z_n))| \\ &- \limsup_{n\to\infty}\Big[(1-|z_n|^2)|\psi'''(z_n)\|F_n'(\psi(z_n))| + 3(1-|z_n|^2)|(\psi'\psi'')(z_n)\|F_n''(\psi(z_n))|\Big]. \end{aligned}$$

Recalling the formulas for F_n', F_n'', and F_n''' following (3.5), we see that
$F_n''(\psi(z_n)) = 0$ and $|F_n'''(\psi(z_n))| = \frac{2}{1-|\psi(z_n)|^2}$. Thus

$$\limsup_{n\to\infty} \frac{2(1-|z_n|^2)|\psi'(z_n)^3|}{1-|\psi(z_n)|^2} \leq Q\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} + \limsup_{n\to\infty}\Big[(1-|z_n|^2)|\psi'''(z_n)\|F_n'(\psi(z_n))|\Big].$$

Since F_n' converge uniformly to 0 on $\overline{\mathbb{D}}$ and

$$(1-|z_n|^2)|\psi'''(z_n)| \leq \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|\psi'''(\omega)| < \infty, \tag{3.7}$$

then

$$\limsup_{n\to\infty} \frac{2(1-|z_n|^2)|\psi'(z_n)^3|}{1-|\psi(z_n)|^2} \leq Q\|C_\psi\|_{e,V_H} \tag{3.8}$$

Again, fix $b \in \mathbb{D}$ such that $1/2 < |b| < 1$ and define the one parameter family of test function G_b as follows. For $\omega \in \mathbb{D}$, let

$$G_b(\omega) = \frac{(1-\overline{b}\omega)^2\Big[15 - 10\log(1-\overline{b}\omega) + 2\log^2(1-\overline{b}\omega)\Big]}{4\overline{b}^2 \log \frac{e}{1-|b|^2}}. \tag{3.9}$$

We have G_b is analytic, and straightforward calculations show that

$$\begin{aligned} G_b'(\omega) &= \frac{(1-\overline{b}\omega)\Big[-5 + 4\log(1-\overline{b}\omega) - \log^2(1-\overline{b}\omega)\Big]}{\overline{b}\log \frac{e}{1-|b|^2}}, \\ G_b''(\omega) &= \Big(\log \frac{e}{1-\overline{b}\omega}\Big)^2 \Big(\log \frac{e}{1-|b|^2}\Big)^{-1}, \\ G_b'''(\omega) &= \frac{2\overline{b}}{1-\overline{b}\omega}\Big(\log \frac{e}{1-\overline{b}\omega}\Big)\Big(\log \frac{e}{1-|b|^2}\Big)^{-1}. \end{aligned}$$

Then

$$\begin{aligned} (1-|\omega|^2)|G_b'''(\omega)| &= (1-|\omega|^2)\frac{2|b|}{|1-\overline{b}\omega|}\Big|\log \frac{e}{1-\overline{b}\omega}\Big|\Big(\log \frac{e}{1-|b|^2}\Big)^{-1} \\ &\leq \frac{2(1-|\omega|^2)}{1-|\omega|}\Big|\log \frac{e}{1-\overline{b}\omega}\Big|\Big(\log \frac{e}{1-|b|^2}\Big)^{-1} \\ &\leq 4\Big(\log \frac{e}{1-|b|^2}\Big)^{-1} \sqrt{\log^2 \frac{e}{|1-\overline{b}\omega|} + \pi^2} \\ &\leq 4\Big(\log \frac{e}{1-|b|^2}\Big)^{-1}\Big(\log \frac{e}{|1-\overline{b}\omega|} + \pi\Big) \\ &\leq 4\Big(\log \frac{e}{1-|b|^2}\Big)^{-1}\Big(\log \frac{e}{1-|b|} + \pi\Big) \\ &\leq 4\Big(\log \frac{e}{1-|b|^2}\Big)^{-1}\Big(\log \frac{2e}{1-|b|^2} + \pi\Big) \\ &= 4\left(1 + \frac{\log 2 + \pi}{\log \frac{e}{1-|b|^2}}\right) \\ &\leq 4(1 + \log 2 + \pi). \end{aligned}$$

Therefore,

$$\sup_{\omega\in\mathbb{D}}(1-|\omega|^2)|G_b'''(\omega)| \leq 4(1+\log 2+\pi).$$

Thus, $G_b \in \mathcal{V}^{\mathcal{H}}$. Moreover,

$$\sup_{1/2<|b|<1} \|G_b\|_{\mathcal{V}^{\mathcal{H}}} < \infty.$$

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in the unit disk such that $|\psi(z_n)| \to 1$ as $n \to \infty$. For $n \in \mathbb{N}$, and $\omega \in \mathbb{D}$, let $G_n = G_{\psi(z_n)}$ as in (3.9), and let $N = \sup_{n\in\mathbb{N}} \|G_n\|_{\mathcal{V}^{\mathcal{H}}}$.

Observing that G_n is a bounded sequence on $\mathcal{V}^{\mathcal{H}}$ and uniformly converges to 0 on $\overline{\mathbb{D}}$, we note additionally that $\lim_{|\omega|\to 1}(1-|\omega|^2)|G_n'''(\omega)| = 0$. Thus, $G_n \in \mathcal{V}_0^{\mathcal{H}}$. Moreover, by Lemma 3.1, G_n converges weakly to 0 in $\mathcal{V}^{\mathcal{H}}$. Assume the operator T be a compact on $\mathcal{V}^{\mathcal{H}}$. Then

$$\lim_{n\to\infty} \|TG_n\|_{\mathcal{V}^{\mathcal{H}}} = 0.$$

It follows that

$$\begin{aligned} N\|C_\psi - T\|_{\mathcal{V}^{\mathcal{H}}\to\mathcal{V}^{\mathcal{H}}} &\geq \limsup_{k\to\infty} \|(C_\psi - T)G_n\|_{\mathcal{V}^{\mathcal{H}}} \\ &\geq \limsup_{n\to\infty} \|C_\psi G_n\|_{\mathcal{V}^{\mathcal{H}}} - \limsup_{n\to\infty} \|TG_n\|_{\mathcal{V}^{\mathcal{H}}} \\ &\geq \limsup_{n\to\infty} \|C_\psi G_n\|_{\mathcal{V}^{\mathcal{H}}}. \end{aligned}$$

Hence

$$N\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \geq \limsup_{n\to\infty} \|C_\psi G_n\|_{\mathcal{V}^{\mathcal{H}}}. \tag{3.10}$$

Therefore

$$\begin{aligned} N\|C_{\theta,\psi}\|_{e,\mathcal{V}^{\mathcal{H}}} \geq &\limsup_{n\to\infty} 3(1-|z_n|^2)|(\psi'\psi'')(z_n)||G_n''(\psi(z_n))| \\ &- \limsup_{n\to\infty} \Big[(1-|z_n|^2)|\psi'''(z_n)||G_n'(\psi(z_n))| + (1-|z_n|^2)|\psi'(z_n)^3||G_n'''(\psi(z_n))|\Big]. \end{aligned}$$

Note that $|G_n''(\psi(z_n))| = \log \frac{e}{1-|\psi(z_n)|^2}$ and $|G_n'''(\psi(z_n))| = \frac{2|\psi(z_n)|}{1-|\psi(z_n)|^2}$. Using the fact that G_n' converges uniformly to 0 on $\overline{\mathbb{D}}$, and by (3.7) and (3.8), we deduce that

$$\begin{aligned} \limsup_{n\to\infty} 3(1-|z_n|^2)|(\psi'\psi'')(z_n)| \log \frac{e}{1-|\psi(z_n)|^2} &\leq N\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} + \limsup_{n\to\infty} \frac{2(1-|z_n|^2)|\psi'(z_n)^3|}{1-|\psi(z_n)|^2} \\ &\preceq \|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}}. \end{aligned} \tag{3.11}$$

As conclusion, by (3.4), (3.8) and (3.11), we obtain

$$\|C_\psi\|_{e,\mathcal{V}^{\mathcal{H}}} \asymp \max\left\{ \limsup_{|\psi(\omega)|\to 1} (1-|\omega|^2)|(\psi'\psi'')(\omega)| \log \frac{e}{1-|\psi(\omega)|^2}, \limsup_{|\psi(\omega)|\to 1} \frac{(1-|\omega|^2)|\psi'(\omega)^3|}{1-|\psi(\omega)|^2} \right\}.$$

□

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References

[1] M. Aljuaid, F. Colonna, Characterizations of Bloch-Type Spaces of Harmonic Mappings, J. Funct. Spaces 2019 (2019), 5687343. https://doi.org/10.1155/2019/5687343.

[2] M. Aljuaid, F. Colonna, On the Harmonic Zygmund Spaces, Bull. Aust. Math. Soc. 101 (2020), 466–476. https://doi.org/10.1017/s0004972720000180.

[3] M. Aljuaid, F. Colonna, Composition Operators on Some Banach Spaces of Harmonic Mappings, J. Funct. Spaces 2020 (2020), 9034387. https://doi.org/10.1155/2020/9034387.

[4] M. Aljuaid, Composition Operators on the Space of Harmonic Mapping $\mathcal{V}^{\mathcal{H}}$, J. Math. Anal. 14 (2023), 25–35. https://doi.org/10.54379/jma-2023-2-3.

[5] C. Pommerenke, J. Clunie, J. Anderson, On Bloch Functions and Normal Functions., J. Reine Angew. Math. 1974 (1974), 12–37. https://doi.org/10.1515/crll.1974.270.12.

[6] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, Springer New York, 2001. https://doi.org/10.1007/978-1-4757-8137-3.

[7] S. Chen, S. Ponnusamy, X. Wang, Landau's Theorem and Marden Constant for Harmonic ν-Bloch Mappings, Bull. Aust. Math. Soc. 84 (2011), 19–32. https://doi.org/10.1017/s0004972711002140.

[8] S. Chen, X. Wang, on Harmonic Bloch Spaces in the Unit Ball of $\mathbb{C}^n$, Bull. Aust. Math. Soc. 84 (2011), 67–78. https://doi.org/10.1017/s0004972711002164.

[9] F. Colonna, The Bloch Constant of Bounded Harmonic Mappings, Indiana Univ. Math. J. 38 (1989), 829–840. https://www.jstor.org/stable/24895370.

[10] F. Colonna, N. Hmidouch, Weighted Composition Operators on Iterated Weighted-Type Banach Spaces of Analytic Functions, Complex Anal. Oper. Theory 13 (2019), 1989–2016. https://doi.org/10.1007/s11785-019-00905-2.

[11] P.L. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.

[12] A.L. Shields, P.L. Duren, B.W. Romberg, Linear Functionals on H^p Spaces with $0 < p < 1$, J. Reine Angew. Math. 238 (1969), 32–60. http://eudml.org/doc/183521.

[13] P.L Duren, Harmonic Mapping in the Plane, Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511546600.

[14] X. Fu, X. Liu, On Characterizations of Bloch Spaces and Besov Spaces of Pluriharmonic Mappings, J. Inequal. Appl. 2015 (2015), 360. https://doi.org/10.1186/s13660-015-0884-0.

[15] A. Kamal, Q-Type Spaces of Harmonic Mappings, J. Math. 2022 (2022), 1342051. https://doi.org/10.1155/2022/1342051.

[16] K. Stroethoff, Harmonic Bergman Spaces, MSRI Publications, (1998).

[17] M. Tjani, Compact Composition Operators on Besov Spaces, Trans. Am. Math. Soc. 355 (2003), 4683–4698. https://doi.org/10.1090/s0002-9947-03-03354-3.

[18] R. Yoneda, A Characterization of the Harmonic Bloch Space and the Harmonic Besov Spaces by an Oscillation, Proc. Edinb. Math. Soc. 45 (2002), 229–239. https://doi.org/10.1017/s001309159900142x.