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# Exploring $\mathfrak{F}$-Khan-Contraction with Mann's Iterative Scheme in Convex $S_b$-Metric Spaces

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**Abstract.** This manuscript presents the concept of $S_b$ metric space for Mann's iteration scheme, which extends the notion of $b$-metric, $G_b$-metric and $S$-metric spaces, respectively. We begin by introducing some improved and interesting properties, specifically regarding the concepts of symmetric and nonsymmetric within the context of $S_b$-metric space provided by examples. Additionally, we expand the notation of convex $S_b$-metric space through a convex Mann's iteration algorithm. Furthermore, we display numerous outcomes of this new type of notion in the literature, with a particular focus on rational-Khan contractions and Wardowski-type contractions. The aim is to establish fixed-point results, accompanied by examples that clarify our findings. Finally, we provide applications to mixed Volterra-Fredholm integral and polynomial equations to support our theorems.

## 1. Introduction

Fixed-point theorems (FPT) represent a significant branch of function analysis with extensive applications, playing a crucial role in nonlinear analysis. Initially, Banach [1] established a highly consequential theorem in 1922 regarding the existence and uniqueness of an FPT in complete metric spaces, for a comprehensive concept termed the "Banach contractive principle". This marked the inception of efforts to expand his theorem, through either generalisations of metric spaces (MS) or improvements of contractions. We also demonstrate an iterative scheme to find the fixed point of a mapping, employing various contractions to provide the existence of solutions for technical model applications. Significant generalisations of MS include b-MS introduced by Aloqaily [2], Bakhtin [3] and Czerwik [4]. Mustafa et al. [5] present an extension of the defined MS, referred to as a $G$-MS. Then, Aghajani et al. [6] presented the $G_b$-MS, which is an extension of $G$-MS and $b$-MS.

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On the other side, Nizar et al. [7] named $S_b$-metric space ($S_b$-MS), whereas $S_b$-MS most generalises the previous notions. A lot of FPT were focused in $S_b$-MS (see, [18,19,21,36]).

Recently, Chen et al. [8] introduced the idea of convex $b$-MS and confirmed some fixed-point (FP) results along with it; also, Gehad M. et al. studied this [9] with different types of Khan contractions. Subsequently, several generalisations of convex $b$-MS were presented, such as convex $G_b$-MS, due to Dong Ji et al. [10], convex $G_b$-MS with $\mathfrak{F}$-contraction, which were initiated by Amna Naz et al. [11]. Before that, since iterative schemes play an important role in finding solutions to FPT problems, Mann [12] presented the Mann iterative scheme for approximating FP mapping, which novelly replaced the Picard iterative scheme of BCP. In 2022, Yildirim [13] redefined the concept of Mann's iterative scheme for FP results. Ji et al. [10] extended Mann's iterative scheme to the new space convex $G_b$-MS. Later on, Naz et al. employed [11] Mann's iterative scheme via $\mathfrak{F}$-contraction. A lot of results have focused on the concepts of convexity in Mann's iterative scheme, Karahan et al. [14], also see [15–17].

In the current paper, we modified the $S_b$-MS by providing some properties endowed with the notation of symmetric and nonsymmetric, generalising all previous to $CS_b$-MS by using Wardowski-contraction equipped with Mann's iterative scheme. Some results are obtained using Khan rational-contraction as a special case. The main goal is to manifest an FPT involving a Khan type of $\mathfrak{F}$-contraction of Mann's iterative scheme, focusing on the results of $CS_b$-MS with examples. It also provides special relationships for convergence sequences. Finally, the paper introduces applications of mixed Volterra-Fredholm integral equations and the $m$th polynomial equation, which support our FPT on these new spaces.

## 2. Preliminaries

This section provides definitions of some basic concepts of $S_b$-MS.

**Definition 2.1.** *[3,4] Let $\mathfrak{D}$ be a nonempty set and $\mu \geq 1$. The mapping $d_b : \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ is said to be a b-metric on $\mathfrak{D}$ if for all $a, b, c \in \mathfrak{D}$ are satisfied:*

(1) $d_b(a, b) = 0$ *if and only if* $a = b$,
(2) $d_b(a, b) = d_b(b, a)$,
(3) $d_b(a, b) \leq \mu \left[d_b(a, c) + d_b(c, b)\right]$.

*The pair $(\mathfrak{D}, d_b)$ is called a b-metric space (bMS).*

**Definition 2.2.** *[6] Let $\mathfrak{D}$ be a nonempty set and $\mathfrak{G} : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ be a mapping, which satisfies the following properties for all $a, b, c \in \mathfrak{D}$:*

(1) $\mathfrak{G}(a, b, c) = 0$ *if* $a = b = c$,
(2) $\mathfrak{G}(a, a, b) > 0$, *for each* $a, b \in \mathfrak{D}$ *and* $a \neq b$,
(3) $\mathfrak{G}(a, a, b) \leq \mathfrak{G}(a, b, c)$, *for each* $a, b, c \in \mathfrak{D}$ *and* $b \neq c$,
(4) $\mathfrak{G}(a, b, c) = \mathfrak{G}(a, c, b) = \cdots$,
(5) $\mathfrak{G}(a, b, c) \leq \mu \left[\mathfrak{G}(a, t, t) + \mathfrak{G}(t, b, c)\right]$, *for each* $a, b, c, t \in \mathfrak{D}$ *and* $\mu \geq 1$.

*The pair $(\mathfrak{D}, \mathfrak{G})$ is called a $G_b$-metric space (GbMS). Clearly, GbMS and bMS are equivalent topologically.*

Next, Souayah et al. [7] initiated several generalisations of bMS below, and differentiating it from GbMS, named it a $S_b$-MS.

**Definition 2.3.** *[7] Let $\mathfrak{D}$ be a nonempty set and $\mu \geq 1$ be a given real number. A mapping $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ is said to be a $S_b$-metric if for all $a, b, c \in \mathfrak{D}$, it satisfied the following:*

(S1) $S_b(a, b, c) = 0$ *if and only if* $a = b = c$,

(S2) $S_b(a, b, c) \leq \mu [S_b(a, a, t) + S_b(b, b, t) + S_b(c, c, t)]$, *for each* $a, b, c, t \in \mathfrak{D}$.

*The pair $(\mathfrak{D}, S_b)$ is called an $S_b$-metric space ($S_b$-MS). Also, we say that $(\mathfrak{D}, S_b)$ is an S-MS with $\mu = 1$. For more, see Refs. [18–20].*

**Example 2.1.** *Let $(\mathfrak{D}, d_b)$ be a bMS and define $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ by*

(1) $S_b(a, b, c) = d_b(a, b) + d_b(b, c) + d_b(a, c)$,

(2) $S_b(a, b, c) = d_b(a, c) + d_b(b, c)$,

(3) $S_b(a, b, c) = [S_1(a, b, c)]^p, p \geq 1$,

*for any $a, b, c \in \mathfrak{D}$. Then it can be easily seen that $S_b$ is an $S_b$-MS on $\mathfrak{D}$.*

**Remark 2.1.** *Inspired by [18], the $S_b$-MS $S_b$ is termed as symmetric if "$S_b(a, a, c) = S_b(c, c, a)$" for any $a, c \in \mathfrak{D}$. In addition, from Example 2.1, we note that all cases are symmetric.*

**Example 2.2.** *[21] Let $\mathfrak{D} = \mathbb{R}$ and the mapping $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ be defined by*
$S_b(0, 0, 1) = 2, S_b(1, 1, 0) = 4$,
$S_b(a, b, c) = 0$ *if* $a = b = c$, $S_b(a, b, c) = 1$ *otherwise, for all $a, b, c \in \mathfrak{D}$. Then the mapping $S_b$ is an $S_b$-MS with $\mu \geq 2$, which is not symmetric.*

In the following example, we show that if we assume that $d_b$ is a quasi-MS, we obtain a non-symmetric case.

**Example 2.3.** *Let $(\mathfrak{D}, d)$ be a quasi-MS, $\mathfrak{D} = \mathbb{N} \cup \{\infty\}$ and define $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ by $S_b(a, b, c) = (d(a, c) + d(b, c))^p, p \geq 1$, where*

$$d(a, c) = \begin{cases} 0 & \text{if all } a = c \\ 0 & \text{if } a \in \mathfrak{D}, c = \infty \\ \frac{1}{c} & \text{if } c \in \mathfrak{D}, a = \infty \\ \frac{1}{a} & \text{otherwise,} \end{cases}$$

*for any $a, b, c \in \mathfrak{D}$. Then it is not difficult that we see $S_b$ is an $S_b$-MS on $\mathfrak{D}$ with $\mu = 2^{2(p-1)}$, but this is not symmetric as $d_b(a, c) \neq d_b(c, a)$.*

**Lemma 2.1.** *[22] Let $(\mathfrak{D}, P)$ be a partial MS. Assume $\bar{P}(a, b) = P(a, b) - (\eta P(a, a) + \zeta P(b, b))$, where $0 \leq \eta + \zeta \leq 1$ and $\eta + \zeta = 1$. Then the following statements always hold:*

(1) $\bar{P}$ *is a quasi-metric,*

(2) $\bar{P}$ *is a metric if and only if* $\eta + \zeta = \frac{1}{2}$,

(3) $\bar{q}$ *is a metric, where* $\bar{q} = \max\{\bar{P}(a,b), \bar{P}(b,a)\}$.

**Remark 2.2.** *Note that we can take any approach of MS to generate the* $S_b$*-MS as well as 3 items above in Lemma 2.1, the space being symmetric or non-symmetric.*

In the following lemma, Sedghi et al. [23] provided some properties of $S_b$-MS. We show with modified cases or generalisations.

**Lemma 2.2.** *[23] In an* $S_b$*-MS, where* $S_b$ *is non-symmetric, we deduce whether* $S_b(a,a,c) \leq \mu S_b(c,c,a)$ *or (not and)* $S_b(c,c,a) \leq \mu S_b(a,a,c)$*. See Example 2.3, in which* $S_b(0,0,1) = 2 \leq 2(S_b(1,1,0) = 4)$*, but the reverse is not always true.*

**Lemma 2.3.** *[23] Let* $(\mathfrak{D}, S_b)$ *be an* $S_b$*-MS. Then*

(1) *If* $S_b$ *is symmetric, then* $S_b(a,a,c) \leq 2\mu S_b(a,a,b) + \mu S_b(b,b,c)$,

(2) *If* $S_b$ *is non-symmetric, then* $S_b(a,a,c) \leq 2\mu S_b(a,a,b) + \mu^2 S_b(b,b,c)$.

The notion of convergence and Cauchy sequences is introduced as in the case of $S_b$-MS.

**Definition 2.4.** *[23] Let* $(\mathfrak{D}, S_b)$ *be an* $S_b$*-MS. Let* $\{z_n\}_{n\geq 0}$ *be a sequence in* $\mathfrak{D}$*. Then,*

(1) $\{z_n\}$ *is said to be convergent with* $z_0$ *in* $\mathfrak{D}$*, if for all* $\epsilon > 0$*, there exists a positive integer, N such that* $S_b(z_n, z_n, z_0) < \epsilon$ *or* $S_b(z_0, z_0, z_n) < \epsilon$ *for each* $n \geq N$*, that is;* $\lim_{n\to\infty} z_n = z_0$.

(2) $\{z_n\}$ *is said to be Cauchy if for all* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $S_b(z_n, z_n, z_m) < \epsilon$ *for each* $n, m \geq N$.

(3) *The space* $(\mathfrak{D}, S_b)$ *is said to be complete if for each Cauchy in* $\mathfrak{D}$ *is convergence in* $\mathfrak{D}$.

**Definition 2.5.** *[23] Let* $(\mathfrak{D}, S_b)$ *and* $(\mathfrak{D}', S_b')$ *be* $S_b$*-MS, and let* $g : \mathfrak{D} \to \mathfrak{D}'$ *be a function. Then g is called continuous at a point* $z_0 \in \mathfrak{D}$ *if and only if for every sequence* $z_n$ *in* $\mathfrak{D}$*,* $S_b(z_n, z_n, z_0) \to 0$ *implies* $S_b'(g(z_n), g(z_n), g(z_0)) \to 0$*. A function g is continuous at* $\mathfrak{D}$ *if and only if it is continuous at all* $z_0 \in \mathfrak{D}$.

In the following lemma, provided by several researchers, [21,23,24,30–36], we attempt to modify it with another proof and focus in case the $S_b$-MS is symmetric or not.

**Lemma 2.4.** *Let* $(\mathfrak{D}, S_b)$ *be a symmetric* $S_b$*-MS, and assume that* $\{z_n\}$ *and* $\{c_n\}$ *converge to* $z_0, c_0$*, respectively. Then, we have the following:*

(1) $\frac{1}{2\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n) \leq \lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq 2\mu^2 S_b(z_0, z_0, c_0)$,

(2) $\frac{1}{\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n) \leq \lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq \mu^2 S_b(z_0, z_0, c_0)$.
*In particular, if* $z_0 = c_0$*, then* $\lim_{n\to\infty} S_b(z_n, z_n, c_n) = 0$*. Moreover, for each* $b \in \mathfrak{D}$ *we deduce*

(3) $\frac{2}{3\mu^2} S_b(z_0, z_0, b) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, b) \leq \lim_{n\to\infty} \sup S_b(z_n, z_n, b) \leq \frac{3}{2}\mu^2 S_b(z_0, z_0, b)$.

*Proof.* Through (S2) in $S_b$-MS, where $S_b$ is symmetric, and by Lemma 2.3 (1), we obtain

$$S_b(z_n, z_n, c_n) \leq 2\mu S_b(z_n, z_n, z_0) + \mu S_b(c_n, c_n, z_0) \tag{2.1}$$

$$\leq 2\mu S_b(z_n, z_n, z_0) + \mu[2\mu S_b(c_n, c_n, c_0) + \mu S_b(z_0, z_0, c_0)]$$

$$= 2\mu S_b(z_n, z_n, z_0) + 2\mu^2 S_b(c_n, c_n, c_0) + \mu^2 S_b(z_0, z_0, c_0) \tag{2.2}$$

Moreover, by (2.1), by the same process, where $S_b(c_n, c_n, z_0) = S_b(z_0, z_0, c_n)$ we have

$$S_b(z_n, z_n, c_n) \leq 2\mu S_b(z_n, z_n, z_0) + 2\mu^2 S_b(z_0, z_0, c_0) + \mu^2 S_b(c_n, c_n, c_0). \tag{2.3}$$

Also,

$$\begin{aligned} S_b(c_n, c_n, z_n) &\leq 2\mu S_b(c_n, c_n, c_0) + \mu S_b(z_n, z_n, c_0) \quad (2.4)\\ &\leq 2\mu S_b(c_n, c_n, c_0) + \mu[2\mu S_b(z_n, z_n, z_0) + \mu S_b(c_0, c_0, z_0)]\\ &\leq 2\mu S_b(c_n, c_n, c_0) + 2\mu^2 S_b(z_n, z_n, z_0) + \mu^2 S_b(z_0, z_0, c_0). \quad (2.5)\end{aligned}$$

Further, by (2.4), we take $S_b(z_n, z_n, c_0) = S_b(c_0, c_0, z_n)$ we have

$$S_b(c_n, c_n, z_n) \leq 2\mu S_b(c_n, c_n, c_0) + 2\mu^2 S_b(z_0, z_0, c_0) + \mu^2 S_b(z_n, z_n, z_0). \tag{2.6}$$

Take the supremum and tend $n$ to infinite, hence some cases are as follows:
Case 1: from (2.2) and (2.5), $\lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq \mu^2 S_b(z_0, z_0, c_0)$,
Case 2: from (2.3) and (2.6), $\lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq 2\mu^2 S_b(z_0, z_0, c_0)$,
Case 3: from additive (2.2) and (2.6) (or (2.3) and (2.5)), $\lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq \frac{3}{2}\mu^2 S_b(z_0, z_0, c_0)$.
By the same way, where instead of $z_n$ to $z_0$ and $c_n$ to $c_0$ with vice versa, we conclude

$$\frac{1}{\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n), \tag{2.7}$$

and

$$\frac{1}{2\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n) \tag{2.8}$$

also

$$\frac{2}{3\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n). \tag{2.9}$$

Therefore, the result holds for several cases, leading to various inequalities (weak-inequality) as follows:

$$\frac{1}{2\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n) \leq \lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq 2\mu^2 S_b(z_0, z_0, c_0).$$

But, for the smallest interval of inequality (strong-inequality), we have

$$\frac{1}{\mu^2} S_b(z_0, z_0, c_0) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n) \leq \lim_{n\to\infty} \sup S_b(z_n, z_n, c_n) \leq \mu^2 S_b(z_0, z_0, c_0).$$

Towards item 3, if instead of $c_n$ to $b$ in (2.1), (2.4) and additive, by assuming that the upper limit tends to infinity, we obtain

$$\lim_{n\to\infty} \sup S_b(z_n, z_n, b) \leq \frac{3}{2}\mu^2 S_b(z_0, z_0, b).$$

Similarly, about the left side, we obtain

$$\frac{2}{3\mu^2} S_b(z_0, z_0, b) \leq \lim_{n\to\infty} \inf S_b(z_n, z_n, b).$$

Thus, the desired results follow. □

**Lemma 2.5.** *Let $(\mathfrak{D}, S_b)$ be a non-symmetric $S_b$-MS, and let us assume that $\{z_n\}$ and $\{c_n\}$ converge to $z_0, c_0$, respectively. Then, we have*

(1) $\frac{1}{\mu^3} S_b(c_0, c_0, z_0) \le \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n)$ *and* $\lim_{n\to\infty} \sup S_b(c_n, c_n, z_n) \le \mu^3 S_b(z_0, z_0, c_0)$,

(2) $\frac{1}{\mu^3} S_b(c_0, c_0, z_0) \le \lim_{n\to\infty} \inf \widetilde{S_b}(c_n, c_n, z_n) \le \lim_{n\to\infty} \sup \widetilde{S_b}(c_n, c_n, z_n) \le \mu^2(\mu+1) S_b(z_0, z_0, c_0)$.
*In particular, if $z_0 = c_0$, then $\lim_{n\to\infty} \sup \widetilde{S_b}(c_n, c_n, z_n) = 0$. Moreover, for each $w \in \mathfrak{D}$ we deduce*

(3) $\frac{2}{2\mu^2+\mu} S_b(c_0, c_0, w) \le \lim_{n\to\infty} \inf S_b(w, w, c_n)$ *and* $\lim_{n\to\infty} \sup S_b(c_n, c_n, w) \le \frac{2\mu^2+\mu}{2} S_b(w, w, c_0)$,

(4) $\frac{2}{2\mu^2+\mu} S_b(c_0, c_0, w) \le \lim_{n\to\infty} \inf \widetilde{S_b}(c_n, c_n, w) \le \lim_{n\to\infty} \sup \widetilde{S_b}(c_n, c_n, w) \le 3\mu S_b(w, w, c_0)$.

*Proof.* Through (S2) in $S_b$-MS, where $S_b$ is not symmetric (assume $S_b(c_0, c_0, z_0) \le \mu S_b(z_0, z_0, c_0)$), we conclude

$$
\begin{aligned}
S_b(c_n, c_n, z_n) &\le 2\mu S_b(c_n, c_n, c_0) + \mu S_b(z_n, z_n, c_0) \\
&\le 2\mu S_b(c_n, c_n, c_0) + \mu[2\mu S_b(z_n, z_n, z_0) + \mu S_b(c_0, c_0, z_0)] \\
&= 2\mu S_b(c_n, c_n, c_0) + 2\mu^2 S_b(z_n, z_n, z_0) + \mu^2 S_b(c_0, c_0, z_0).
\end{aligned}
$$

By Lemma 2.2, we have

$$
S_b(c_n, c_n, z_n) \le 2\mu S_b(c_n, c_n, c_0) + 2\mu^2 S_b(z_n, z_n, z_0) + \mu^3 S_b(z_0, z_0, c_0). \tag{2.10}
$$

Now, taking $S_b^*(c_n, c_n, z_n) = S_b(z_n, z_n, c_n)$ and repeating the process (2.10), we deduce

$$
S_b^*(c_n, c_n, z_n) = S_b(z_n, z_n, c_n) \le 2\mu S_b(z_n, z_n, z_0) + 2\mu^2 S_b(c_n, c_n, c_0) + \mu^2 S_b(z_0, z_0, c_0). \tag{2.11}
$$

Since $S_b(c_n, c_n, z_n) \le \mu S_b(z_n, z_n, c_n)$ for all $n$, with inequality (2.11), we obtain that

$$
S_b(c_n, c_n, z_n) \le 2\mu^2 S_b(z_n, z_n, z_0) + 2\mu^3 S_b(c_n, c_n, c_0) + \mu^3 S_b(z_0, z_0, c_0). \tag{2.12}
$$

By additive (2.10) and (2.12), and assuming that the upper limit tends to infinite, we deduce

$$
\lim_{n\to\infty} \sup S_b(c_n, c_n, z_n) \le \mu^3 S_b(z_0, z_0, c_0). \tag{2.13}
$$

On the other hand, if instead of $c_n$ by $c_0$ and $z_n$ by $z_0$ with vice versa in (2.10), we conclude

$$
S_b(c_0, c_0, z_0) \le 2\mu S_b(c_0, c_0, c_n) + 2\mu^2 S_b(z_0, z_0, z_n) + \mu^3 S_b(z_n, z_n, c_n), \tag{2.14}
$$

also

$$
\begin{aligned}
S_b(c_0, c_0, z_0) &\le \mu S_b(z_0, z_0, c_0) \\
&\le 2\mu^2 S_b(z_0, z_0, z_n) + 2\mu^3 S_b(c_0, c_0, c_n) + \mu^3 S_b(z_n, z_n, c_n).
\end{aligned} \tag{2.15}
$$

By additive (2.14) and (2.15), and assuming that the lowest limit tends to infinite, we deduce

$$
\frac{1}{\mu^3} S_b(c_0, c_0, z_0) \le \lim_{n\to\infty} \inf S_b(z_n, z_n, c_n). \tag{2.16}
$$

Let $S_b^*(c_0, c_0, z_0) = S_b(z_0, z_0, c_0)$, we can get a symmetric form by

$$
\widetilde{S_b}(c_0, c_0, z_0) = \max\{S_b(c_0, c_0, z_0), S_b^*(c_0, c_0, z_0)\} = \widetilde{S_b}(z_0, z_0, c_0).
$$

From (2.10) and (2.11), we have

$$\lim_{n\to\infty}\sup \widetilde{S_b}(c_n,c_n,z_n) \le \lim_{n\to\infty}\sup \max\{2\mu S_b(c_n,c_n,c_0)+2\mu^2 S_b(z_n,z_n,z_0) + \mu^3 S_b(z_0,z_0,c_0), 2\mu S_b(z_n,z_n,z_0)+2\mu^2 S_b(c_n,c_n,c_0)+\mu^2 S_b(z_0,z_0,c_0)\} \le \mu^2(\mu+1)S_b(z_0,z_0,c_0). \text{ (In fact, } \max\{a,b\}\le a+b \text{ for all } a,b\ge 0). \tag{2.17}$$

Again, by (2.16), we get

$$\frac{1}{\mu^3}S_b(c_0,c_0,z_0) \le \lim_{n\to\infty}\inf S_b(z_n,z_n,c_n) = \lim_{n\to\infty}\inf S_b^*(c_n,c_n,z_n) \le \lim_{n\to\infty}\inf \widetilde{S_b}(c_n,c_n,z_n) \le \lim_{n\to\infty}\sup \widetilde{S_b}(c_n,c_n,z_n) \le \mu^2(\mu+1)S_b(z_0,z_0,c_0).$$

Moreover, for each $w\in\mathfrak{D}$, we obtain by (S2)

$$S_b(c_n,c_n,w) \le 2\mu S_b(c_n,c_n,c_0)+\mu S_b(w,w,c_0) \tag{2.18}$$

$$S_b^*(c_n,c_n,w) = S_b(w,w,c_n) \le 2\mu S_b(w,w,c_0)+\mu S_b(c_n,c_n,c_0). \tag{2.19}$$

By hypothesis $S_b(c_n,c_n,w)\le \mu S_b(w,w,c_n)$, for each $w\in\mathfrak{D}$ and $n\in\mathbb{N}$ utilised (2.19), we have

$$S_b(c_n,c_n,w) \le \mu S_b(w,w,c_n) \le 2\mu^2 S_b(w,w,c_0)+\mu^2 S_b(c_n,c_n,c_0). \tag{2.20}$$

By additive (2.18) and (2.20), it leads to

$$\lim_{n\to\infty}\sup S_b(c_n,c_n,w) \le \frac{2\mu^2+\mu}{2}S_b(w,w,c_0).$$

On the other hand, we get

$$S_b(c_0,c_0,w) \le 2\mu S_b(c_0,c_0,c_n)+\mu S_b(w,w,c_n), \tag{2.21}$$

and

$$S_b^*(c_0,c_0,w) = S_b(w,w,c_0) \le 2\mu S_b(w,w,c_n)+\mu S_b(c_0,c_0,c_n),$$

so that

$$S_b(c_0,c_0,w) \le \mu S_b(w,w,c_0) \le 2\mu^2 S_b(w,w,c_n)+\mu^2 S_b(c_0,c_0,c_n). \tag{2.22}$$

Additive (2.21) and (2.22) with the infimum, leads to

$$\frac{2}{2\mu^2+\mu}S_b(c_0,c_0,w) \le \lim_{n\to\infty}\inf S_b(w,w,c_n). \tag{2.23}$$

Furthermore, $S_b^*(c_n,c_n,w)=S_b(w,w,c_n)$ for all $n\in\mathbb{N}$, so that

$$\widetilde{S_b}(c_n,c_n,w) = \max\{S_b(c_n,c_n,w), S_b^*(c_n,c_n,w)\}.$$

From Eq. (2.18) and (2.19), we undergo

$$\lim_{n\to\infty}\sup \widetilde{S_b}(c_n,c_n,w) \le \lim_{n\to\infty}\sup \max\{2\mu S_b(c_n,c_n,c_0)+\mu S_b(w,w,c_0), 2\mu S_b(w,w,c_0)+\mu S_b(c_n,c_n,c_0)\} \le 3\mu S_b(w,w,c_0).$$

Thus,

$$\lim_{n\to\infty} \sup \widetilde{S_b}(c_n, c_n, w) \leq 3\mu S_b(w, w, c_0).$$

Also, from inequality (2.23) and the notation of $\widetilde{S_b}(c_n, c_n, w)$, we obtain

$$\frac{2}{2\mu^2+\mu} S_b(c_0, c_0, w) \leq \lim_{n\to\infty} \widetilde{S_b}(c_n, c_n, w).$$

Therefore, the desired results follow. □

The following is a generalised revised version of the lemma in [25].

**Lemma 2.6.** *Let $(\mathfrak{D}, S_b)$ be a symmetric $S_b$-MS and let $\{c_n\}$ be a sequence in it such that $\lim_{n\to\infty} S_b(c_{n+1}, c_{n+1}, c_n) = 0$. If $\{c_n\}$ is not a Cauchy, then there exists $\epsilon > 0$ and a two-sequence $\{m_k\}$ and $\{n_k\}$ where $n_k > m_k > k$ of positive integers such that $S_b(c_{n_k}, c_{n_k}, c_{m_k}) \geq \epsilon$ and $S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_k}) < \epsilon$. Moreover*

(1) $\frac{1}{\mu^2}\epsilon \leq \lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_k}) \leq \mu^2\epsilon$,
(2) $\frac{1}{\mu^2}\epsilon \leq \lim_{k\to+\infty} \sup S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_k}) \leq \mu^2\epsilon$,
(3) $\frac{1}{\mu^2}\epsilon \leq \lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k-1}}) \leq \mu^2\epsilon$,
(4) $\frac{1}{\mu^2}\epsilon \leq \lim_{k\to+\infty} \sup S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k-1}}) \leq \mu^2\epsilon$,

**Remark 2.3.** *Note in Lemma 2.6 about $(\mathfrak{D}, S_b)$ being non-symmetric $S_b$-MS. It leads to some difference with respect to Lemma 2.5 in all cases.*

**Definition 2.6.** *Let $\mathfrak{D}$ be a nonempty set and $0 \leq \mathfrak{A} \leq 1$. Define the mapping $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ as a continuous function $\mathcal{W} : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{A} \to \mathfrak{D}$. Then $\mathcal{W}$ is the convex form on $\mathfrak{D}$ if $S_b(c_0, c_0, \mathcal{W}(\eta_1, \eta_2; \beta)) \leq \mathcal{W}(S_b(c_0, c_0, \eta_1), S_b(c_0, c_0, \eta_2); \beta)$ holds for each $c_0 \in \mathfrak{D}$ and $(\eta_1, \eta_2, \beta) \in \mathfrak{D} \times \mathfrak{D} \times \mathfrak{A}$. Let taking in adhear paper $\mathcal{W}(S_b(c_0, c_0, \eta_1), S_b(c_0, c_0, \eta_2); \beta) = \beta S_b(c_0, c_0, \eta_1) + (1-\beta) S_b(c_0, c_0, \eta_2)$.*

**Definition 2.7.** *Let the function $\mathcal{W} : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{A} \to \mathfrak{D}$ be a convex form on a $S_b$-MS $(\mathfrak{D}, S_b)$ with constant $\mu \geq 1$ and $0 \leq \mathfrak{A} \leq 1$. Then $(\mathfrak{D}, S_b, \mathcal{W})$ is termed a convex $S_b$-MS (short, $CS_b$-MS).*

Let $(\mathfrak{D}, S_b, \mathcal{W})$ be a $CS_b$-MS and a mapping $\Gamma : \mathfrak{D} \to \mathfrak{D}$. Generalisation of Mann's iteration scheme into $CS_b$-MS is as $c_{n+1} := \mathcal{W}(c_n, \Gamma c_n; \beta_n)$, where $c_n \in \mathfrak{D}$ and $0 \leq \beta_n \leq 1, n \in \mathbb{N}$. The sequence $\{c_n\}$ is called a Mann's iteration sequence for $\Gamma$.

**Example 2.4.** *Let $\mathfrak{D} = \mathbb{R}^m$, and define a $S_b$-MS as $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ for each $a, b, c \in \mathfrak{D}$ by*

$$S_b(a, b, c) = \sum_{j=1}^{m} \Big(|a_j - c_j| + |a_j + c_j - 2b_j|\Big)^2,$$

*where $a = (a_1, a_2, \cdots, a_m) \in \mathfrak{D}$, $b = (b_1, b_2, \cdots, b_m) \in \mathfrak{D}$ and $c = (c_1, c_2, \cdots, c_m) \in \mathfrak{D}$ and let the mapping $\mathcal{W} : \mathfrak{D} \times \mathfrak{D} \times [0, 1] \to \mathfrak{D}$ as*

$$\mathcal{W}\Big(a, b; \frac{1}{2}\Big) = \frac{a+b}{2}.$$

*Then, $(\mathfrak{D}, S_b, \mathcal{W})$ is a $CS_b$-MS with $\mu = 2$.*

**Example 2.5.** *Let $\mathfrak{D} = \mathbb{R}$, and define a $S_b$-MS as $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, \infty)$ for each $a, b, c \in \mathfrak{D}$ by*

$$S_b(a, b, c) = \left[\frac{1}{2}(|a - c| + |b - c|)\right]^2,$$

*and also, the mapping $\mathcal{W} : \mathfrak{D} \times \mathfrak{D} \times [0, 1] \to \mathfrak{D}$ as $\mathcal{W}(a, b; \beta) = \beta a + (1 - \beta)b$. It is not difficult to see that $(\mathfrak{D}, S_b, \mathcal{W})$ is a $CS_b$-MS with $\mu = 2$.*

We summarise the most important and very useful lemmas and results in the main section of the paper.

**Lemma 2.7.** *Let $(\mathfrak{D}, S_b, \mathcal{W})$ be $CS_b$-MS. If $\beta \in (0, 1)$, then $S_b$-MS is symmetric.*

*Proof.* Obviously, $S_b(a, b, b) = S_b(b, a, a)$ is satisfied, where $a = b$. Assume that $a \neq b$. Since $\beta < 1$, it is not difficult to see that $a \neq \mathcal{W}(a, b; \beta)$ and $b \neq \mathcal{W}(a, b; \beta)$. Indeed, if $a = \mathcal{W}(a, b; \beta)$, we undergo

$$S_b(a, b, b) = S_b(\mathcal{W}(a, b; \beta), b, b) \leq \beta S_b(a, b, b),$$

a contradiction. Therefore, $a \neq \mathcal{W}(a, b; \beta)$. Utilising similar arguments, we deduce that $b \neq \mathcal{W}(a, b; \beta)$.
Now, consider

$$S_b(a, b, b) \leq S_b(a, \mathcal{W}(a, b; \beta), b) \leq \beta S_b(a, a, b) + (1 - \beta)S_b(a, b, b).$$

This implies that

$$S_b(a, b, b) \leq S_b(a, a, b). \tag{2.24}$$

In addition

$$S_b(a, a, b) \leq S_b(a, \mathcal{W}(a, b; \beta), b) \leq \beta S_b(a, a, b) + (1 - \beta)S_b(a, b, b),$$

so that

$$S_b(a, a, b) \leq S_b(a, b, b). \tag{2.25}$$

By induction, we get

$$S_b(a, a, b) = S_b(a, b, b). \tag{2.26}$$

What remains is to show that $S_b(a, b, b) = S_b(b, b, a)$ (or $S_b(a, a, b) = S_b(b, a, a)$).

$$\begin{aligned} S_b(a, b, b) &\leq S_b(a, b, \mathcal{W}(a, b; 1 - \beta)) = S_b(a, b, \mathcal{W}(b, a; \beta)) \\ &\leq \beta S_b(a, b, b) + (1 - \beta)S_b(a, b, a). \end{aligned}$$

This yields

$$S_b(a, b, b) \leq S_b(a, b, a). \tag{2.27}$$

Also,

$$\begin{aligned} S_b(b, b, a) &\leq S_b(\mathcal{W}(b, a; \gamma), b, a) = S_b(\mathcal{W}(a, b; 1 - \gamma), b, a), \text{(where } \gamma = 1 - \beta) \\ &\leq (1 - \gamma)S_b(a, b, a) + \gamma S_b(b, b, a). \end{aligned}$$

So that,

$$S_b(b, b, a) \leq S_b(a, b, a). \tag{2.28}$$

On the other side, we conclude that

$$\begin{aligned} S_b(a,b,a) &\le S_b(\mathcal{W}(a,b;\beta),b,a) = S_b(\mathcal{W}(b,a;1-\beta),b,a) \\ &\le (1-\beta)S_b(b,b,a) + \beta S_b(a,b,a), \end{aligned}$$

hence

$$S_b(a,b,a) \le S_b(b,b,a). \tag{2.29}$$

Again,

$$S_b(a,b,a) \le S_b(a,b,\mathcal{W}(a,b;\beta)) \le \beta S_b(a,b,a) + (1-\beta)S_b(a,b,b),$$

implies that

$$S_b(a,b,a) \le S_b(a,b,b). \tag{2.30}$$

Therefore, from (2.27), (2.28), (2.29) and (2.30), we have $S_b(a,b,b) = S_b(a,b,a) = S_b(b,b,a)$. □

## 3. Main Results

This section deals with some FP results in the framework of $CS_b$-MS. Our first theorem will be analogous to the contraction of Khan type by Mann's iteration for complete $CS_b$-MS of an FPT.

Before proposing our theorems, we will utilise the idea of $\mathfrak{F}$-contraction, which is due to Wardowski [26] involved in this context.

Consider a mapping $\mathfrak{F} : (0,\infty) \to \mathbb{R}$ satisfying,

(i) $\mathfrak{F}$ is strictly increasing,
(ii) For all sequences $\{\gamma_q\}$ for positive numbers $\lim_{q\to\infty} \gamma_q = 0$ if and only if $\lim_{q\to\infty} \mathfrak{F}(\gamma_q) = -\infty$,
(iii) There exists $r \in (0,1)$ such that $\lim_{\gamma\to 0} \gamma^r \mathfrak{F}(\gamma) = 0$.

Recall that this mapping of a $\mathfrak{F}$-contraction if there exists $\tau > 0$ such that $d(\Gamma c, \Gamma z) > 0$ implies that

$$\tau + \mathfrak{F}(d(\Gamma c, \Gamma z)) \le \mathfrak{F}(d(c,z)),$$

For each $c, z \in \mathfrak{D}$, $(\mathfrak{D}, d)$ is an MS. Moreover, the authors in [26, 27] presented the property of the function $\mathfrak{F}$, which is the axiom (i):

Whenever all points $h \in (0,\infty)$, there exist their left and right limits as $\lim_{\gamma\to h^-} \mathfrak{F}(\gamma) = \mathfrak{F}(h^-)$ and $\lim_{\gamma\to h^+} \mathfrak{F}(\gamma) = \mathfrak{F}(h^+)$. Further, for the function $\mathfrak{F}$, one of the two axioms holds: $\mathfrak{F}(0^+) = c \in \mathbb{R}$ or $\mathfrak{F}(0^+) = -\infty$.

In 2021, Huang et al. [29] introduced the idea of a convex $\mathfrak{F}$-contraction in b MS.

**Definition 3.1.** *[29] Let $(\mathfrak{D}, d_b)$ be a b-MS. Define $\Gamma : \mathfrak{D} \to \mathfrak{D}$ a self-mapping is referred to as a convex $\mathfrak{F}$-contraction if there exists a function $\mathfrak{F} : (0,\infty) \to \mathbb{R}$ such that it satisfies the conditions above (i), (ii), (iii), and the condition*

(iv) *There exists $\tau > 0$ and $\beta \in [0,1)$ such that $\tau + \mathfrak{F}(c_q) \le \mathfrak{F}(\beta c_q + (1-\beta)c_q)$, for each $c_q > 0, q \in \mathbb{N}$.*

**Theorem 3.1.** *Let $(\mathfrak{D}, S_b, \mathcal{W})$ be a complete $CS_b$-MS and $\Gamma : \mathfrak{D} \to \mathfrak{D}$ satisfy the condition: For all $c, z \in \mathfrak{D}$, such that $S_b(\Gamma c, \Gamma c, \Gamma z) > 0$ implies*

$$\tau + \mathfrak{F}(S_b(\Gamma c, \Gamma c, \Gamma z)) \leq \mathfrak{F}\left(\rho \frac{S_b(c, c, \Gamma c) S_b(c, c, \Gamma z) + S_b(z, z, \Gamma z) S_b(z, z, \Gamma c)}{\max\{S_b(c, c, \Gamma z), S_b(z, z, \Gamma c)\}}\right). \tag{3.1}$$

*Let $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$. If $\rho < \frac{1}{3\mu^2}$ and $\beta_{n-1} \in [0, \rho]$ for $n \in \mathbb{N}$, then there is a unique fixed point of $\Gamma$ in $\mathfrak{D}$.*

*Proof.* Assume that $\Gamma c \neq z$ and $\Gamma z \neq c$. For $n \in \mathbb{N}$, we get

$$\begin{aligned} S_b(c_n, c_n, \Gamma c_{n-1}) = S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, c_n) &= S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})) \\ &\leq \beta_{n-1} S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, c_{n-1}). \end{aligned}$$

If $c_n = c_{n-1}$, then

$$S_b(c_{n-1}, \Gamma c_{n-1}, \Gamma c_{n-1}) = S_b(c_n, \Gamma c_{n-1}, \Gamma c_{n-1}) \leq \beta_{n-1} S_b(c_{n-1}, \Gamma c_{n-1}, \Gamma c_{n-1}),$$

it implies that the mapping $\Gamma$ is a fixed point. Thus, assume that $c_n \neq c_{n-1}$ and $c_n \neq \Gamma c_n$.
Through (S2), we deduce

$$\begin{aligned} \mathfrak{F}(S_b(c_n, c_n, \Gamma c_n)) &\leq \tau + \mathfrak{F}(S_b(c_n, c_n, \Gamma c_n)) \\ &\leq \mathfrak{F}(2\mu S_b(c_n, c_n, \Gamma c_{n-1}) + \mu S_b(\Gamma c_n, \Gamma c_n, \Gamma c_{n-1})) \\ &= \mathfrak{F}(2\mu S_b(c_n, c_n, \Gamma c_{n-1}) + \mu S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)). \end{aligned} \tag{3.2}$$

From the eq. (3.1), we obtain

$$\begin{aligned} &\tau + \mathfrak{F}(S_b(\Gamma c_n, \Gamma c_n, \Gamma c_{n-1})) = \tau + \mathfrak{F}(S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)) \\ &\leq \mathfrak{F}\left(\rho \frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_{n-1})}{\max\{S_b(c_{n-1}, c_{n-1}, \Gamma c_n), S_b(c_n, c_n, \Gamma c_{n-1})\}}\right). \end{aligned}$$

Since

$$\begin{aligned} S_b(c_{n-1}, c_{n-1}, c_n) &= S_b(c_{n-1}, c_{n-1}, \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})) \\ &\leq (1 - \beta_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}), \end{aligned}$$

and, if $S_b(c_{n-1}, c_{n-1}, \Gamma c_n) < S_b(c_n, c_n, \Gamma c_{n-1})$, then

$$\begin{aligned} &\tau + \mathfrak{F}(S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)) \\ &\leq \mathfrak{F}\left(\frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_{n-1})}{S_b(c_n, c_n, \Gamma c_{n-1})}\right) \\ &\leq \mathfrak{F}\left(\rho \left[\frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_n)}{S_b(c_n, c_n, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)}\right]\right) \\ &\leq \mathfrak{F}\left(\rho \left[S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)\right]\right). \end{aligned} \tag{3.3}$$

But, if $S_b(c_n, c_n, \Gamma c_{n-1}) < S_b(c_{n-1}, c_{n-1}, \Gamma c_n)$, then

$$\begin{aligned}
&\tau + \mathfrak{F}\left(S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)\right)\\
&\leq \mathfrak{F}\left(\rho \frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_{n-1})}{S_b(c_{n-1}, c_{n-1}, \Gamma c_n)}\right)\\
&\leq \mathfrak{F}\left(\rho\left[S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)\right]\right).
\end{aligned}$$

Then, utilising (i), we obtain

$$S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n) \leq \rho\left[S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)\right]. \tag{3.4}$$

Hence, from (3.2) and (3.4),

$$\begin{aligned}
\mathfrak{F}(S_b(c_n, c_n, \Gamma c_n)) \leq &\mathfrak{F}\left(2\mu\beta_{n-1} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1})\right.\\
&\left.+\mu\rho[S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)\right).
\end{aligned}$$

Thus, utilising (i), we obtain

$$S_b(c_n, c_n, \Gamma c_n) \leq \frac{2\mu\beta_{n-1} + \mu\rho}{1 - \mu\rho} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}),$$

putting $\gamma_{n-1} = \frac{2\mu\beta_{n-1}+\mu\rho}{1-\mu\rho}$, and note that $\beta_{n-1} \in [0, \rho]$, $\rho < \frac{1}{3\mu^2}$ for $n \in \mathbb{N}$, we deduce

$$S_b(c_n, c_n, \Gamma c_n) \leq \gamma_{n-1} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) < \frac{1}{\mu} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}).$$

Showing that $\{S_b(c_n, c_n, \Gamma c_n)\}$ is a decreasing sequence of non-negative reals, then there is $\lambda \geq 0$, where $\lim_{n\to+\infty} S_b(c_n, c_n, \Gamma c_n) = \lambda$. Let $\lambda > 0$, and $n$ tend to infinite, leads to $\lambda < \lambda$ a contradiction, so that $\lambda = 0$, and

$$\begin{aligned}
S_b(c_n, c_n, c_{n+1}) = &S_b(c_n, c_n, \mathcal{W}(c_n, \Gamma c_n; \beta_n)) \leq (1 - \beta_n) S_b(c_n, c_n, \Gamma c_n) < S_b(c_n, c_n, \Gamma c_n),\\
&\lim_{n\to+\infty} S_b(c_n, c_n, c_{n+1}) = 0.
\end{aligned}$$

Next, we prove that $\{c_n\}$ is a Cauchy in $\mathfrak{D}$. On the contrary, if the claim is that $\{c_n\}$ is not Cauchy, then by Lemma 2.6, there exist $\epsilon > 0$ and subsequence $\{c_{m_k}\}$ and $\{c_{n_k}\}$ of $\{c_n\}$ positive integers such that $\lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k-1}}) \leq \frac{\epsilon}{\mu^2}$, and

$$\begin{aligned}
&S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) = S_b(c_{m_{k+1}}, c_{m_{k+1}}, \mathcal{W}(c_{n_{k-1}}, \Gamma c_{n_{k-1}}; \beta_{n_{k-1}}))\\
&\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}}) S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, c_{m_{k+1}})\\
&\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}})\\
&+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})].
\end{aligned}$$

By the same process above with (3.4) from (3.1), with respect to the two subsequences, we obtain

$$S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \leq \rho[S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})], \tag{3.5}$$

because

$$\begin{aligned}S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) &\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}})\\ &+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})]\\ &< \beta_{n_{k-1}}\mu[2S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_k}) + S_b(c_{m_k}, c_{m_k}, c_{m_{k+1}})]\\ &+ (1-\beta_{n_{k-1}})\mu[2\rho[S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}})\\ &+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})].\end{aligned}$$

Taking $k \to +\infty$, we deduce $\lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) \leq \frac{\epsilon}{\mu^2}$; this is a contradiction, so $\{c_n\}$ is a Cauchy sequence in $\mathfrak{D}$. By completeness, there is $c_0 \in \mathfrak{D}$ so that $\lim_{n\to+\infty} c_n = c_0$. To prove that $c_0$ is a fixed point of $\Gamma$, that is

$$\begin{aligned}S_b(c_0, c_0, \Gamma c_0) &\leq \mu[2S_b(c_0, c_0, c_n) + S_b(\Gamma c_0, \Gamma c_0, c_n)]\\ &\leq 2\mu S_b(c_0, c_0, c_n) + 2\mu^2 S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n) + \mu^2 S_b(\Gamma c_n, \Gamma c_n, c_n),\end{aligned} \tag{3.6}$$

and

$$\tau + \mathfrak{F}(S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n)) \leq \mathfrak{F}\left(\rho \frac{S_b(c_0, c_0, \Gamma c_0) S_b(c_0, c_0, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_0)}{\max\{S_b(c_0, c_0, \Gamma c_n), S_b(c_n, c_n, \Gamma c_0)\}}\right),$$

if $S_b(c_0, c_0, \Gamma c_n) < S_b(c_n, c_n, \Gamma c_0)$, then

$$\tau + \mathfrak{F}(S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n)) \leq \mathfrak{F}(\rho[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)]). \tag{3.7}$$

Similarly, if we take $S_b(c_n, c_n, \Gamma c_0) < S_b(c_0, c_0, \Gamma c_n)$, we get (3.7). Therefore, by (i), we undergo

$$S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n) \leq \rho[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)],$$

Subsequently,

$$S_b(c_0, c_0, \Gamma c_0) \leq 2\mu S_b(c_0, c_0, c_n) + 2\mu^2\rho[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)] + \mu^2 S_b(\Gamma c_n, \Gamma c_n, c_n).$$

Letting $n \to +\infty$, we get, $S_b(c_0, c_0, \Gamma c_0) < \mu^2(4\rho+1)S_b(c_0, c_0, \Gamma c_0)$, so go to $\Gamma c_0 = c_0$. The uniqueness is in letting $c_0$ and $\widehat{c_0}$ be different fixed points of $\Gamma$. Thus, by (3.1), $S_b(c_0, c_0, \widehat{c_0}) = S_b(\Gamma c_0, \Gamma c_0, \Gamma\widehat{c_0}) leq 0$, hence, $S_b(c_0, c_0, \widehat{c_0}) = 0$, that is, $c_0 = \widehat{c_0}$. □

Now, we show the second kind of $\mathfrak{F}$-Khan-contraction with Mann's iteration in complete $CS_b$-MS.

**Theorem 3.2.** *Suppose $(\mathfrak{D}, S_b, \mathcal{W})$ is a complete $CS_b$-MS and $\Gamma : \mathfrak{D} \to \mathfrak{D}$ satisfies the condition: For all $c, z \in \mathfrak{D}$, such that $S_b(\Gamma c, \Gamma c, \Gamma z) > 0$ implies*

$$\tau + \mathfrak{F}(S_b(\Gamma c, \Gamma c, \Gamma z)) \leq \mathfrak{F}\left(\alpha_1 S_b(c, c, z) + \alpha_2 \frac{S_b(c, c, \Gamma c) S_b(c, c, \Gamma z) + S_b(z, z, \Gamma z) S_b(z, z, \Gamma c)}{S_b(c, c, \Gamma z) + S_b(z, z, \Gamma c)}\right). \tag{3.8}$$

*Let $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$. If $\alpha_1 < \frac{1}{\mu^3}, \alpha_2 < 1$ and $0 < \beta_{n-1} < \frac{1-\mu[\alpha_2(\mu+1)-2\alpha_1\mu^2]}{\mu^2(2+\alpha_1\mu)}$, for $n \in \mathbb{N}$, then there is a unique fixed point of $\Gamma$ in $\mathfrak{D}$.*

*Proof.* Considering the same construction of the previous result, if we conclude from the Eq. (3.8), we obtain

$$\begin{aligned}
&\tau+\mathfrak{F}(S_b(\Gamma c_n,\Gamma c_n,\Gamma c_{n-1}))=\tau+\mathfrak{F}(S_b(\Gamma c_{n-1},\Gamma c_{n-1},\Gamma c_n))\\
&\le \mathfrak{F}\Big(\alpha_1 S_b(c_{n-1},c_{n-1},c_n)\\
&+\alpha_2\frac{S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})S_b(c_{n-1},c_{n-1},\Gamma c_n)+S_b(c_n,c_n,\Gamma c_n)S_b(c_n,c_n,\Gamma c_{n-1})}{S_b(c_{n-1},c_{n-1},\Gamma c_n)+S_b(c_n,c_n,\Gamma c_{n-1})}\Big).
\end{aligned}$$

Since

$$\begin{aligned}
S_b(c_{n-1},c_{n-1},c_n)&=S_b(c_{n-1},c_{n-1},\mathcal{W}(c_{n-1},\Gamma c_{n-1};\beta_{n-1}))\\
&\le(1-\beta_{n-1})S_b(c_{n-1},c_{n-1},\Gamma c_{n-1}),
\end{aligned}$$

and,

$$\begin{aligned}
&\tau+\mathfrak{F}(S_b(\Gamma c_{n-1},\Gamma c_{n-1},\Gamma c_n))\\
&\le\mathfrak{F}\Big(\alpha_1 S_b(c_{n-1},c_{n-1},c_n)\\
&+\alpha_2\frac{S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})S_b(c_{n-1},c_{n-1},\Gamma c_n)+S_b(c_n,c_n,\Gamma c_n)S_b(c_n,c_n,\Gamma c_{n-1})}{S_b(c_{n-1},c_{n-1},\Gamma c_n)+S_b(c_n,c_n,\Gamma c_{n-1})}\Big)\\
&\le\mathfrak{F}\Big(\alpha_1[2\mu S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\mu S_b(c_n,c_n,\Gamma c_{n-1})]+\alpha_2[S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})\\
&+S_b(c_n,c_n,\Gamma c_n)]\Big)\\
&\le\mathfrak{F}((2\mu\alpha_1+\alpha_2)S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\alpha_1\mu S_b(c_n,c_n,\Gamma c_{n-1})+\alpha_2 S_b(c_n,c_n,\Gamma c_n)). \qquad (3.9)
\end{aligned}$$

Then, utilising (i), we obtain

$$\begin{aligned}
S_b(\Gamma c_{n-1},\Gamma c_{n-1},\Gamma c_n)\le{}&(2\mu\alpha_1+\alpha_2)S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\alpha_1\mu S_b(c_n,c_n,\Gamma c_{n-1})\\
&+\alpha_2 S_b(c_n,c_n,\Gamma c_n). \qquad (3.10)
\end{aligned}$$

Hence, from (3.2) and (3.10),

$$\begin{aligned}
\mathfrak{F}(S_b(c_n,c_n,\Gamma c_n))\le{}&\mathfrak{F}(2\mu\beta_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\mu(2\mu\alpha_1+\alpha_2)S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})\\
&+\alpha_1\mu^2\alpha_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\alpha_2\mu S_b(c_n,c_n,\Gamma c_n)).
\end{aligned}$$

Utilising (i), we conclude

$$S_b(c_n,c_n,\Gamma c_n)\le\frac{\mu[(2+\alpha_1\mu)(\beta_{n-1}+2)+\alpha_2-4]}{1-\alpha_2\mu}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1}),$$

putting $\gamma_{n-1}=\frac{\mu[(2+\alpha_1\mu)(\beta_{n-1}+2)+\alpha_2-4]}{1-\alpha_2\mu}$, and note that $\alpha_1<\frac{1}{\mu^3},\alpha_2<1$ and $0<\beta_{n-1}<\frac{1-\mu[\alpha_2(\mu+1)-2\alpha_1\mu^2]}{\mu^2(2+\alpha_1\mu)}$ for $n\in\mathbb{N}$, we deduce

$$S_b(c_n,c_n,\Gamma c_n)\le\gamma_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})<\frac{1}{\mu}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1}).$$

Showing that $\{S_b(c_n, c_n, \Gamma c_n)\}$ is a decreasing sequence of nonnegative reals, then, there is $\lambda \geq 0$, where $\lim_{n\to+\infty} S_b(c_n, c_n, \Gamma c_n) = \lambda$. Let $\lambda > 0$, and $n$ tend to infinite, which leads to $\lambda < \lambda$ a contradiction, so that $\lambda = 0$, and

$$S_b(c_n, c_n, c_{n+1}) = S_b(c_n, c_n, \mathcal{W}(c_n, \Gamma c_n; \beta_n)) \leq (1-\beta_n) S_b(c_n, c_n, \Gamma c_n) < S_b(c_n, c_n, \Gamma c_n),$$

$$\lim_{n\to+\infty} S_b(c_n, c_n, c_{n+1}) = 0.$$

Next, we prove that $\{c_n\}$ is a Cauchy in $\mathfrak{D}$. On the contrary, if the claim is that $\{c_n\}$ is not Cauchy, then by Lemma 2.6, there exist $\epsilon > 0$ and subsequence $\{c_{m_k}\}$ and $\{c_{n_k}\}$ of $\{c_n\}$ positive integers such that $\lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k-1}}) \leq \frac{\epsilon}{\mu^2}$, and

$$\begin{aligned}
S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) &= S_b(c_{m_{k+1}}, c_{m_{k+1}}, \mathcal{W}(c_{n_{k-1}}, \Gamma c_{n_{k-1}}; \beta_{n_{k-1}})) \\
&\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}}) S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, c_{m_{k+1}}) \\
&\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}}) \mu [2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \\
&+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})]
\end{aligned}$$

By the same process above with (3.10) from (3.8), with respect to the equations, we obtain

$$\begin{aligned}
S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \leq \alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + \alpha_2 [S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\
+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})],
\end{aligned} \tag{3.11}$$

because

$$\begin{aligned}
& S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) \leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}}) \mu [2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \\
& + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\
& < \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}}) \mu [2\alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) \\
& + 2\alpha_2 [S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] + 2S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\
& < (\beta_{n_{k-1}} + 2\alpha_1 \mu (1-\beta_{n_{k-1}})) S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + 2\mu\alpha_2 (1-\beta_{n_{k-1}}) [S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\
& + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] + 2\mu (1-\beta_{n_{k-1}}) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}) \\
& < (\beta_{n_{k-1}} + 2\alpha_1 \mu (1-\beta_{n_{k-1}})) \mu [2S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{n_k}) + S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}})] \\
& + 2\mu\alpha_2 (1-\beta_{n_{k-1}}) [S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\
& + 2\mu (1-\beta_{n_{k-1}}) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}) \\
& < 2\mu^2 \alpha_1 [2S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{n_k}) + S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}})] + 2\mu\alpha_2 (1-\beta_{n_{k-1}}) [S_b(c_{(k-1}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\
& + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] + 2\mu (1-\beta_{n_{k-1}}) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}).
\end{aligned}$$

Taking $k \to +\infty$, we deduce $\lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) \leq \frac{\epsilon}{\mu^2}$, which is a contradiction, so $\{c_n\}$ is a Cauchy sequence in $\mathfrak{D}$. By completeness, there is $c_0 \in \mathfrak{D}$ so that $\lim_{n\to+\infty} c_n = c_0$. To prove that $c_0$ is a fixed point of $\Gamma$, from (3.6) and

$$\begin{aligned}\tau + \mathfrak{F}\Big(S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n)\big) &\leq \mathfrak{F}\big(\alpha_1 S_b(c_0, c_0, c_n) \\ &\quad + \alpha_2 \frac{S_b(c_0, c_0, \Gamma c_0) S_b(c_0, c_0, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_0)}{S_b(c_0, c_0, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_0)}\Big) \\ &\leq \mathfrak{F}(\alpha_1 S_b(c_0, c_0, c_n) + \alpha_2[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)]),\end{aligned} \tag{3.12}$$

Therefore, by (i), we undergo

$$S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n) \leq \alpha_1 S_b(c_0, c_0, c_n) + \alpha_2[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)],$$

Subsequently,

$$\begin{aligned}S_b(c_0, c_0, \Gamma c_0) &\leq 2\mu S_b(c_0, c_0, c_n) + 2\mu^2\Big[\alpha_1 S_b(c_0, c_0, c_n) \\ &\quad + \alpha_2[S_b(c_0, c_0, \Gamma c_0) + S_b(c_n, c_n, \Gamma c_n)]\Big] + \mu^2 S_b(\Gamma c_n, \Gamma c_n, c_n).\end{aligned}$$

Letting $n \to +\infty$, we get, $S_b(c_0, c_0, \Gamma c_0) < \mu^2(4\alpha_2 + 1) S_b(c_0, c_0, \Gamma c_0)$, so go to $\Gamma c_0 = c_0$. The uniqueness is in letting $c_0$ and $\widehat{c_0}$ be different fixed points of $\Gamma$. Thus, by (3.8), $S_b(c_0, c_0, \widehat{c_0}) = S_b(\Gamma c_0, \Gamma c_0, \Gamma \widehat{c_0}) \leq \alpha_1 S_b(c_0, c_0, \widehat{c_0})$, hence, $S_b(c_0, c_0, \widehat{c_0}) = 0$, that is, $c_0 = \widehat{c_0}$. □

**Theorem 3.3.** *Suppose $(\mathfrak{D}, S_b, \mathcal{W})$ is a complete $CS_b$-MS and $\Gamma : \mathfrak{D} \to \mathfrak{D}$ satisfies the condition: For all $c, z \in \mathfrak{D}$, such that $S_b(\Gamma c, \Gamma c, \Gamma z) > 0$ implies*

$$\begin{aligned}\tau + \mathfrak{F}(S_b(\Gamma c, \Gamma c, \Gamma z)) &\leq \mathfrak{F}\Big(\alpha_1 S_b(c, c, z) + \alpha_2 \frac{S_b(c, c, \Gamma c) S_b(c, c, \Gamma z) + S_b(z, z, \Gamma z) S_b(z, z, \Gamma c)}{S_b(c, c, \Gamma z) + S_b(z, z, \Gamma c)} \\ &\quad + \alpha_3 \frac{S_b(c, c, \Gamma c) S_b(z, z, \Gamma c) + S_b(z, z, \Gamma z) S_b(c, c, \Gamma z)}{S_b(c, c, \Gamma z) + S_b(z, z, \Gamma c)}\Big).\end{aligned} \tag{3.13}$$

*Let $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$. If $\alpha_1 < \frac{1}{\mu^3}$, $\alpha_2 + \alpha_3 < 1$ and $0 < \beta_{n-1} < \frac{1-\mu[(\alpha_2+\alpha_3)(\mu+1)-2\alpha_1\mu^2]}{\mu^2(2+\alpha_1\mu)}$, for $n \in \mathbb{N}$, then there is a unique fixed point of $\Gamma$ in $\mathfrak{D}$.*

*Proof.* Considering the same construction of the previous result, if we conclude

$$\begin{aligned}&\tau + \mathfrak{F}(S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)) \leq \mathfrak{F}(\alpha_1 S_b(c_{n-1}, c_{n-1}, c_n) \\ &+ \alpha_2 \frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_n) S_b(c_n, c_n, \Gamma c_{n-1})}{S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_{n-1})} \\ &+ \alpha_3 \frac{S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) S_b(c_n, c_n, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n) S_b(c_{n-1}, c_{n-1}, \Gamma c_n)}{S_b(c_{n-1}, c_{n-1}, \Gamma c_n) + S_b(c_n, c_n, \Gamma c_{n-1})}\Big) \\ &\leq \mathfrak{F}(\alpha_1[2\mu S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + \mu S_b(c_n, c_n, \Gamma c_{n-1})] \\ &+ (\alpha_2 + \alpha_3)[S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + S_b(c_n, c_n, \Gamma c_n)]) \\ &\leq \mathfrak{F}\big((2\mu\alpha_1 + \alpha_2 + \alpha_3) S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + \alpha_1 \mu S_b(c_n, c_n, \Gamma c_{n-1}) + (\alpha_2 + \alpha_3) S_b(c_n, c_n, \Gamma c_n)\big).\end{aligned} \tag{3.14}$$

Then, utilising (i), we obtain

$$S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n) \leq (2\mu\alpha_1 + \alpha_2 + \alpha_3) S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + \alpha_1 \mu S_b(c_n, c_n, \Gamma c_{n-1}) + (\alpha_2 + \alpha_3) S_b(c_n, c_n, \Gamma c_n), \tag{3.15}$$

implying, from (3.2) and (3.15),

$$\mathfrak{F}(S_b(c_n, c_n, \Gamma c_n)) \leq \mathfrak{F}\Big(2\mu\beta_{n-1} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + \mu(2\mu\alpha_1 + \alpha_2 + \alpha_3) S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + \alpha_1 \mu^2 \beta_{n-1} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) + (\alpha_2 + \alpha_3)\mu S_b(c_n, c_n, \Gamma c_n)\Big).$$

Utilising (i), we conclude

$$S_b(c_n, c_n, \Gamma c_n) \leq \frac{\mu[(2 + \alpha_1 \mu)(\beta_{n-1} + 2) + \alpha_2 + \alpha_3 - 4]}{1 - \alpha_2 \mu - \alpha_3 \mu} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}).$$

Plugging $\gamma_{n-1} = \frac{\mu[(2+\alpha_1\mu)(\beta_{n-1}+2)+\alpha_2+\alpha_3-4]}{1-\alpha_2\mu-\alpha_3\mu}$, and note that $\alpha_1 < \frac{1}{\mu^3}, \alpha_2 + \alpha_3 < 1$, and $0 < \beta_{n-1} < \frac{1-\mu[(\alpha_2+\alpha_3)(\mu+1)-2\alpha_1\mu^2]}{\mu^2(2+\alpha_1\mu)}$ for $n \in \mathbb{N}$, we deduce

$$S_b(c_n, c_n, \Gamma c_n) \leq \gamma_{n-1} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}) < \frac{1}{\mu} S_b(c_{n-1}, c_{n-1}, \Gamma c_{n-1}).$$

Proceeding in the same way, we can show that $\{S_b(c_n, c_n, \Gamma c_n)\}$ is a decreasing sequence of non-negative reals, then, $\lim_{n\to+\infty} S_b(c_n, c_n, c_{n+1}) = 0$.

Moreover, we prove that $\{c_n\}$ is a Cauchy in $\mathfrak{D}$. On the contrary, if the claim is that $\{c_n\}$ is not Cauchy, then by Lemma 2.6, there exist $\epsilon > 0$ and subsequence $\{c_{m_k}\}$ and $\{c_{n_k}\}$ of $\{c_n\}$ positive integers such that $\lim_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k-1}}) \leq \frac{\epsilon}{\mu^2}$, and

$$\begin{aligned} S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) &= S_b(c_{m_{k+1}}, c_{m_{k+1}}, \mathcal{W}(c_{n_{k-1}}, \Gamma c_{n_{k-1}}; \beta_{n_{k-1}})) \\ &\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}}) S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, c_{m_{k+1}}) \\ &\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \\ &+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})]. \end{aligned}$$

By the same process above with (3.15) from (3.13), with respect to the two subsequences, we obtain

$$S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \leq \alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (\alpha_2 + \alpha_3)[S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})], \tag{3.16}$$

because

$$\begin{aligned} &S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) \leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \\ &+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\ &< \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1 - \beta_{n_{k-1}})\mu[2\alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) \\ &+ 2(\alpha_2 + \alpha_3)[S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] + 2S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\ &< (\beta_{n_{k-1}} + 2\alpha_1 \mu(1 - \beta_{n_{k-1}})) S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) \end{aligned}$$

$$
\begin{aligned}
&+2\mu(\alpha_2+\alpha_3)(1-\beta_{n_{k-1}})[S_b(c_{n_{k-1}},c_{n_{k-1}},\Gamma c_{n_{k-1}})+S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}})]\\
&+2\mu(1-\beta_{n_{k-1}})S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}})\\
&<(\beta_{n_{k-1}}+2\alpha_1\mu(1-\beta_{n_{k-1}}))\mu[2S_b(c_{n_{k-1}},c_{n_{k-1}},c_{n_k})+S_b(c_{n_k},c_{n_k},c_{m_{k+1}})]\\
&+2\mu(\alpha_2+\alpha_3)(1-\beta_{n_{k-1}})[S_b(c_{n_{k-1}},c_{n_{k-1}},\Gamma c_{n_{k-1}})+S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}})]\\
&+2\mu(1-\beta_{n_{k-1}})S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}})\\
&<2\mu^2\alpha_1[2S_b(c_{n_{k-1}},c_{n_{k-1}},c_{n_k})+S_b(c_{n_k},c_{n_k},c_{m_{k+1}})]\\
&+2\mu(\alpha_2+\alpha_3)(1-\beta_{n_{k-1}})[S_b(c_{n_{k-1}},c_{n_{k-1}},\Gamma c_{n_{k-1}})+S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}})]\\
&+2\mu(1-\beta_{n_{k-1}})S_b(c_{m_{k+1}},c_{m_{k+1}},\Gamma c_{m_{k+1}}).
\end{aligned}
$$

Taking $k\to+\infty$, we deduce $\lim_{k\to+\infty}\sup S_b(c_{n_k},c_{n_k},c_{m_{k+1}})\le\frac{\epsilon}{\mu^2}$, which is a contradiction, so $\{c_n\}$ is Cauchy sequence in $\mathfrak{D}$. By completeness, there is $c_0\in\mathfrak{D}$ so that $\lim_{n\to+\infty}c_n=c_0$. To prove that $c_0$ is a fixed point of $\Gamma$, from Eq. (3.13) and

$$
\begin{aligned}
&\tau+\mathfrak{F}\Big(S_b(\Gamma c_0,\Gamma c_0,\Gamma c_n)\Big)\le\mathfrak{F}\Big(\alpha_1S_b(c_0,c_0,c_n)\\
&+\alpha_2\frac{S_b(c_0,c_0,\Gamma c_0)S_b(c_0,c_0,\Gamma c_n)+S_b(c_n,c_n,\Gamma c_n)S_b(c_n,c_n,\Gamma c_0)}{S_b(c_0,c_0,\Gamma c_n)+S_b(c_n,c_n,\Gamma c_0)}\\
&+\alpha_3\frac{S_b(c_0,c_0,\Gamma c_0)S_b(c_n,c_n,\Gamma c_0)+S_b(c_n,c_n,\Gamma c_n)S_b(c_0,c_0,\Gamma c_n)}{S_b(c_0,c_0,\Gamma c_n)+S_b(c_n,c_n,\Gamma c_0)}\Big)\\
&\le\mathfrak{F}(\alpha_1S_b(c_0,c_0,c_n)+(\alpha_2+\alpha_3)[S_b(c_0,c_0,\Gamma c_0)+S_b(c_n,c_n,\Gamma c_n)]),
\end{aligned}
\tag{3.17}
$$

Therefore, by (i), we undergo

$$
S_b(\Gamma c_0,\Gamma c_0,\Gamma c_n)\le\alpha_1S_b(c_0,c_0,c_n)+(\alpha_2+\alpha_3)[S_b(c_0,c_0,\Gamma c_0)+S_b(c_n,c_n,\Gamma c_n)].
$$

Subsequently,

$$
\begin{aligned}
S_b(c_0,c_0,\Gamma c_0)\le{}&2\mu S_b(c_0,c_0,c_n)+2\mu^2[\alpha_1S_b(c_0,c_0,c_n)\\
&+(\alpha_2+\alpha_3)\,[S_b(c_0,c_0,\Gamma c_0)+S_b(c_n,c_n,\Gamma c_n)]]+\mu^2S_b(\Gamma c_n,\Gamma c_n,c_n).
\end{aligned}
$$

Letting $n\to+\infty$, we get, $S_b(c_0,c_0,\Gamma c_0)<\mu^2(4\alpha_2+4\alpha_3+1)S_b(c_0,c_0,\Gamma c_0)$, so go to $\Gamma c_0=c_0$. The uniqueness is in, letting $c_0$ and $\widehat{c_0}$ be different fixed points of $\Gamma$. Thus, by (3.13), $S_b(c_0,c_0,\widehat{c_0})=S_b(\Gamma c_0,\Gamma c_0,\Gamma\widehat{c_0})\le\alpha_1S_b(c_0,c_0,\widehat{c_0})$. Thus, $c_0=\widehat{c_0}$ and we get the desired result. □

**Example 3.1.** *Suppose $\mathfrak{D}=\mathbb{R}$ and define $S_b$ by (see Example 2.1 (2))*

$$
S_b(a,b,c)=d_b(a,c)+d_b(b,c).
$$

*Then let $(\mathfrak{D},S_b,\mathcal{W})$ be a complete $CS_b$-MS with $\mu=2$, where $d_b(a,b)=(a-b)^2$, for each $a,b\in\mathfrak{D}$. Now, consider the map $\Gamma:\mathfrak{D}\to\mathfrak{D}$ by*

$$
\Gamma(c)=\frac{-c}{26}.
$$

*Here* $\mathcal{W} : \mathfrak{D} \times \mathfrak{D} \times [0,1] \to \mathfrak{D}$ *is a mapping such that* $\mathcal{W}(a,b;\beta) \leq a\beta + (1-\beta)b$ *for* $a,b \in \mathfrak{D}$ *and* $\beta \in [0,1]$. *Set* $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$ *and* $\beta_{n-1} = \frac{1}{14} < \frac{1}{13} = \rho$ *in Theorem 3.1.*

$$\begin{aligned} S_b(\Gamma c, \Gamma c, \Gamma z) &= (\Gamma c - \Gamma c)^2 + (\Gamma c - \Gamma z)^2 \\ &= \frac{1}{26^2}(c-z)^2 \leq (\frac{1}{26}c)^2 + (\frac{1}{26}z)^2 \\ &\leq \frac{1}{26}(c + \frac{1}{26}c)^2 + \frac{1}{26}(z + \frac{1}{26}z)^2 \\ &= \frac{1}{26}\frac{S_b(c,c,\Gamma c)S_b(c,c,\Gamma z) + S_b(z,z,\Gamma z)S_b(z,z,\Gamma c)}{\max\{S_b(c,c,\Gamma z), S_b(z,z,\Gamma c)\}}. \end{aligned}$$

*Taking* $F(c) = \ln(c)$ *in* $\mathfrak{F}$, *we deduce that*

$$\ln(2) + \ln(S_b(\Gamma c, \Gamma c, \Gamma z)) \leq \ln\left(\frac{1}{13}\frac{S_b(c,c,\Gamma c)S_b(c,c,\Gamma z) + S_b(z,z,\Gamma z)S_b(z,z,\Gamma c)}{\max\{S_b(c,c,\Gamma z), S_b(z,z,\Gamma c)\}}\right).$$

*Hence* $\mathfrak{F}$*-contraction axiom is satisfied.*

*Choose* $c_0 \in \mathfrak{D}, c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1}), \beta_{n-1} = \frac{1}{14}$ *and* $\Gamma(c) = \frac{-c}{26}$, *implies that* $c_n = \frac{1}{14}c_{n-1} + \frac{13}{14}\left(\frac{-c_{n-1}}{26}\right) = \frac{1}{28}c_{n-1}$, *repeating in the same manner, we get* $c_n = (\frac{1}{28})^n c_0$. *Thus,* $c_n \to 0$ *as* $n$ *tends to* $\infty$. *So* $0$ *is a fixed point of* $\Gamma$. *Similarly, the results hold if we take* $S_b$ *in Example 2.5.*

**Example 3.2.** *Let* $\{c_n\}_{n>0}$ *be a sequence such that:* $c_n = \frac{n(n+1)}{2}$. *Let* $\mathfrak{D} = \{c_n : n \in \mathbb{N}\}$ *and take* $S_b$ *in Example 2.5. Then let* $(\mathfrak{D}, S_b, \mathcal{W})$ *be a complete* $CS_b$*-MS with* $\mu = 2$, *such that a mapping* $\mathcal{W}(a,b;\beta) \leq a\beta + (1-\beta)b$ *for* $a,b \in \mathfrak{D}$ *and* $\beta \in [0,1]$. *Define the mapping* $\Gamma : \mathfrak{D} \to \mathfrak{D}$ *by* $\Gamma(c_1) = c_1 = 1$ *and* $\Gamma(c_n) = c_{n-1}$ *for* $n > 1$. *Therefore,* $c_1$ *is a fixed point of* $\Gamma$ *with* $\mathfrak{F}$*-contraction, which is* $F(c) = c + \ln c$ *and* $\tau = e^{-1}$ *see Ref. [19]. Moreover, taking* $\alpha_1 = \frac{1}{9}, \alpha_2 = 0$ *and* $\beta_{n-1} < \frac{1+16\alpha_1}{8(1+\alpha_1)}$ *in Theorem 3.2. It is not difficult to see that* $\Gamma$ *satisfies the desired result.*

Next, we show the special rational kind of $\mathfrak{F}$-contraction with Mann's iteration in complete $CS_b$-MS.

**Theorem 3.4.** *Let* $(\mathfrak{D}, S_b, \mathcal{W})$ *be a complete* $CS_b$*-MS and* $\Gamma : \mathfrak{D} \to \mathfrak{D}$ *satisfies the condition: For all* $c, z \in \mathfrak{D}$, *such that* $S_b(\Gamma c, \Gamma c, \Gamma z) > 0$ *implies*

$$\begin{aligned} \tau + \mathfrak{F}(S_b(\Gamma c, \Gamma c, \Gamma z)) \leq \mathfrak{F}\Big(&\alpha_1 S_b(c,c,z) + \alpha_2 \frac{S_b^2(c,c,\Gamma c)}{\mu[2S_b(z,z,\Gamma c) + S_b(z,z,c)]} \\ &+ \alpha_3 \frac{S_b^2(z,z,\Gamma z)}{\mu[2S_b(c,c,z) + S_b(c,c,\Gamma z)]}\Big). \end{aligned} \tag{3.18}$$

*Let* $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$. *If* $\alpha_1 < \frac{1}{2\mu^3}, \alpha_2 < \frac{1}{\mu^2}, \alpha_3 < 1$, *and* $0 < \beta_{n-1} < \frac{(1-\alpha_3)-\mu[\alpha_2+2\alpha_1\mu]}{\mu^2(2+\alpha_1\mu)}$, *for* $n \in \mathbb{N}$, *then there is a unique fixed point of* $\Gamma$ *in* $\mathfrak{D}$.

*Proof.* Considering the same construction of the previous result, if we conclude

$$\begin{aligned} \tau + \mathfrak{F}(S_b(\Gamma c_{n-1}, \Gamma c_{n-1}, \Gamma c_n)) \leq \mathfrak{F}\Big(&\alpha_1 S_b(c_{n-1}, c_{n-1}, c_n) \\ &+ \alpha_2 \frac{S_b^2(c_{n-1}, c_{n-1}, \Gamma c_{n-1})}{\mu[2S_b(c_n, c_n, \Gamma c_{n-1}) + S_b(c_n, c_n, c_{n-1})]} \end{aligned}$$

$$+\alpha_3\frac{S_b^2(c_n,c_n,\Gamma c_n)}{\mu[2S_b(c_{n-1},c_{n-1},c_n)+S_b(c_{n-1},c_{n-1},\Gamma c_n)]}\Big).$$

Since

$$S_b(c_n,c_n,\Gamma c_n)\le 2\mu S_b(c_n,c_n,c_{n-1})+\mu S_b(c_{n-1},c_{n-1},\Gamma c_n).$$

Hence,

$$\begin{aligned}\tau+\mathfrak{F}(S_b(\Gamma c_{n-1},\Gamma c_{n-1},\Gamma c_n))&\le\mathfrak{F}\Big(\alpha_1S_b(c_{n-1},c_{n-1},c_n)\\&+\alpha_2\frac{[2\mu S_b(c_{n-1},c_{n-1},c_n)+\mu S_b(c_n,c_n,\Gamma c_{n-1})]S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})}{\mu[2S_b(c_n,c_n,\Gamma c_{n-1})+S_b(c_n,c_n,c_{n-1})]}\\&+\alpha_3\frac{[2\mu S_b(c_{n-1},c_{n-1},c_n)+\mu S_b(c_{n-1},c_{n-1},\Gamma c_n)]S_b(c_n,c_n,\Gamma c_n)}{\mu[2S_b(c_{n-1},c_{n-1},c_n)+S_b(c_{n-1},c_{n-1},\Gamma c_n)}\Big)\\&=\mathfrak{F}(\alpha_1[2\mu S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\mu S_b(c_n,c_n,\Gamma c_{n-1})]\\&+\alpha_2S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\alpha_3S_b(c_n,c_n,\Gamma c_n)).\end{aligned}\tag{3.19}$$

Then, utilising (i), we obtain

$$\begin{aligned}S_b(\Gamma c_{n-1},\Gamma c_{n-1},\Gamma c_n)&\le(2\mu\alpha_1+\alpha_2)S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})\\&+\mu\alpha_1S_b(c_n,c_n,\Gamma c_{n-1})+\alpha_3S_b(c_n,c_n,\Gamma c_n),\end{aligned}\tag{3.20}$$

implies, from (3.2) and (3.20),

$$\begin{aligned}\mathfrak{F}(S_b(c_n,c_n,\Gamma c_n))\le\mathfrak{F}(&2\mu\beta_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+(2\mu\alpha_1+\alpha_2)S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})\\&+\mu\alpha_1\beta_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})+\alpha_3S_b(c_n,c_n,\Gamma c_n)).\end{aligned}$$

Utilising (i), we conclude

$$S_b(c_n,c_n,\Gamma c_n)\le\frac{\mu\beta_{n-1}(2+\alpha_1)+2\mu\alpha_1+\alpha_2}{1-\alpha_3}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1}).$$

Plugging $\gamma_{n-1}=\frac{\mu\beta_{n-1}(2+\alpha_1)+2\mu\alpha_1+\alpha_2}{1-\alpha_3}$, and note that $\alpha_1<\frac{1}{2\mu^3},\alpha_2<\frac{1}{\mu^2},\alpha_3<1$, and $0<\beta_{n-1}<\frac{(1-\alpha_3)-\mu[\alpha_2+2\alpha_1\mu]}{\mu^2(2+\alpha_1\mu)}$ for $n\in\mathbb{N}$, we deduce

$$S_b(c_n,c_n,\Gamma c_n)\le\gamma_{n-1}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1})<\frac{1}{\mu}S_b(c_{n-1},c_{n-1},\Gamma c_{n-1}).$$

Proceeding in the same way, we can show that $\{S_b(c_n,c_n,\Gamma c_n)\}$ is a decreasing sequence of non-negative reals, then, $\lim_{n\to+\infty}S_b(c_n,c_n,c_{n+1})=0$.
Moreover, we prove that $\{c_n\}$ is a Cauchy in $\mathfrak{D}$. On the contrary, if claim that $\{c_n\}$ is not a Cauchy, then by Lemma 2.6, there exist $\epsilon>0$ and subsequence $\{c_{m_k}\}$ and $\{c_{n_k}\}$ of $\{c_n\}$ positive integers such that $\lim_{k\to+\infty}\sup S_b(c_{n_k},c_{n_k},c_{m_{k-1}})\le\frac{\epsilon}{\mu^2}$, and

$$\begin{aligned}S_b(c_{n_k},c_{n_k},c_{m_{k+1}})&=S_b(c_{m_{k+1}},c_{m_{k+1}},\mathcal{W}(c_{n_{k-1}},\Gamma c_{n_{k-1}});\beta_{n_{k-1}}))\\&\le\beta_{n_{k-1}}S_b(c_{n_{k-1}},c_{n_{k-1}},c_{m_{k+1}})+(1-\beta_{n_{k-1}})S_b(\Gamma c_{n_{k-1}},\Gamma c_{n_{k-1}},c_{m_{k+1}})\\&\le\beta_{n_{k-1}}S_b(c_{n_{k-1}},c_{n_{k-1}},c_{m_{k+1}})+(1-\beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}},\Gamma c_{n_{k-1}},\Gamma c_{m_{k+1}})\end{aligned}$$

$$+ S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})].$$

By same process above with (3.20) from (3.18), with respect to the two subsequences, we obtain

$$\begin{aligned} S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) &\leq \alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + \alpha_2 S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\ &\quad + \alpha_3 S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}), \end{aligned} \tag{3.21}$$

because

$$\begin{aligned} S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) &\leq \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}})\mu[2S_b(\Gamma c_{n_{k-1}}, \Gamma c_{n_{k-1}}, \Gamma c_{m_{k+1}}) \\ &\quad + S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\ &< \beta_{n_{k-1}} S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) + (1-\beta_{n_{k-1}})\mu[2\alpha_1 S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) \\ &\quad + 2\alpha_2 S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) + (2\alpha_3+1) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}})] \\ &< (\beta_{n_{k-1}} + 2\alpha_1\mu(1-\beta_{n_{k-1}})) S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{m_{k+1}}) \\ &\quad + 2\mu\alpha_2(1-\beta_{n_{k-1}}) S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\ &\quad + \mu(1-\beta_{n_{k-1}})(2\alpha_3+1) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}) \\ &< (\beta_{n_{k-1}} + 2\alpha_1\mu(1-\beta_{n_{k-1}}))\mu[2S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{n_k}) + S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}})] \\ &\quad + 2\mu\alpha_2(1-\beta_{n_{k-1}}) S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\ &\quad + \mu(1-\beta_{n_{k-1}})(2\alpha_3+1) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}) \\ &< 2\mu^2\alpha_1[2S_b(c_{n_{k-1}}, c_{n_{k-1}}, c_{n_k}) + S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}})] \\ &\quad + 2\mu\alpha_2(1-\beta_{n_{k-1}}) S_b(c_{n_{k-1}}, c_{n_{k-1}}, \Gamma c_{n_{k-1}}) \\ &\quad + \mu(1-\beta_{n_{k-1}})(2\alpha_3+1) S_b(c_{m_{k+1}}, c_{m_{k+1}}, \Gamma c_{m_{k+1}}). \end{aligned}$$

Taking $k \to +\infty$, we deduce $\lim\limits_{k\to+\infty} \sup S_b(c_{n_k}, c_{n_k}, c_{m_{k+1}}) \leq \frac{\epsilon}{\mu^2}$, which is a contradiction, so $\{c_n\}$ is a Cauchy sequence in $\mathfrak{D}$. By completeness, there is $c_0 \in \mathfrak{D}$ so that $\lim\limits_{n\to+\infty} c_n = c_0$. To prove that $c_0$ is a FP of $\Gamma$, from (3.18)

$$\begin{aligned} \tau + \mathfrak{F}(S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n)) &\leq \mathfrak{F}\Big(\alpha_1 S_b(c_0, c_0, c_n) + \alpha_2 \frac{S_b^2(c_0, c_0, \Gamma c_0)}{\mu[2S_b(c_n, c_n, \Gamma c_0) + S_b(c_n, c_n, c_0)]} \\ &\quad + \alpha_3 \frac{S_b^2(c_n, c_n, \Gamma c_n)}{\mu[2S_b(c_0, c_0, c_n) + S_b(c_0, c_0, \Gamma c_n)]}\Big) \\ &\leq \mathfrak{F}(\alpha_1 S_b(c_0, c_0, c_n) + \alpha_2 S_b(c_0, c_0, \Gamma c_0) + \alpha_3 S_b(c_n, c_n, \Gamma c_n)). \end{aligned} \tag{3.22}$$

Therefore, by (i), we undergo

$$S_b(\Gamma c_0, \Gamma c_0, \Gamma c_n) \leq \alpha_1 S_b(c_0, c_0, c_n) + \alpha_2 S_b(c_0, c_0, \Gamma c_0) + \alpha_3 S_b(c_n, c_n, \Gamma c_n).$$

Subsequently,

$$\begin{aligned} S_b(c_0, c_0, \Gamma c_0) &\leq 2\mu S_b(c_0, c_0, c_n) + 2\mu^2[\alpha_1 S_b(c_0, c_0, c_n) + \alpha_2 S_b(c_0, c_0, \Gamma c_0) \\ &\quad + \alpha_3 S_b(c_n, c_n, \Gamma c_n)] + \mu^2 S_b(\Gamma c_n, \Gamma c_n, c_n). \end{aligned}$$

Letting $n \to +\infty$, we get, $S_b(c_0, c_0, \Gamma c_0) < \mu^2(2\alpha_2 + 2\alpha_3 + 1)S_b(c_0, c_0, \Gamma c_0)$, so go to $\Gamma c_0 = c_0$. The uniqueness is in, letting $c_0$ and $\widehat{c_0}$ be different FP of $\Gamma$. Thus, by (3.18), $S_b(c_0, c_0, \widehat{c_0}) = S_b(\Gamma c_0, \Gamma c_0, \Gamma\widehat{c_0}) \leq \alpha_1 S_b(c_0, c_0, \widehat{c_0}) < S_b(c_0, c_0, \widehat{c_0})$. Thus, $c_0 = \widehat{c_0}$ and we get the desired result. □

## 4. Applications

The concepts of existence and uniqueness have become attractive for researchers in nonlinear analysis, particularly for solving differential equations, integral equations, and fractional differential equations, among others. This situation has improved the applications of FP techniques.

4.1. **Mixed Volterra-Fredholm integral:** In this part, we apply Theorem 3.2 The mixed Volterra-Fredholm integral (MVFI) is given by

$$z(\eta) = h(\eta) + \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z(t) dt dr, \ \eta \in [a,b] = I, \tag{4.1}$$

where $h(\eta)$ and $\mathfrak{H}(r,t)$ are continuous on $[a,b]$ and $\mathcal{K} = \{(r,t) : t, r \leq \eta \in I\}$, but not necessary if $z(\eta)$ is a continuous.

**Theorem 4.1.** *Let Eq.* (4.1) *be satisfied and* $\mathfrak{H}(r,t)$ *be a bounded function, if* $|\alpha| < \frac{1}{3k^3\xi(b-a)^2}, k > 1$ *be a constant number, then* (4.1) *admits a unique solution on I. Moreover, we can write the solution as*

$$z_n(\eta) = h(\eta) + \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z_n(t) dt dr, \ \eta \in I,$$

*for n tend to* $+\infty$, *such* $z_0(\eta) = z_0, 0 < \beta_{n-1} < \frac{1-2[3\alpha_2 - 8\alpha_1]}{8(1+\alpha_1}$, *and* $z_n(\eta) = \beta_{n-1} z_{n-1}(\eta) + (1 - \beta_{n-1})\Gamma z_{n-1}(\eta)$.

*Proof.* Consider the space $\mathfrak{D} = C[a,b]$, and the mapping $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ be defined as $S_b(z_1, z_2, z_3) = \max_{\eta \in I}(|z_1(\eta) - z_3(\eta)| + |z_1(\eta) + z_3(\eta) - 2z_2(\eta)|)$, and the operator of $\Gamma$ on $\mathfrak{D}$ by the form

$$\Gamma(z) = h(\eta) + \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z(t) dt dr,$$

$$\begin{aligned} z_n(\eta) &= \mathcal{W}(z_{n-1}(\eta), \Gamma z_{n-1}(\eta); \beta_{n-1}) = \beta_{n-1} z_{n-1}(\eta) + (1 - \beta_{n-1})\Gamma z_{n-1}(\eta) \\ &= \beta_{n-1} z_{n-1}(\eta) + (1 - \beta_{n-1})\left(h(\eta) + \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z_n(t) dt dr\right), \end{aligned}$$

for $n \geq 1$. Obviously, $(\mathfrak{D}, S_b, \mathcal{W})$ is a complete $CS_b$-MS. So that

$$\begin{aligned} S_b(\Gamma z_1, \Gamma z_1, \Gamma z_2) &= 2 \max_{\eta \in I} |\Gamma z_1(\eta) - \Gamma z_2(\eta)| \\ &= 2 \max_{\eta \in I} \left| \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z_1(t) dt dr - \alpha \int_a^\eta \int_a^b \mathfrak{H}(r,t) z_2(t) dt dr \right| \\ &\leq 2|\alpha| \max_{\eta \in I} \int_a^\eta \int_a^b |\mathfrak{H}(r,t)| \max_{\eta \in I} |z_1(\eta) - z_2(\eta)| dt dr \end{aligned}$$

$$\leq 2|\alpha| S_b(z_1, z_1, z_2) \max_{\eta \in I} \int_a^\eta \int_a^b |\mathfrak{H}(r,t)| dtdr,$$

known that $\mathfrak{H}(r,t)$ is bounded, there is $\xi > 0$ such that $|\mathfrak{H}(r,t)| \leq \xi$, hence

$$\begin{aligned} S_b(\Gamma z_1, \Gamma z_1, \Gamma z_2) &\leq 2|\alpha| \xi S_b(z_1, z_1, z_2) \max_{\eta \in I} \int_a^\eta \int_a^b dtdr \\ &\leq 2|\alpha| \xi S_b(z_1, z_1, z_2)(b-a)^2 \\ &\leq \frac{2\xi(b-a)^2}{3k^3 \xi (b-a)^2} S_b(z_1, z_1, z_2) = \frac{2}{3k^3} S_b(z_1, z_1, z_2) \\ &\leq \frac{2}{3}\Bigg[\alpha_1 S_b(z_1, z_1, z_2) \\ &+ \alpha_2 \frac{S_b(z_1, z_1, \Gamma z_1) S_b(z_1, z_1, \Gamma z_2) + S_b(z_2, z_2, \Gamma z_2) S_b(z_2, z_2, \Gamma z_1)}{S_b(z_1, z_1, \Gamma z_2) + S_b(z_2, z_2, \Gamma z_1)}\Bigg]. \end{aligned}$$

Taking $\alpha_1 = \frac{1}{k^3}$, and both sides $F(\eta) = \ln(\eta)$. Therefore, the desired results hold for the (3.8), where $\tau = \ln(1.5)$. Finally, showing the uniqueness, there is $z(\eta)$, which implies $z_n(\eta) \to z(\eta)$, for $n \geq 0$, and $\Gamma z(\eta) = z(\eta)$, we obtain

$$\lim_{n \to +\infty} \int_a^\eta \int_a^b \mathfrak{H}(r,t) z_n(t) dtdr = \int_a^\eta \int_a^b \mathfrak{H}(r,t) z(t) dtdr.$$

Rearranging terms and taking the supremum for both sides, we obtain

$$\begin{aligned} \lim_{n \to +\infty} \sup \int_a^\eta \int_a^b \mathfrak{H}(r,t)(z_n(t) - z(t)) dtdr &\leq \lim_{n \to +\infty} \sup \int_a^\eta \int_a^b |\mathfrak{H}(r,t)||z_n(t) - z(t)| dtdr \\ &\leq \xi(b-a)^2 \lim_{n \to +\infty} \sup \max |z_n(t) - z(t)| = 0. \end{aligned}$$

Then, the MVFI inclusion (4.1) has a unique solution. □

4.2. **Polynomial Equation:** In this part, we demonstrate the $m$th degree polynomial by FPT with convexity. It can be solved in numerous ways; however, we try to have a unique solution using FP techniques.

**Theorem 4.2.** *Assume $\mathfrak{D} = [-1, 1]$, and let $m \geq 3$ be an arbitrary number in $\mathbb{N}$. Then*

$$c^m - (m^4 - 1)c^{m+1} - m^4 c + 1 = 0, \tag{4.2}$$

*admits a unique solution in $\mathfrak{D}$.*

*Proof.* Consider the space $\mathfrak{D} = [-1, 1]$, such that without the space $\mathfrak{D}$ in the Eq. (4.2), it has no solution. Thus the mapping $S_b : \mathfrak{D} \times \mathfrak{D} \times \mathfrak{D} \to [0, +\infty)$ is defined as $S_b(c_1, c_2, c_3) = \Big(|c_1 - c_3| + |c_1 + c_3 - 2c_2|\Big)^2$, and the mapping $\Gamma$ on $\mathfrak{D}$ as

$$\Gamma(c) = \frac{c^m + 1}{(m^4 - 1)c^m + m^4},$$

also the mapping is defined as:
By hypothesis $m \geq 3$, taking $m = 3$ is an easy computation, and by not utilising this method, it leads to the results that hold for each $m \geq 3$. We get

$$\Gamma(c) = \frac{c^3+1}{80c^3+81}, \tag{4.3}$$

therefore, the $\Gamma$ is satisfying the $\mathfrak{F}$-contraction, where $F(c) = \ln(c)$, as follows:

$$\begin{aligned} S_b(\Gamma c_1, \Gamma c_1, \Gamma c_2) &= 4|\Gamma c_1 - \Gamma c_2|^2 \\ &= 4\left|\frac{c_1^m+1}{(m^4-1)c_1^m+m^4} - \frac{c_2^m+1}{(m^4-1)c_2^m+m^4}\right|^2 \\ &= 4\left|\frac{c_1^3-c_2^3}{(80c_1^3+81)(80c_2^3+81)}\right|^2 \leq \left(\frac{2}{81}\right)^2 |c_1-c_2|^2. \end{aligned}$$

Hence

$$\ln(81) + \ln(S_b(\Gamma c_1, \Gamma c_1, \Gamma c_2)) \leq \ln\left(\frac{1}{81}\Big(|c_1-c_2| + |c_1-c_2|\Big)^2\right).$$

We note that $\tau = \ln(81) > 0$, in (4.3). Then, $\Gamma$ is $\mathfrak{F}$-contraction mapping of Theorem 3.2.
Lastly, let $c_0 \in \mathfrak{D}$; according to $c_n = \mathcal{W}(c_{n-1}, \Gamma c_{n-1}; \beta_{n-1})$, so that $c_n = \frac{1}{11}c_{n-1} + \frac{10}{11}\Gamma c_{n-1}$, where $\alpha_1 = \frac{1}{81}, \alpha_2 = 0$ and $\beta_{n-1} = \frac{1}{11}$ as in Theorem 3.2, we deduce that

$$c_1 = \frac{1}{11}c_0 + \frac{10}{11}\left[\frac{c_0^3+1}{80c_0^3+81}\right].$$

Continuing in the same process, observe that $c_n \to 0$ as $n \to +\infty$. Also, $c_n \in \mathfrak{D}$. Therefore, all the conditions of Theorem 3.2 are fulfilled. We have the desired result. □

## 5. Conclusions

This paper aims to explore $\mathfrak{F}$-Khan Contraction in the context of $CS_b$-MS. We introduced the concept of $CS_b$-MS endowed with Mann's iterative Scheme in $\mathfrak{F}$-Contraction (Wardowski's), which expands to $b$-MS and GbMS. We also introduced some improved results within the framework of $S_b$-MS with some intresting examples. Moreover, this work establishes the flexibility of various $\mathfrak{F}$-Khan Contractions in the context of MVFI inclusion and polynomial $m$th degree, providing special applications for these convex and nonlinear analysis concepts. This task was achieved by further weakening the conditions of Wardowski-Khan Contractions. Examples were provided to support our work. In the future, authors can use refined contractions or extended MS in broader literature. Furthermore, the result of G. M. Abd-Elhamed is a special case of these theorems.

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