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Generating a New Family of Distributions for Reliability Control

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Abstract. Recently, many researchers have been searching for a new family of distributions that improves the fit to a specific dataset and has attractive properties. In this paper, we present a flexible method for constructing continuous distributions by controlling their reliability. Our new technique incorporates a single parameter into the chosen continuous distribution, but it is a novel method and independent of any previously used methods. We present two special cases of this family. The new modified family offers significant improvements in fit and applicability in many fields. Furthermore, the maximum likelihood method is used to estimate the parameters in the proposed family. Real data analysis is performed to demonstrate the suitability of the new family.

1. Introduction

In distribution theory, one may be interested in generating new distributions from an old one to guarantee some desirable properties. The novel work on the exponentiated method, given in Mudholkar and Srivastava [1] was on the exponentiated Weibull family with cumulative distribution function (CDF) defined as follows:

$$G(y) = (F(y))^{\alpha},$$

where F(y) is the CDF of the Weibull distribution. Marshall and Olkin [2] presented a new method for expanding a given distribution that led to produce a generalization of the exponential distribution that was competitive with the widely used two-parameter families of lifetime distributions such as the gamma, Weibull, and lognormal distributions. Azzalini [3] introduced a family of skew-normal distributions depending on a new shape parameter. Alzaatreh et al. [4] introduced a method for generating families of continuous probability distributions as follows. Assuming that $u(t)$ is the density function of a random variable $T \in [c, d]$, for $-\infty \leq c < d \leq \infty$ and $V(F(x))$ is

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some function of the cumulative distribution function (CDF) of any random variable X such that $V(F(x))$ satisfies the following conditions:

- $V(F(x)) \in [c, d]$;
- $V(F(x))$ is differentiable and non decreasing;
- $V(F(x)) \to c$ as $X \to -\infty$ and $V(F(x)) \to d$ as $x \to \infty$.

A new family of distributions has CDF defined as:

$$G(x) = \int_{c}^{V(F(x))} u(t)\, dt.$$

Recently, Ahmad and Amleh [5] introduced a novel method for generating new distributions called the $\alpha - R$ family. This technique is supported by the addition of a new shape parameter that expands any old CDF into a more flexible one as follows. If a random variable X has a CDF $F(x)$ and a survival function $R(x)$, then the new CDF family is given by $F_\alpha(x) = 1 - R(x) \cdot \alpha^{F(x)}$, where $\alpha \in (0, e]$. They showed how this method can generate new shapes of distributions that fit (in some cases) better than the original distribution. They also showed that the new family of distributions increases the survival function for the same values of the new parameter α and decreases it for other values of α.

In this article, we introduce a new method for generating probability distributions with the flexibility to increase or decease the reliability function of a random variable by adding a new parameter that changes the shape of the distribution, but does not affect the maximum likelihood estimators (MLEs) of other parameters in the model.

This novel method can be applied to any continuous random variable X easily. The formulation of the new method does not change the original probability density function (PDF) of X, but it divides it into two parts. Consequently, the reliability function of X can be controlled, which leads also to affect the hazard rate function of X.

The rest of the paper is organized as follows. In Section 2, we introduce the new method and derive the related reliability function, hazard rate function, quantile function, the density of the k^{th} order statistic and the Shannon entropy. In Section 3, we apply the new method to two well-known distributions, namely, the exponential and Weibull distributions. In Section 4, the new parameter α is estimated using the maximum likelihood method. In Section 5, a real data set is used to compare the new model with other distributions using different criteria. Finally, the conclusion and rationale of this article are given in Section 6.

2. A New Method for Generating Distributions

In this section, we present a new approach to create a new family of probability distributions. The new family is obtained by expanding the CDF of a given distribution after a certain point and shrinking it before that point, or vice versa. This point is chosen to be the median of the random variable while maintaining the conditions of the CDF. The distribution following this new family will be called the modified distribution by controlling the reliability (MDCR).

Definition 2.1. *Let X be a continuous random variable having CDF $F(x)$, then the CDF of the MDCR is given by:*

$$F_\alpha(x) = \begin{cases} (1-\alpha)F(x), & x \le Q_2 \\ (1+\alpha)F(x) - \alpha, & x > Q_2 \end{cases} \tag{2.1}$$

where Q_2 is the median of X and $\alpha \in [-1, 1]$.

Clearly, when $\alpha = 0$, then $F_o(X) = F(x)$. For $\alpha = 1$, we have $F_1(x) = 2F(x) - 1$ with $x > Q_2$. On the other hand, for $\alpha = -1$, we get $F_{-1}(x) = \begin{cases} 2\,F(x), & x \le Q_2 \\ 1, & x > Q_2 \end{cases}$.

Now, provided that the PDF of X is $f(x)$, the reliability function of X is $R(x)$ and the hazard rate function (HRF) of X is $h(x)$, so the corresponding PDF, reliability function and HRF of the new family are expressed, respectively, as:

$$f_\alpha(x) = \begin{cases} (1-\alpha)f(x), & x \le Q_2 \\ (1+\alpha)f(x), & x > Q_2 \end{cases}, \ -1 \le \alpha \le 1; \tag{2.2}$$

$$R_\alpha(x) = \begin{cases} (1-\alpha)R(x) + \alpha, & x \le Q_2 \\ (1+\alpha)R(x), & x > Q_2 \end{cases}, \ -1 \le \alpha \le 1; \tag{2.3}$$

and

$$h_\alpha(x) = \begin{cases} \frac{(1-\alpha)f(x)}{(1-\alpha)R(x)+\alpha}, & x \le Q_2 \\ h(x), & x > Q_2 \end{cases}, \ -1 \le \alpha \le 1. \tag{2.4}$$

Proposition 2.1. *Let X be a continuous random variable with reliability function $R(x)$, HRF $h(x)$, PDF $f(x)$ and CDF $F(x)$, then the following properties hold:*

(i) $R_\alpha(x) \ge R(x)$ *if* $\alpha \ge 0$, *and* $R_\alpha(x) < R(x)$ *given* $\alpha < 0$.

(ii) If $x \le Q_2$, $h_\alpha(x) \le h(x)$ *for* $\alpha \ge 0$ *and* $h_\alpha(x) > h(x)$ *for* $\alpha < 0$.

(iii) The quantile function of the MDCR is given by:

$$Q_\alpha(p) = \begin{cases} F^{-1}\left(\frac{p}{1-\alpha}\right), & 0 < p \le \frac{1-\alpha}{2} \\ F^{-1}\left(\frac{p+\alpha}{1+\alpha}\right), & \frac{1-\alpha}{2} < p \le 1 \end{cases} \tag{2.5}$$

(iv) The density of the k^{th} order statistic, $1 \le k \le n$, of a sample of size n taken from the MDCR is given by:

$$g_{k,\alpha}(y) = \begin{cases} \frac{n!}{(k-1)!(n-k)!}(1-\alpha)^k[F(y)]^{k-1}[(1-\alpha)R(y)+\alpha]^{n-k}f(y), & y \le Q_2 \\ \frac{n!}{(k-1)!(n-k)!}[(1+\alpha)F(y)-\alpha]^{k-1}(1+\alpha)^{n-k+1}[R(y)]^{n-k}f(y), & y > Q_2 \end{cases} \tag{2.6}$$

Proof. (i) Let $0 \le \alpha \le 1$. Then for $x \le Q_2$, we have:

$$R_\alpha(x) = (1-\alpha)R(x) + \alpha$$

$$R_\alpha(x) = R(x) + \alpha F(x).$$

Since $\alpha F(x) \geq 0$, then

$$R_\alpha(x) \geq R(x).$$

For $x > Q_2$, we have:

$$0 \leq \alpha \leq 1$$

$$\Rightarrow 1 \leq \alpha + 1 \leq 2$$

$$\Rightarrow R(x) \leq (1+\alpha)R(x) \leq 2R(x)$$

$$\therefore R(x) \leq R_\alpha(x)$$

Thus, for $0 \leq \alpha \leq 1$, we have:

$$R_\alpha(x) \geq R(x), \forall x$$

Now, for $-1 \leq \alpha \leq 0$, we have:

$$0 \leq (1+\alpha) < 1$$

$$0 \leq (1+\alpha)R(x) < R(x)$$

$$\therefore R_\alpha(x) < R(x), \text{ for } x < Q_2$$

Also, $R_\alpha(x) = R(x) + \alpha F(x)$ for $x \leq Q_2$, but $\alpha F(x) < 0$, and hence $R_\alpha(x) < R(x)$.

That is, for $-1 \leq \alpha < 0$, $R_\alpha(x) < R(x), \ \forall x$. □

(ii) For $x \leq Q_2$, we have:

$$h_\alpha(x) = \frac{(1-\alpha)f(x)}{(1-\alpha)R(x)+\alpha} = \frac{(1-\alpha)f(x)}{R_\alpha(x)}.$$

If $0 \leq \alpha \leq 1, \ R_\alpha(x) \geq R(x)$

$$\Rightarrow \frac{1}{R_\alpha(x)} \leq \frac{1}{R(x)}$$

$$\Rightarrow \frac{(1-\alpha)f(x)}{R_\alpha(x)} \leq \frac{(1-\alpha)f(x)}{R(x)} \leq \frac{f(x)}{R(x)}$$

$$\therefore h_\alpha(x) \leq h(x).$$

If $-1 \leq x < 0 \Rightarrow 1 < 1-\alpha \leq 2$, and

$$R_\alpha(x) < R(x)$$

$$\frac{1}{R_\alpha(x)} > \frac{1}{R(x)}$$

$$\frac{(1-\alpha)f(x)}{R_\alpha(x)} > \frac{(1-\alpha)f(x)}{R(x)} > \frac{f(x)}{R(x)}$$

$$\therefore h_\alpha(x) > h(x)$$

□

(iii) For $x \leq Q_2 \Rightarrow F(x) \leq \frac{1}{2}$, but $F_\alpha(x) = (1-\alpha)F(x)$. If $0 \leq \alpha \leq 1 \Rightarrow$

$$0 \leq 1-\alpha \leq 1$$
$$0 \leq F(x) \leq \frac{1}{2}$$
$$0 \leq (1-\alpha)F(x) \leq \frac{1-\alpha}{2}$$
$$0 \leq F_\alpha(x) \leq \frac{1-\alpha}{2}$$
$$0 \leq p \leq \frac{1-\alpha}{2}.$$

If $-1 \leq \alpha < 0 \Rightarrow$

$$1 < 1-\alpha \leq 2$$
$$0 \leq F(x) \leq \frac{1}{2}$$
$$0 \leq (1-\alpha)F(x) \leq \frac{1-\alpha}{2}$$
$$0 \leq F_\alpha(x) \leq \frac{1-\alpha}{2}$$
$$\therefore 0 \leq p \leq \frac{1-\alpha}{2}.$$

For $x > Q_2 \Rightarrow 1 \geq F(x) > \frac{1}{2}$. If $0 \leq \alpha \leq 1 \Rightarrow$

$$0 \leq 1-\alpha \leq 1$$
$$\frac{1}{2} < F(x) \leq 1$$
$$\frac{1-\alpha}{2} < (1-\alpha)F(x) \leq 1-\alpha \leq 1$$
$$\frac{1-\alpha}{2} < F_\alpha(x) \leq 1$$
$$\frac{1-\alpha}{2} < p \leq 1.$$

If $-1 \leq \alpha < 0 \Rightarrow$

$$1 < 1-\alpha \leq 2$$
$$\frac{1-\alpha}{2} < (1-\alpha)F(x) \leq 1-\alpha \leq 1$$
$$\frac{1-\alpha}{2} < p \leq 1.$$

(iv) The proof of this part is straightforward.

□

Moving on to another aspect, the Shannon entropy of a continuous random variable X is given explicitly as:

$$S(x) = -E[\ln(f(x)], \tag{2.7}$$

where f(x) is the pdf of X. Here, we examine the relationship between the Shannon entropies of the original and modified distributions.

Theorem 2.1. *If the Shannon entropy of the original random variable X is $S(x)$, then the Shannon entropy of the MDCR is provided as:*

$$S_\alpha(x) = S(x) - \frac{1}{2}\ln(1-\alpha^2),\ -1 \le \alpha \le 1. \tag{2.8}$$

Proof.

$$\begin{aligned}
S_\alpha(x) &= -\int_{-\infty}^{\infty} \ln(f_\alpha(x)) f_\alpha(x)\, dx \\
&= -\int_{-\infty}^{Q_2} \ln[(1-\alpha)f(x)] f(x)\, dx - \int_{Q_2}^{\infty} \ln[(1+\alpha)f(x)] f(x)\, dx \\
&= -\frac{\ln(1-\alpha)}{2} - \int_{-\infty}^{Q_2} \ln(f(x)) f(x)\, dx - \frac{\ln(1+\alpha)}{2} - \int_{Q_2}^{\infty} \ln(f(x)) f(x)\, dx \\
&= -\frac{1}{2}\ln(1-\alpha^2) - \left[\int_{-\infty}^{Q_2} \ln(f(x)) f(x)\, dx + \int_{Q_2}^{\infty} \ln(f(x)) f(x)\, dx\right] \\
&= S(x) - \frac{1}{2}\ln(1-\alpha^2)
\end{aligned}$$

□

Corollary 2.1. $S_\alpha(x) \ge S(x)$ *for all* $\alpha \in [-1, 1]$.

3. Special Cases

Now, we present examples of the proposed modified family applied on different standard distributions, namely for exponential and Weibull distributions.

3.1. A Modified Exponential Distribution.

If Y has an exponential distribution having CDF given by:

$$H(y) = 1 - e^{-\theta y},\ y > 0,\ \theta > 0.$$

So, the CDF of the modified exponential distribution by controlling the reliability (*MEDCR*) is given by:

$$G(y) = \begin{cases} (1-\alpha)(1-e^{-\theta y}), & 0 < y \le \frac{\ln 2}{\theta} \\ 1-(1+\alpha)e^{-\theta y}, & y > \frac{\ln 2}{\theta} \end{cases},\ -1 \le \alpha \le 1,\ \theta > 0.$$

The corresponding PDF of the *MEDCR* is given by:

$$g(y) = \begin{cases} (1-\alpha)\,\theta\, e^{-\theta y}, & 0 < y \le \frac{\ln 2}{\theta} \\ (1+\alpha)\,\theta\, e^{-\theta y}, & y > \frac{\ln 2}{\theta} \end{cases}\quad -1 \le \alpha \le 1,\ \theta > 0.$$

Here, we denote the modified distribution as $MEDCR(\alpha,\ \theta)$. The quantile function of the $MEDCR(\alpha,\ \theta)$ is defined as:

$$Q(p) = \begin{cases} \ln\left(\frac{1-\alpha}{1-\alpha-p}\right)^{\frac{1}{\theta}}, & 0 \le p \le \frac{1-\alpha}{2} \\ \ln\left(\frac{1+\alpha}{1-p}\right)^{\frac{1}{\theta}}, & \frac{1-\alpha}{2} < p \le 1 \end{cases}.$$

Proposition 3.1. *Let* $Y \sim MEDCR(\alpha,\ \theta)$*, then:*

a. $E(Y) = \frac{1+\alpha \ln 2}{\theta}$.
b. $Var(Y) = \frac{1+(\alpha-\alpha^2)(\ln 2)^2}{\theta^2}$.
c. The Shannon entropy of Y *is obtained as:*

$$S(y) = 1 - \ln(\theta\sqrt{1-\alpha^2}).$$

Proof. a. $f(y) = \theta e^{-\theta y},\ y > 0$, so

$$f_\alpha(y) = \begin{cases} (1-\alpha)\,\theta\, e^{-\theta y}, & 0 < y \le Q_2 \\ (1+\alpha)\,\theta\, e^{-\theta y}, & Q_2 < y \end{cases}$$

$$\begin{aligned} E(Y) &= \int_0^{Q_2} (1-\alpha)\theta y e^{-\theta y} dy \ + \int_{Q_2}^{\infty} (1+\alpha)\theta y e^{-\theta y} dy \\ &= (1-\alpha)\theta\left[\frac{ye^{-\theta y}}{-\theta} - \frac{e^{-\theta y}}{\theta^2}\right]_0^{Q_2} + (1+\alpha)\theta\left[\frac{ye^{-\theta y}}{-\theta} - \frac{e^{-\theta y}}{\theta^2}\right]_{Q_2}^{\infty} \\ &= (1-\alpha)\theta\left[\frac{-Q_2e^{-\theta Q_2}}{\theta} - \frac{e^{-\theta Q_2}}{\theta^2} + \frac{1}{\theta^2}\right] + (1+\alpha)\theta\left[\frac{Q_2e^{-\theta Q_2}}{\theta} + \frac{e^{-\theta Q_2}}{\theta^2}\right] \\ &= \frac{1-\alpha}{\theta} - (1-\alpha)\left(Q_2 + \frac{1}{\theta}\right)e^{-\theta Q_2} + (1+\alpha)\left(Q_2 + \frac{1}{\theta}\right)e^{-\theta Q_2} \\ &= (1-\alpha)\theta + \left(Q_2 + \frac{1}{\theta}\right)e^{-\theta Q_2}(2\alpha), \end{aligned}$$

but

$$\begin{aligned} 1 - e^{-\theta Q_2} &= \frac{1}{2} \\ e^{-\theta Q_2} &= \frac{1}{2} \\ -\theta Q_2 &= \ln\left(\frac{1}{2}\right) \\ Q_2 &= \frac{\ln(2)}{\theta} \end{aligned}$$

So,

$$\begin{aligned} E(Y) &= \frac{1-\alpha}{\theta} + \left(\frac{\ln(2)+1}{\theta}\right)(2\alpha)\left(\frac{1}{2}\right) \\ &= \frac{1-\alpha+\alpha+\alpha\ln(2)}{\theta} \\ &= \frac{1+\alpha\ln(2)}{\theta} \end{aligned}$$

□

b. $Var(Y) = E(Y^2) - (E(Y))^2$

$$\begin{aligned} E(Y^2) &= \int_0^{Q_2} (1-\alpha)\theta y^2 e^{-\theta y} dy \ + \int_{Q_2}^{\infty} (1+\alpha)\theta y^2 e^{-\theta y} dy \\ &= (1-\alpha)\left[\left(\frac{-y^2}{\theta} - \frac{2y}{\theta^2} - \frac{2}{\theta^3}\right)e^{-\theta y}\right]_0^{Q_2} + (1+\alpha)\theta\left[\left(\frac{-y^2}{\theta} - \frac{2y}{\theta^2} - \frac{2}{\theta^3}\right)e^{-\theta y}\right]_{Q_2}^{\infty} \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha)\theta\left[\frac{2}{\theta^3}-\left(\frac{Q_2^2}{\theta}+\frac{2Q_2}{\theta^2}+\frac{2}{\theta^3}\right)e^{-\theta Q_2}\right]+(1+\alpha)\theta\left[\frac{Q_2^2}{\theta}+\frac{2Q_2}{\theta^2}+\frac{2}{\theta^3}\right]e^{-\theta Q_2}\\
&= \frac{2(1-\alpha)}{\theta^2}+\left(\frac{Q_2^2}{\theta}+\frac{2Q_2}{\theta^2}+\frac{2}{\theta^3}\right)e^{-\theta Q_2}\theta\left[(1+\alpha)-(1-\alpha)\right]\\
&= \frac{2(1-\alpha)}{\theta^2}+\left(Q_2^2+\frac{2Q_2}{\theta}+\frac{2}{\theta^2}\right)e^{-\theta Q_2}(2\alpha)\\
&= \frac{2(1-\alpha)}{\theta^2}+(2\alpha)\left(\frac{1}{2}\right)\left(\frac{\ln^2(2)}{\theta^2}+\frac{2\ln(2)}{\theta^2}+\frac{2}{\theta^2}\right)\\
&= \frac{2-2\alpha+\alpha(\ln^2(2)+2\ln(2)+2)}{\theta^2}\\
Var(Y) &= \frac{2-2\alpha+\alpha(\ln^2(2)+2\ln(2)+2)-(1+\alpha\ln(2))^2}{\theta^2}\\
&= \frac{2-2\alpha+\alpha(\ln^2(2)+2)-(1+2\alpha\ln(2)+\alpha^2\ln^2(2))}{\theta^2}\\
&= \frac{1+(\ln(2))^2(\alpha-\alpha^2)}{\theta^2}
\end{aligned}
$$

□

3.2. **A Modified Weibull Distribution.** One of the most widely used models in reliability engineering is the Weibull distribution, see [6] and [7]. Now, assuming that Y has a Weibull distribution having CDF:

$$H(y) = 1 - e^{-\lambda y^{\beta}},\ y > 0,\ \lambda, \beta > 0.$$

Therefore, the CDF of the modified Weibull distribution by controlling the reliability (*MWDCR*) under the proposed family is given by:

$$G(y) = \begin{cases} (1-\alpha)(1-e^{-\lambda y^{\beta}}), & 0 < y \le [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \\ 1-(1+\alpha)\,e^{-\lambda y^{\beta}}, & y > [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \end{cases},\ -1 \le \alpha \le 1,\ \lambda > 0,\ \beta > 0. \tag{3.1}$$

The PDF corresponding to Eq.(3.1) is defined as:

$$g(x) = \begin{cases} (1-\alpha)\,\beta\,\lambda\,x^{\beta-1}e^{-\lambda x^{\beta}}, & 0 < x \le [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \\ (1+\alpha)\,\beta\,\lambda\,x^{\beta-1}e^{-\lambda x^{\beta}}, & x > [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \end{cases},\ -1 \le \alpha \le 1,\ \lambda > 0,\ \beta > 0. \tag{3.2}$$

The quantile function of the *MWDCR* is given as:

$$Q(p) = \begin{cases} \left[\frac{1}{\lambda}\ln\left(\frac{1-\alpha}{1-\alpha-p}\right)\right]^{\frac{1}{\beta}}, & 0 \le p \le \frac{1-\alpha}{2} \\ \left[\frac{1}{\lambda}\ln\left(\frac{1+\alpha}{1-p}\right)\right]^{\frac{1}{\beta}}, & \frac{1-\alpha}{2} < p \le 1 \end{cases}. \tag{3.3}$$

Such new distribution will be denoted as $MWDCR(\alpha,\ \lambda,\ \beta)$. The flexibility of *MWDCR* can be observed in Figure 1, which displays PDF curves of the new distribution. Further, Figure 2 displays the CDF curves of the *MWDCR* for different parameter values.

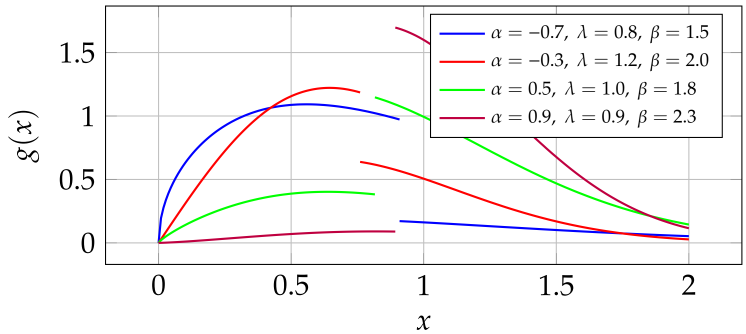

Figure 1. The PDF plots for different values of $\alpha,\ \lambda,$ and β

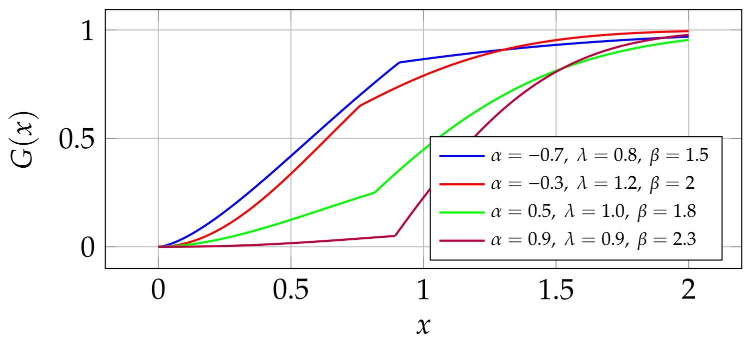

Figure 2. The CDF curves for different values of $\alpha,\ \lambda,$ and β

Proposition 3.2. *Let $Y \sim MWDCR(\alpha,\ \lambda,\ \beta)$, then:*

a. The reliability function of Y is given by:

$$R(y) = \begin{cases} (1-\alpha)\, e^{-\lambda y^{\beta}} + \alpha, & 0 < y \leq [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \\ (1+\alpha)\, e^{-\lambda y^{\beta}}, & y > [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \end{cases}.$$

b. The HRF of Y is defined as:

$$z(y) = \begin{cases} \frac{(1-\alpha)\,\lambda\,\beta\, y^{\beta-1}\, e^{-\lambda y^{\beta}}}{(1-\alpha)\, e^{-\lambda y^{\beta}}+\alpha}, & 0 < y \leq [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \\ \lambda\,\beta\, y^{\beta-1}, & y > [\frac{1}{\lambda}\ln 2]^{\frac{1}{\beta}} \end{cases}.$$

c.

$$E(Y) = \frac{(1-\alpha)}{\sqrt[\beta]{\lambda}}\,\gamma\left(\frac{1}{\beta}+1,\ \ln 2\right) + \frac{(1+\alpha)}{\sqrt[\beta]{\lambda}}\,\Gamma\left(\frac{1}{\beta}+1,\ \ln 2\right), \tag{3.4}$$

where $\gamma(s,x) = \int_0^x w^{s-1}\, e^{-w}\, dw$ represents the lower incomplete gamma function while $\Gamma(s,x) = \int_x^\infty w^{s-1}\, e^{-w}\, dw$ is the upper incomplete gamma function.

4. Parameter Estimation

Here, we discuss the maximum likelihood method for estimating the original parameters of the baseline model in addition to the new parameter α.

Assuming that we have a random sample of the MDCR as described in Eq.(2.1) and Eq.(2.2), we may write this random sample as:

$$Y_1,\ Y_2,\ ...,\ Y_{n_1},\ Y_{n_1+1},\ Y_{n_1+2},\ ...,\ Y_n, \tag{4.1}$$

where n_1 represents the number of random variables in the sample in such a way that $Y_i \leq Q_2$, $i = 1,\ 2,\ 3,\ ...,\ n_1$. Moreover, let n_2 denote the number of random variables of the sample such that $Y_1 > Q_2$, therefore, we get $n_1 + n_2 = n$.

Theorem 4.1. *Assume that $Y_1,\ Y_2,\ ...,\ Y_n$ is a random sample following a MDCR with PDF $f_\alpha(y)$, then:*

(i) The MLE of α, $\hat{\alpha}$, is given by:

$$\hat{\alpha} = \frac{n_2 - n_1}{n} \tag{4.2}$$

(ii) If $\hat{\theta}$ is the MLE for any parameter θ of the original family, then $\hat{\theta}$ is also the MLE of θ regarding the corresponding new family.

Proof. (i) We may write the likelihood function of the above sample as:

$$\begin{aligned} L(\alpha) &= \prod_{i=1}^{n} f_\alpha(x_i) \\ &= \binom{n}{n_1}(1-\alpha)^{n_1}(1+\alpha)^{n_2}\prod_{i=1}^{n} f(x_i), \end{aligned} \tag{4.3}$$

where $f(x)$ represents the PDF of the baseline distribution. The log $-$ likelihood function corresponding to (15) is given as:

$$l(\alpha) \propto\ n_1 \ln(1-\alpha) + n_2 \ln(1+\alpha) + \sum_{i=1}^{n} \ln f(x_i) \tag{4.4}$$

Differentiating Eq.(4.4) partially with respect to α and equating the result to zero gives:

$$\frac{\partial l}{\partial \alpha} = -\frac{n_1}{1-\alpha} + \frac{n_2}{1+\alpha} = 0$$

Hence, we get:

$$\hat{\alpha} = \frac{n_2 - n_1}{n}.$$

□

(ii) It can be shown, using the log-likelihood function obtained above, that

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \ln f(x_i),$$

□

which ends the proof.

Corollary 4.1. *If $\hat{\theta}$ is the UMVUE for any parameter θ of a given distribution, then $\hat{\theta}$ is also the UMVUE of θ for the MDCR.*

5. Application

To demonstrate the effectiveness of the MDCR proposed in this article and its superiority over the original family, we perform a comparison study using some data from previous studies. In particular, we use the *MWDCR* model, which was presented in Section 3. Furthermore, in order to evaluate how well the proposed distribution compares to some other competing distributions, we use different measures of model fit, such as the Kolmogorov-Smirnov statistic (KSS) along with its p- value, Anderson-Darling (AD) and Cramer-von Mises (CVM), in addition to other measures such as Bayesian information (BI) and Akaike information (AI). All measures of model fit are considered when comparing the fit of all models. We use the R software and the MLE method to estimate the parameters of the specified distributions and evaluate the model fit measures.

The goodness of fit of the proposed distribution is compared with the following distributions: Standard Weibull distribution [8], with CDF:

$$F_{Wei}(y) = 1 - e^{-\lambda y^{\beta}}, \; y > 0, \; \lambda > 0, \; \beta > 0.$$

Exponentiated Weibull distribution [1], with PDF:

$$F_{EW}(y) = \left(1 - e^{-\lambda y^{\beta}}\right)^{\theta}, \; y > 0, \; \lambda > 0, \; \beta > 0, \; \theta > 0.$$

The data set represents the lifetimes of 72 guinea pigs infected with virulent tuberculosis bacilli. These data have previously been used by several authors; for example [9]. The times are reported as follows:

Data

0.10	0.33	0.44	0.56	0.59	0.72	0.74	0.77	0.92
0.93	0.96	1.00	1.00	1.02	1.05	1.07	1.07	1.08
1.08	1.09	1.12	1.13	1.15	1.16	1.20	1.21	1.22
1.22	1.24	1.30	1.34	1.36	1.39	1.44	1.46	1.53
1.59	1.60	1.63	1.63	1.68	1.71	1.72	1.76	1.83
1.95	1.96	1.97	2.02	2.13	2.15	2.16	2.22	2.30
2.31	2.40	2.45	2.51	2.53	2.54	2.54	2.78	2.93
3.27	3.42	3.47	3.61	4.02	4.32	4.58	5.55	

Table 1. K-S, p-values, AI, CVM, BI and AD of the fitted distributions

Distribution	**KSS**	***p*-value**	**AI**	**CVM**	**BI**	**AD**
MWDCR	0.0844	0.7144	187.58	0.0632	194.41	0.3957
Weibull	0.1048	0.4079	195.59	0.1679	200.13	1.0068
Exponentiated Weibull	0.0891	0.6599	194.17	0.0893	200.99	0.538

The results are given in Table 1. It can be seen that among all distributions that have been applied to the dataset, *MWDCR* shows the lowest values for the KSS, AI, CVM, BI and AD statistics. In addition, the associated $p - value$ is the largest value among the other distributions.

Consequently, the *MWDCR* distribution is more adequate to fit this real data.

In Table 2, we present the MLEs of the parameters of the *MWDCR* and the other existing distributions with their corresponding standard error.

Table 2. The Parameter Estimates and Standard Errors of the distributions considered.

Distribution	Estimates of the parameters	Std Error
MWDCR	$\alpha = 0.722$	0.0873
	$\lambda = 0.729$	0.0015
	$\beta = 1.221$	0.0016
Weibull	$\lambda = 0.2832$	0.1363
	$\beta = 1.8255$	0.1587
Exponentiated Weibull	$\lambda = 0.8684$	0.4457
	$\beta = 1.161$	0.3095
	$\theta = 2.653$	1.5388

Figure 3 and Figure 4 show the PDF and CDF of the *MWDCR* distribution along with the alternative models in the data set under study, respectively. Based on the plotted data, it can be clearly noticed that the *MWDCR* model provides a better applicability to the data than the other distributions.

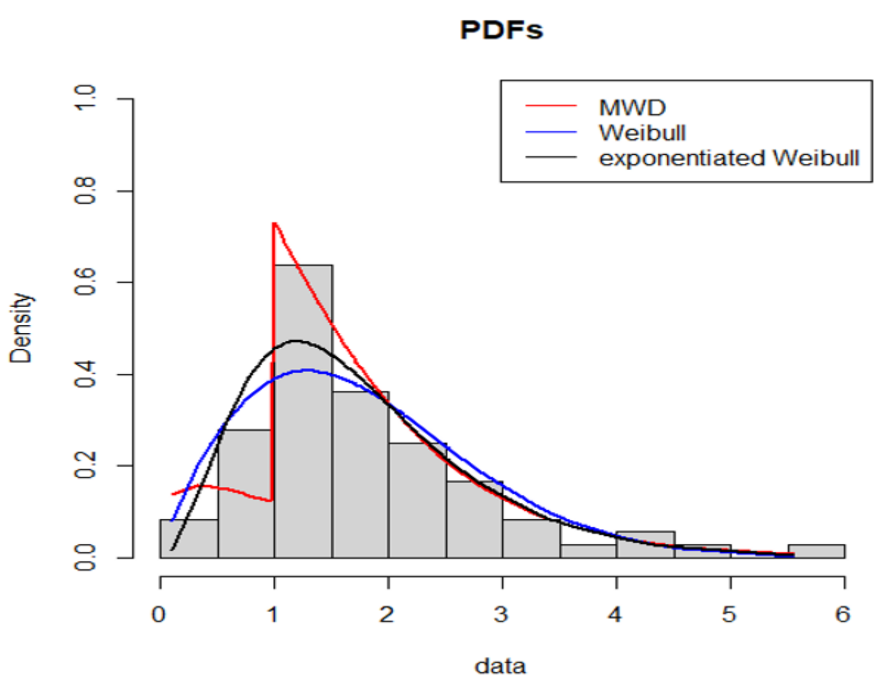

Figure 3. Plots of the estimated PDFs with the data set.

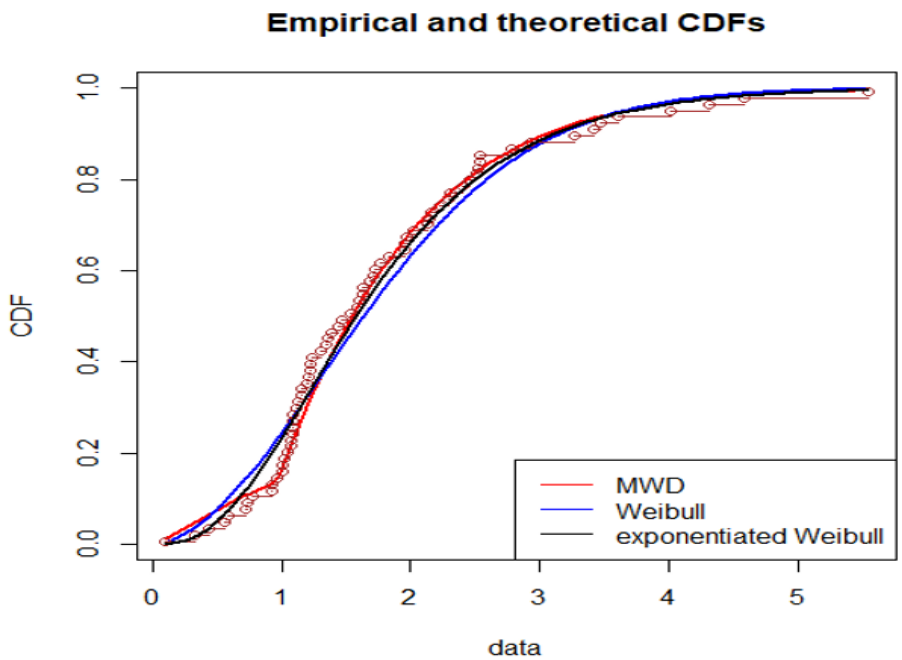

Figure 4. The CDFs for the four distribution functions.

6. Conclusion

We presented a novel generalized family of univariate distributions called MDCR, a family that was not constructed using any previous technique. This modified family relies on increasing or decreasing the reliability function by adding a new parameter that changes the shape of the distribution without affecting the MLE of the original parameters. A real data set was used to demonstrate the effectiveness of one special case of the MDCR and its superiority over other modified distributions. Further research can be conducted using the same sense, including multivariate expansion and applying this family to other distributions and in more diverse fields.

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