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A Novel Approach to D-Stability and Additive D-Stability of Economic Models

Mutti-Ur Rehman[1,2,*], Saima Akram[3], Nozima Giyazova[4], Madina Sayfullaeva[4], Sitora Xasanova[4], Narzillo Ochilov[4]

[1]Center of Research and Innovation, Asia International University, Yangiobod MFY, G'ijduvon Street, House 74, Bukhara, Uzbekistan

[2]Department of Mathematics and Informatics, Al-Khwarizmi University, 220100, Khorezm, Uzbekistan

[3]Department of Mathematics, Government College Women University Faisalabad, Faisalabad, 38000, Pakistan

[4]Department of Economics, Bukhara State University, 200117, Uzbekistan

*Corresponding author: mutti.abbasi@gmail.com

Abstract. The study of D-stability in mathematical analysis is crucial for understanding and ensuring the stability of linear dynamical systems. This article introduces novel findings on the characterization of D-stability, along with its connections to additive D-stability concerning speed and coordinate transformations in linear dynamical systems with n degrees of freedom

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0,\ \tau \in \mathbb{R},\ \tau > 0,$$

Consider the stiffness, mass, and damping matrices $A, B, C \in \mathcal{M}^{n\times n}$, and let $\mu(\tau) \in \mathbb{R}^n$ denote the vector of generalized coordinates with $\frac{d\mu(\tau)}{d\tau}$ representing its corresponding velocity vector. This work derives new theoretical insights into D-stability, additive D-stability with respect to velocity, and additive D-stability concerning coordinate transformations. These results are established using techniques from linear algebra, matrix theory, dynamical systems, and their connections to structured singular value computations. Additionally, numerical investigations of the spectrum, singular values, and pseudospectra of the coefficient matrices $A, B, C \in \mathcal{M}^{n\times n}$ are conducted using EigTool, providing further validation of the theoretical framework.

1. Introduction

The notion of D-stability was first introduced in the seminal works of Arrow and McManus [18] and Enthoven and Arrow [1], where they examined this property for a specific class of structured matrices. The study of D-stability emerged from their analysis of equilibrium stability in competitive market dynamics, particularly in the context of optimal steady-state behavior. The analysis on D-stability and its utility in various mathematical problems has been found across the large scale dynamical systems as well as in the multi-parameter singular perturbations,

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see [8, 10, 11, 14, 31]. The characterization of D-stability is an important but becomes very difficult specially when sizes n of matrices are large, that is, $n > 3$, see [6, 7]. This lack of specification for D-stability led to the establishment and introduction of 13 sufficient requirements for D-stability by Johnson [6].

Numerous applications in the field of mathematical economics heavily rely on the computation and analysis of D-stable matrices, see [2, 3]. The vast amount of literature had been developed on the analysis of D-stable matrices, and some of the classical literature is [12, 15, 19, 24, 26, 32]. A positive diagonal matrix D must exist for a given n-dimensional real-valued matrix A to be D-stable, implying that the real part of each of the matrix-product eigenvalues of DA or AD is strictly positive (or strictly negative in literature). While D-stability of a matrix guarantees its stability, the converse does not necessarily hold.

In control theory, the structured singular value (commonly denoted as μ-values) serves as a powerful and versatile tool. It plays a crucial role in analyzing stability, performance, robustness, and synthesis problems for linear time-invariant systems subjected to structured perturbations or uncertainties with block-diagonal structure, as discussed in [4]. Block-diagonal acceptable perturbations or uncertainties fall into one of two classes: the number of whole blocks or the number of repeating scalar blocks. The repeating scalar blocks may consist of complex blocks, real blocks, or a combination of complex and real blocks. On the other hand, the number of full blocks are either dense complex matrices or dense real-valued matrices.

The accurate evaluation of the structured singular values is an NP-hard problem for the different classes of admissible real or complex perturbations, see [27]. Various mathematical approaches have been developed to estimate both lower and upper bounds for structured singular values, as demonstrated in [5, 9, 20, 25]. The estimation of structured singular values against the real or complex or a mixture of both real and complex uncertainties were studied in [4, 5, 9, 23], while for the pure complex uncertainties it was first studied and analyzed by Doyle [17].

The study in [29] investigated the gaps in characterizing structured singular values under different constraints and their connection to D-stability. The proposed framework highlights the deep connections between structured H-stability, $D(\alpha)$-structured stability, and the evaluation of structured singular values. For matrices with specific structures, this methodology allows for a unified treatment of stability analysis, encompassing D-stability, H-stability, $D(\alpha)$-stability, and structured singular values within a single analytical approach.

Since diagonal D-stability imposes stricter requirements than additive D-stability, a matrix A is considered additively D-stable if $A - D$ retains its Hurwitz property for any positive diagonal matrix D (refer to [13]). This concept plays a crucial role in the study of reaction-diffusion systems, where the matrix A corresponds to the linearized reaction terms evaluated at equilibrium. According to Casten and Holland [28], the stability analysis of reaction-diffusion partial differential equations (PDEs) reduces to verifying the stability of the family of matrices $A - \lambda_i D$, with D being a positive diagonal matrix and $\{\lambda_i\}$ denoting the non-zero eigenvalues of the Laplace operator subject to Neumann boundary conditions. Additionally, the authors of [22] explored the relationship between additive D-stability and reaction-diffusion systems, deriving conditions to assess stability or instability in the presence of diffusion effects.

This paper presents novel theoretical results for linear dynamical systems with n degrees of freedom, where the coefficient matrices A, B, and C represent stiffness, mass, and damping properties respectively. Our investigation of such systems involves the construction and analysis of specialized matrix products that capture the essential dynamics. Specifically, we develop

$$\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}; \begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} -A^{-1}B & -A^{-1}C+2D \\ I & 0 \end{bmatrix}.$$

We aim to construct and prove theoretical results concerning the D-stability, additive D-stability in relation to coordinates and additive D-stability in relation to speed, respectively. Our analytical methodology integrates spectral decomposition (eigenvalue calculation), singular value assessment, and the evaluation of structured singular values (μ-analysis), while investigating their intrinsic relationships with D-stability theory.

Overview of article: In section 2, we give basic concepts, definitions, and observations on structured singular values, D-stability, additive D-stability in relation to speed, and additive D-stability in relation to coordinates. We recall 13 sufficient conditions for D-stability as well. The problem statement is formulated and presented in section 3. We provide new results on D-stable, additive D-stability in relation to speed, and additive D-stability in relation to coordinates in section 4. The core approach for deriving and validating new results relies on analyzing eigenvalues, singular values, and structured singular values, along with their interplay with D-stability and strong D-stability. Numerical testing for structured matrices, for instance, Toeplitz matrices and transportation matrices, is presented in section 5, and in section 6, we finally bring our effort to a close in the form of conclusion.

2. Preliminaries

In this section, we present a foundational overview of additive D-stable and D-stable matrices, along with computational aspects of μ-values, by revisiting key definitions. Additionally, we survey existing results concerning the connections between D-stable and structured singular value matrices.

Definition 2.1. *Let $\mathbb{B}_1$ be the collection of block-diagonal matrices having complex and real uncertainty or perturbations, such that*

$$\mathbb{B}_1 := \{diag\left(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \cdots, \Delta_F\right) : \delta_i \in \mathbb{K},\ \Delta_j \in \mathbb{K}^{m_j, m_j},\ i = 1:S,\ j = 1:F\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.2. *[4] For a specific $M \in \mathbb{C}^{n,n}$, the structured singular value is denoted with $\mu_{\mathbb{B}_1}(M)$, and is defined by*

$$\mu_{\mathbb{B}_1}(M) := \begin{cases} 0, \ \ if \ det(I - M\Delta) \neq 0, \ \forall \Delta \in \mathbb{B}_1 \\ \Big(min\{\|\Delta\|_2 : \ det(I - M\Delta) = 0, \ \forall \Delta \in \mathbb{B}_1\}\Big),^{-1} \quad else \end{cases}$$

*where **min** is taken over all $\Delta \in \mathbb{B}_1$.*

Remark 2.1. *[4] The $\mathbb{B}_1$ also represent a multi-index of integers. This further implies that the approximation to μ-values are depended on a given matrix and set of block diagonal structure.*

Remark 2.2. *[4] The number of the full blocks (real or complex) in $\mathbb{B}_1$, are equivalent to rank-1 matrices, or dyads.*

Remark 2.3. *[4] From the definition of μ-value it is evident that to each $\alpha \in \mathbb{C}$, one can have that $\mu_{\mathbb{B}_1}(\alpha M) = \alpha \mu_{\mathbb{B}_1}(M)$.*

The Lemma 2.1 gives an alternate way for the approximation of $\mu_{\mathbb{B}_1}(M)$.

Lemma 2.1. *[4] Let $M \in \mathbb{C}^{n,n}$, and for all $\Delta \in \mathbb{B}_1$, we have $\mu_{\mathbb{B}_1}(M) = max \ \rho(\Delta M)$, where $\rho(\cdot)$ denotes the spectral radius, and **max** is over all $\Delta \in \mathbb{B}_1$.*

Definition 2.3. *[6] A matrix M is defined as D-stable if it satisfies two conditions: (i) all its eigenvalues lie strictly in the left half-plane of the complex domain, and (ii) both products DM and MD maintain this eigenvalue property for every positive diagonal matrix D. This ensures stability under arbitrary positive diagonal scaling of the matrix.*

The following four observations are taken from [6] and they holds true for the *D*-stable matrices.

Observation 1. *The requirements that must be met for D-stability remains the preserved under positive diagonal multiplication.*

Observation 2. *Let $M \in \mathbb{C}^{n,n}$ such that none of its eigenvalues are 0, and matrix-product DM is stable for a positive diagonal matrix D. The M-1 matrix is invertible, and further $\hat{D}^T M \hat{D}$, $\hat{D}MD$, M^T are all D-stable matrices, with $\hat{D}$ with a positive diagonal structure.*

Observation 3. *If $M \in \mathbb{C}^{n,n}$ such that the matrix-product DM., then the $r \times r$ principal sub-matrix of given matrix M is from the euclidean closure of $r \times r$ D-stable matrices.*

Observation 4. *If and only if $det(M \pm iD)$ is not precisely equal to zero, the supplied matrix M for a positive diagonal matrix D is a D-stable matrix.*

[21] is the source of the definitions $4 - 7$ that follow.

Definition 2.4. *Consider a linear dynamical system with n-degrees of freedom described by*

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0, \quad t \in \mathbb{R}, \ t > 0,$$

where $\mu(\tau) \in \mathbb{R}^n$ *represents the displacement vector,* $\frac{d\mu(\tau)}{d\tau}$ *denotes the generalized velocity vector, and* $A, B, C \in \mathcal{M}^{n\times n}$ *are the stiffness, mass, and damping matrices, respectively.*

The system is referred to as D-stable if, for any positive diagonal matrix D, the modified system

$$A\frac{d^2\mu(\tau)}{d\tau^2} + D\left(B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau)\right) = 0,$$

exhibits asymptotic (Lyapunov) stability.

Definition 2.5. *Consider a linear system with n-degrees of freedom described by the second-order differential equation*

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0, \quad t \in \mathbb{R},\ t > 0,$$

where $A, B, C \in \mathcal{M}^{n\times n}$ *denote the stiffness, mass, and damping matrices, respectively. Here,* $\mu(\tau) \in \mathbb{R}^n$ *represents the displacement vector, and* $\frac{d\mu(\tau)}{d\tau}$ *corresponds to the generalized velocity vector.*

This system is said to be additive D-stable *with respect to its coordinates if, for every non-negative diagonal matrix D, the modified system*

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + (C + D)\mu(\tau) = 0,$$

remains asymptotically stable in the sense of Lyapunov.

Definition 2.6. *Consider a linear dynamical system with n degrees of freedom, governed by the second-order differential equation*

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0, \quad t \in \mathbb{R},\ t > 0,$$

where $A, B, C \in \mathcal{M}^{n\times n}$ *denote the stiffness, mass, and damping matrices, respectively,* $\mu(\tau) \in \mathbb{R}^n$ *represents the displacement vector, and* $\frac{d\mu(\tau)}{d\tau}$ *corresponds to the generalized velocity vector.*

The system is defined as additively D-stable *with respect to velocity if, for any non-negative diagonal matrix D, the modified system*

$$A\frac{d^2\mu(\tau)}{d\tau^2} + (B + D)\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0,$$

remains (Lyapunov) asymptotically stable.

Definition 2.7. *Consider a linear dynamical system with n-degrees of freedom described by the second-order differential equation*

$$A\frac{d^2\mathbf{u}(t)}{d\tau^2} + B\frac{d\mathbf{u}(t)}{d\tau} + C\mathbf{u}(t) = \mathbf{0}, \quad t \in \mathbb{R},\ t > 0,$$

where $A, B, C \in \mathcal{M}^{n\times n}$ *denote the stiffness, mass, and damping matrices, respectively. The vector* $\mathbf{u}(t) \in \mathbb{R}^n$ *represents the generalized coordinates, while its time derivative,* $\frac{d\mathbf{u}(t)}{d\tau}$*, corresponds to the generalized velocity. The system is classified as additively D-stable if this property holds for both velocities and coordinates.*

2.1. **Sufficient conditions for D-stability:** Next, we examine the 13 sufficient conditions for the D-stability of an n-dimensional real matrix M, as established by C.R. Johnson [6]. These conditions are outlined below:

C_1 : For all eigenvalues λ_i of $\left(DM + M^t D\right)$, $\lambda_i > 0$, where $D = \text{diag}\{d_{ii}\}$ with $d_{ii} > 0$ for all $i = 1, 2, \ldots, n$.

C_2 : The matrix $M \in \mathbb{R}^{n\times n}$ is an M-matrix, meaning its principal minors are positive and its off-diagonal entries are non-positive.

C_3 : There exists a diagonal matrix $D = \text{diag}\{d_{ii}\}$, where $d_{ii} > 0$ for all i, such that $MD = B = (b_{ij})$ satisfies:

$$\text{Re}(b_{ii}) > \sum_{\substack{j=1 \\ j\neq i}}^{n} |b_{ij}|, \quad i = 1, \ldots, n.$$

C_4 : The matrix $M \in \mathbb{R}^{n\times n}$ is triangular, and the real parts of its diagonal entries m_{ii} are strictly positive.

C_5 : The matrix $M \in \mathbb{R}^{n\times n}$ is sign stable.

C_6 : Every principal minor of $M \in \mathbb{R}^{n\times n}$ is positive, and M has a tri-diagonal structure.

C_7 : The matrix $M \in \mathbb{R}^{n\times n}$ is oscillatory.

C_8 : For every non-zero vector $x \in \mathbb{R}^{n\times 1}$, there exists a diagonal matrix $D = \text{diag}\{d_{ii}\}$ with $d_{ii} > 0$ such that $\text{Re}(x^t DMx) > 0$.

C_9 : For any positive definite matrix P, the Hadamard product of P and $M \in \mathbb{R}^{n\times n}$ results in a stable matrix.

C_{10} : The matrix $M \in \mathbb{R}^{n\times n}$ is strictly sign symmetric, and all its principal minors are positive.

C_{11} : The matrix $M \in \mathbb{R}^{n\times n}$ belongs to the class $\mathbb{R}^{2\times 2} \cap P_0^+$.

C_{12} : The matrix $M \in \mathbb{R}^{n\times n}$ is in $\mathbb{R}^{3\times 3} \cap P_0^+$ and has the form:

$$M = \begin{bmatrix} x & a & b \\ \alpha & y & c \\ \beta & \alpha & z \end{bmatrix}.$$

C_{13} : The matrix $M \in \mathbb{R}^{n\times n}$ satisfies the P_0^+ condition and meets the GKK criterion for $n \leq 4$.

3. Problem Formulation

We consider the linear system with n-degrees of freedom with following mathematical formulation

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0, \; \tau \in \mathbb{R}, \; \tau > 0,$$

with $A, B, C \in \mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau) \in \mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed. We consider the following three problems:

Problem-I: To construct some new results on D-**stability** of the linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

Problem-II: To construct some new results on **additive** D-**stability in relation to coordinates** of the linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

Problem-III: To construct some new results on **additive** D-**stability in relation to speeds** of the linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

4. New Results

This section includes new findings on D-stability, coordinate-dependent additive D-stability, and additive D-stability in relation to speeds for the linear system with n-degrees of freedom of the form

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed. The following Theorem 4.1 shows that the linear system is D-stable if, for a positive diagonal matrix D, the real component of each eigenvalue of $\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}$ is smaller than 0.

Theorem 4.1. *The linear system with n-degrees of freedom*

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed, is D-stable if real part of all the eigenvalues of matrix product

$$\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}$$

is less than 0 *for a positive diagonal matrix D.*

Proof. The linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ t\in\mathbb{R},\ t\geq 0$$

equivalent form can be written as

$$\frac{dx(t)}{d\tau}=\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}x(t),\ x(t)=\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}.$$

It is enough to show that for

$$\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}$$

the coefficient matrix in $\frac{dx(t)}{d\tau}$ is such that the real part of all the eigenvalues of

$$\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}$$

is strictly less than 0.

Let $\alpha\in\mathbb{N}$, the set of natural numbers, and let $x(t)\in\mathbb{R}^{n,1}$, $t\in\mathbb{R},\ t\geq 0$ such that $x[\alpha]\neq 0$. Now, consider that

$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}[\alpha]<0.$$

$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}<0.$$

Also,

$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\operatorname{Re}\left[\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}\right)\right]\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}<0,\ \forall i.$$

This further can be rewritten as

$$\operatorname{Re}\left[\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}\right)\right]\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}<0,\ \forall i.$$

Since, $\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}=1$. Thus, $\operatorname{Re}\left[\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & I\end{bmatrix}\right)\right]<0,\ \forall i.$ □

The following Theorem 4.2 shows that the linear system is additive D-stable in relation to coordinates if the real part of all the eigenvalues of $\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}$ is less than 0 for a positive diagonal matrix D.

Theorem 4.2. *The linear system as described in Theorem 1 is additive D-stable in relation to coordinates, if*

$$Re\left[\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)\right]<0,\ \forall i.$$

Proof. Let $\alpha \in \mathbb{N}$, the set of natural numbers, and let

$$x(t)=\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}\in\mathbb{R}^{n,1},\ t\in\mathbb{R},\ t\geq 0 \text{ such that } \begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}[\alpha]\neq 0.$$

Consider that

$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t[\alpha]\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)[\alpha]\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}[\alpha]$$
$$=$$
$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}<0.$$

Furthermore, we have that

$$\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t \mathrm{Re}\left(\lambda_i\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\lambda_i\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}\right)<0,\ \forall i.$$

Also,

$$\mathrm{Re}\left(\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\lambda_i\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}\right)<0,\ \forall i.$$

Since, $\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}^t\begin{bmatrix}\frac{d\mu(\tau)}{d\tau}\\ \mu(\tau)\end{bmatrix}=1$. Thus, $\mathrm{Re}\left(\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\lambda_i\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)\right)<0,\ \forall i$. This implies that

$$\mathrm{Re}\left[\lambda_i\left(\begin{bmatrix}-A^{-1}B & -A^{-1}C+D\\ I & 0\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\right)\right]<0,\ \forall i.$$

□

The following Theorem 4.3 shows that the linear system is additive D-stable in relation to speeds if the real part of all the eigenvalues of $\begin{bmatrix}-A^{-1}B+D & -A^{-1}C\\ I & 0\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}$ is less than 0 for a positive diagonal matrix D.

Theorem 4.3. *The linear system as described in theorem 1 is additive D-stable in relation to speeds, if the real part of all the eigenvalues of matrix sum*

$$\begin{bmatrix}-A^{-1}B+D & -A^{-1}C\\ I & 0\end{bmatrix}+\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}$$

is strictly less than 0 *for a positive diagonal matrix D.*

Proof. Let $\alpha \in \mathbb{N}$, the set of natural numbers, and let $x(t) = \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} \in \mathbb{R}^{n,1},\ \tau \in \mathbb{R},\ \tau > 0$ such that $\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}[\alpha] \neq 0.$

Further, we consider that

$$\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}^t [\alpha] \left(\begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) [\alpha] \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} [\alpha]$$

$$=$$

$$\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}^t \left(\begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} < 0.$$

Also, we have that

$$\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}^t Re \left[\lambda_i \begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \lambda_i \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} < 0,\ \forall i.$$

The last inequality can be rewritten as

$$Re \left(\lambda_i \begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \lambda_i \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}^t \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} \right) < 0,\ \forall i.$$

Since, $\begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix}^t \begin{bmatrix} \frac{d\mu(\tau)}{d\tau} \\ \mu(\tau) \end{bmatrix} = 1$. In turn, $Re \left[\lambda_i \begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \lambda_i \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right] < 0,\ \forall i$. This can further take the form $\mathrm{Re} \left[\lambda_i \begin{bmatrix} -A^{-1}B + D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right] < 0,\ \forall i.$ □

The following Theorem 4.4 shows that the linear system is D-stable if the structured singular value of the reciprocal of $\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$ with respect to set of block-diagonal structure $\mathbb{B}_1$ is greater than or equal to 0 and less than 1 for a positive diagonal matrix D.

Theorem 4.4. *The linear system as described in theorem is D-stable if*

$$0 \leq \mu_{\mathbb{B}_1} \left(\left[\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} \right]^{-1} \right) < 1.$$

Proof. For the D-stability of $\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$, we have to show that the real part of all the eigenvalues must be strictly less than 1. This further ensures that for a positive diagonal matrix

$\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$, we have that

$$\prod_i \text{Re}\left[\lambda_i\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2 + \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2\right)\right] \neq 0,\ \forall i.$$

To prove last inequality, we have

$$\prod_i Re\left[\lambda_i\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2 - \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2 \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^{-1}\right)\right)$$

$$\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\Bigg] \neq 0,\ \forall i.$$

This further reduces to

$$\prod_i \text{Re}\left[\lambda_i\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^{-1}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right] \neq 0,\ \forall i,$$

where $\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}$ has a block-diagonal structure. Thus, we have that

$$\prod_i \text{Re}\left[\lambda_i\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^{-2}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right] \neq 0,\ \forall i.$$

Finally,

$$0 \leq \mu_{\mathbb{B}_1}\left[\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right]^{-1} < 1.$$

□

The subsequent Theorem 4.5 demonstrates that the linear system is additive D-stable in relation to coordinates if the structured singular value of the reciprocal of $\begin{bmatrix} -A^{-1}B & -A^{-1}C + D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ in relation to set of block-diagonal structure $\mathbb{B}_1$ is greater than or equal to 0 and less than 1 for a positive diagonal matrix D.

Theorem 4.5. *The linear system as described in theorem-1 is additive D-stable in relation to coordinates if*

$$0 \leq \mu_{\mathbb{B}_1}\left[\begin{bmatrix} -A^{-1}B & -A^{-1}C + D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1} < 1.$$

Proof. To prove, we consider a positive block-diagonal structured matrix $\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$ this means that

$$\prod_i Re\left[\lambda_i\left(\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^2+\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2\right)\right)\right] \neq 0,\ \forall i.$$

To prove above inequality, we have

$$\prod_i Re\left[\lambda_i\left(\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^2-\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2\right.\right.\right.$$

$$\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}\right.\right.$$

$$\left.\left.\left.\left.+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)\right)\right)\right] \neq 0,\ \forall i.$$

This further reduces to

$$\prod_i Re\left[\lambda_i\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}-\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right] \neq 0,\ \forall i.$$

Hence, $\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}$ has a block-diagonal structure and is equal to $\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$. Thus, we have that

$$\prod_i Re\left[\lambda_i\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}-\left(\left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-2}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right] \neq 0,\ \forall i.$$

Finally,

$$0 \leq \mu_{\mathbb{B}_1}\left[\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1} < 1.$$

□

The following Theorem 4.6 shows that the linear system is additive D-stable with respect to coordinates if the structured singular value of the reciprocal of

$$\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to set of block-diagonal structure $\mathbb{B}_1$ is greater than or equal to 0 and less than 1 for a positive diagonal matrix D.

Theorem 4.6. *The linear system as described in theorem 1, and is additive D-stable in relation to speeds, if*

$$0 \le \mu_{\mathbb{B}_1}\left[\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1} < 1.$$

Proof. Suppose that $\begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$ has a block-diagonal structure so that

$$\prod_i Re\left[\lambda_i\left(\left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^2 + \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2\right)\right] \ne 0,\ \forall i.$$

In order to prove last inequality, we have

$$\prod_i Re\left[\left(\left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}^2\left(\left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix}\right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)\right)\right)\right] \ne 0,\ \forall i.$$

Furthermore,

$$\prod_i Re\left[\lambda_i\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right] \ne 0,\ \forall i.$$

Here $\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}$. Thus,

$$\prod_i Re\left[\lambda_i\left(\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^{-2}\begin{bmatrix} \tilde{D}_{11} & 0 \\ 0 & \tilde{D}_{22} \end{bmatrix}\right)\right)\right] \ne 0,\ \forall i.$$

Finally,

$$0 \le \mu_{\mathbb{B}_1}\left[\begin{bmatrix} -A^{-1}B+D & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right]^{-1} < 1.$$

□

The following Theorem 4.7 ensures the D-stability of $\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$ if the structured singular values of the matrix $\left(\begin{bmatrix} I & O \\ O & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & O \end{bmatrix}\begin{bmatrix} D & O \\ O & I \end{bmatrix}\right)^2$ in relation to set of block-diagonal matrices $\mathbb{B}_1$ is greater than or equal to zero and strictly less than one.

Theorem 4.7. *The linear system as described in theorem-1 is D-stable if matrix product*

$$\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$$

is stable, and

$$0 \leq \mu_{\mathbb{B}_1}\left(\frac{1}{\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2}\right) < 1.$$

Proof. The matrix $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$ is a D-stable matrix iff it is a stable matrix, and satisfying the condition that the product of all the eigenvalues of following matrix

$$\begin{bmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} & -\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \\ \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} & \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} \end{bmatrix}$$

is not equal to zero, here the diagonal matrix $\begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$ is a positive. This further can be written as that the product of all the eigenvalues of the matix

$$\left[\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2 - \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} X \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - Y\right)\right] \neq 0.$$

where $X = \left(\frac{1}{\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)}\right)$, $Y = \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}$.

As,

$$\prod_{i=1}^{n} \lambda_i \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \frac{1}{\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2}\begin{bmatrix} \tilde{P_{11}} & 0 \\ 0 & \tilde{P_{22}} \end{bmatrix}\right) \neq 0.$$

This finally yields required result of structural singular value, that is,

$$0 \le \mu_{\mathbb{B}_1}\left(\frac{1}{\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)^2}\right) < 1.$$

□

The following Theorem 4.8 ensures the additive D-stability of

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)$$

if its structured singular values with respect to set of block-diagonal matrices $\mathbb{B}_1$ is greater than or equal to zero and strictly less than one.

Theorem 4.8. *The linear system as described in theorem-1 is additive D-stable if the real part of all the eigenvalues of the matrix*

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right) < 0,$$

the structured singular values of the square of the required above matrix is greater than or equal to zero and strictly less than one. The additive D-stability is in relation to coordinates.

Proof. To show that the structured singular values of the square of the reciprocal of

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)$$

is greater than or equal to zero and strictly less than one. For this purpose, it is enough to show that the product of all the eigenvalues of

$$\begin{bmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix} & -\begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \\ \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} & \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \end{bmatrix}$$

is not equal to zero for positive diagonal matrix $\begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$. This further implies that the product of all the eigenvalues of

$$\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C+D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)^2\right) - \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}\tilde{X}\begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$$

$$\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)$$

is not exactly equal to zero, where $\tilde{X}$ is given as

$$\tilde{X} := \frac{1}{\left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}\right)\right]}.$$

Furthermore, we have that

$$\prod_{i}^{n} \lambda_i \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \frac{1}{\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)\right)^2} \begin{bmatrix} \tilde{P}_{11} & 0 \\ 0 & \tilde{P}_{22} \end{bmatrix} \right),$$

being not exactly equal to zero. Thus, finally, we have that the structured singular values of square of reciprocal of

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} -A^{-1}B & -A^{-1}C + D \\ I & 0 \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\right)$$

are greater than or equal to zero and strictly less than one. □

5. Numerical Experimentation

In this phase, we demonstrate the approximation and visualization of eigenvalues, singular values, structured singular values, and pseudo-spectra for second order linear systems with coefficient matrices A, B, and C as the mass, damping and stiffness matrices. We show the level sets which corresponds to the resolvent norm for the pseudo-spectrum in the complex plane and mathematically the resolvent norm is computed by $\|(M - zI)^{-1}\|$.

Example 1. We consider second order linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0,\ \tau \in \mathbb{R},\ \tau > 0,$$

with $A, B, C \in \mathcal{M}^{3\times 3}$, are the stiffness, mass, and damping matrices (Toeplitz), $\mu(\tau) \in \mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}; B = \begin{bmatrix} 4 & 4 & 5 \\ 5 & 4 & 4 \\ 6 & 5 & 4 \end{bmatrix}; C = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}.$$

The matrix M_1 is

$$M_1 = \begin{bmatrix} -A^{-1}B & -A^{-1}C \\ I_3 & 0_3 \end{bmatrix} = \begin{bmatrix} -1.7895 & -1.3421 & -0.8684 & -0.1842 & -0.0263 & -0.5789 \\ -0.1579 & -0.3684 & -0.4737 & -0.7368 & -0.1053 & -0.3158 \\ -0.3158 & -0.2368 & -0.4474 & 0.0263 & -0.7105 & -0.6316 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 1 displays the spectral features of matrix M_1, including the computation of spectrum, singular values, structured singular values, and pseudo-spectrum. In Figure 2, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

Figure 2 shows the plotted value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The pseudomode displays the right singular vector that corresponds to the least singular value in the matrix $(zI_6 - M_1)$.

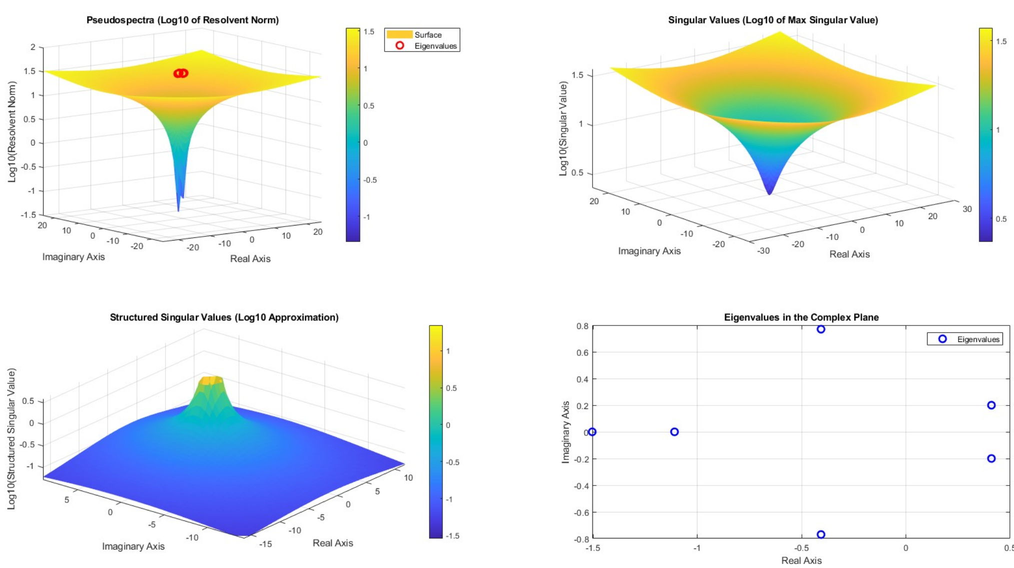

FIGURE 1. Spectral properties of matrix M_1 in Example-1.

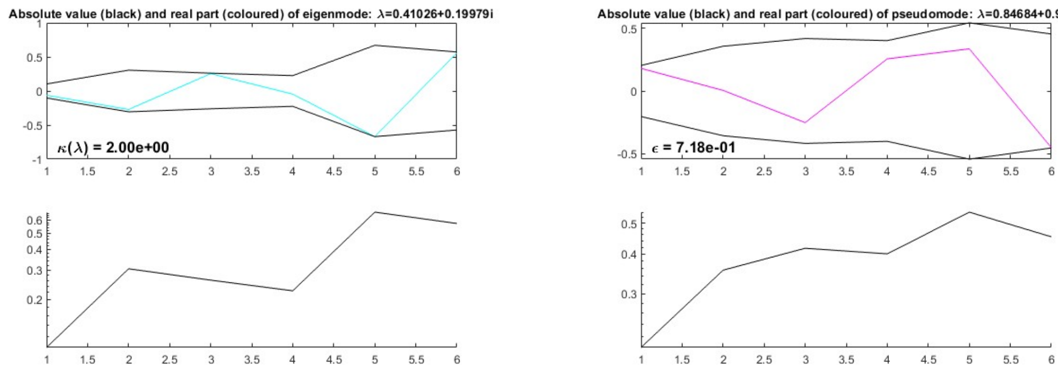

Figure 2. Eigenmode (left) and inverse of resolvent norm (right) of matrix M_1 in Example-1

Example 2. We consider second order linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0,\ \tau \in \mathbb{R},\ \tau > 0,$$

with $A, B, C \in \mathcal{M}^{3\times 3}$, are the stiffness, mass, and damping matrices taken from [30], $\mu(\tau) \in \mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

$$A = \begin{bmatrix} 0.4000 & 0 & 0 & 0 & 0 \\ 0 & 0.8000 & 0 & 0 & 0 \\ 0 & 0 & 1.2000 & 0 & 0 \\ 0 & 0 & 0 & 1.6000 & 0 \\ 0 & 0 & 0 & 0 & 2.0000 \end{bmatrix};\ B = \begin{bmatrix} 12 & -12 & 0 & 0 & 0 \\ -12 & 24 & 12 & 0 & 0 \\ 0 & -12 & 24 & -12 & 0 \\ 0 & 0 & -12 & 24 & 12 \\ 0 & 0 & 0 & -12 & 24 \end{bmatrix};$$

$$C = \begin{bmatrix} 36000 & -36000 & 0 & 0 & 0 \\ -36000 & 72000 & -36000 & 0 & 0 \\ 0 & -36000 & 72000 & -36000 & 0 \\ 0 & 0 & -36000 & 72000 & -36000 \\ 0 & 0 & 0 & -36000 & 72000 \end{bmatrix}.$$

The matrix $M_2 = \begin{bmatrix} -A^{-1}B & -A^{-1}C + 2D \\ I_5 & 0_5 \end{bmatrix}$ is multiple of $1.0e + 04$ and is given as bellow:

$$\begin{bmatrix} -0.0030 & 0.0030 & 0 & 0 & 0 & -8.9998 & 9.0000 & 0 & 0 & 0 \\ 0.0015 & -0.0030 & -0.0015 & 0 & 0 & 4.5000 & -8.9992 & 4.5000 & 0 & 0 \\ 0 & 0.0010 & -0.0020 & 0.0010 & 0 & 0 & 3.0000 & -5.9998 & 3.0000 & 0 \\ 0 & 0 & 0.0008 & -0.0015 & -0.0008 & 0 & 0 & 2.2500 & -4.4996 & 2.2500 \\ 0 & 0 & 0 & 0.0006 & -0.0012 & 0 & 0 & 0 & 1.8000 & -3.5982 \\ 0.0001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0001 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0001 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 3 displays the spectral features of matrix M_1, including the computation of spectrum, singular values, structured singular values, and pseudo-spectrum. In Figure 4, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 4, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The pseudomode displays the right singular vector that corresponds to the least singular value in the matrix $(zI_{10} - M_2)$.

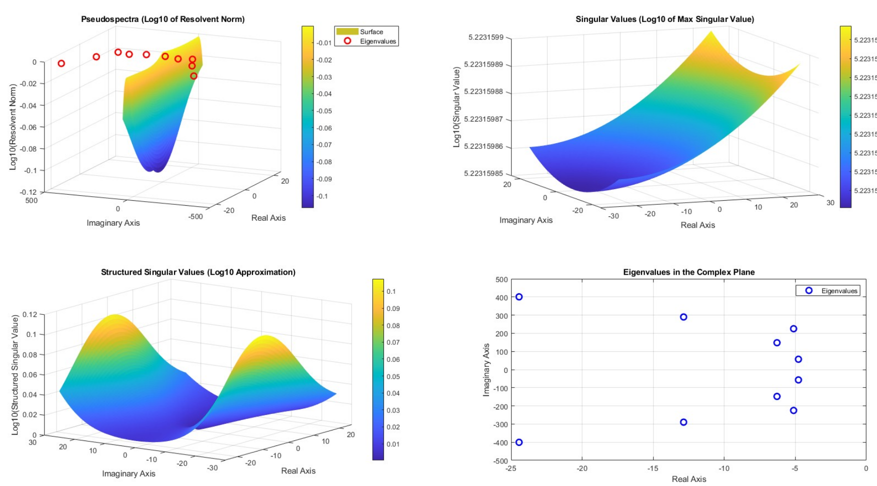

FIGURE 3. Spectral properties of matrix M_2 in Example-2.

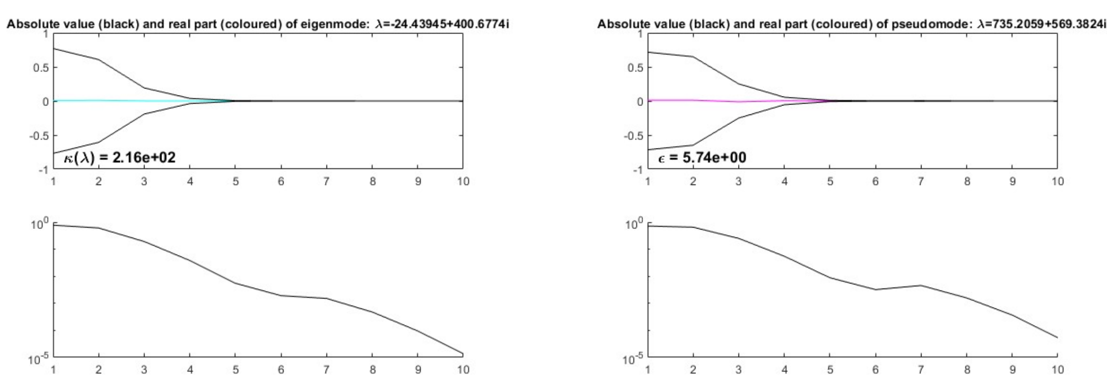

FIGURE 4. Eigenmode (left) and inverse of resolvent norm (right) of matrix M_2 in Example-2

Example 3. We consider second order linear system

$$A\frac{d^2\mu(\tau)}{d\tau^2}+B\frac{d\mu(\tau)}{d\tau}+C\mu(\tau)=0,\ \tau\in\mathbb{R},\ \tau>0,$$

with $A,B,C\in\mathcal{M}^{3\times3}$, are the stiffness, mass, and damping matrices taken from [16], $\mu(\tau)\in\mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed.

$$A=\begin{bmatrix}5&0&0&0\\0&10&0&0\\0&0&10&0\\0&0&0&5\end{bmatrix};\ B=\begin{bmatrix}0.02&-0.01&0&0\\-0.01&0.02&-0.01&0\\0&-0.01&0.02&-0.01\\0&0&-0.01&0.01\end{bmatrix};C=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix};D=\begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&10&0\\0&0&0&50\end{bmatrix}.$$

The matrix $M_3=\begin{bmatrix}-A^{-1}B+2D & -A^{-1}C\\ I_4 & 0_4\end{bmatrix}$ is multiple of $1.0e+04$ and is given as bellow:

$$\begin{bmatrix}3.9960&0.0020&0&0&-0.4000&0.2000&0&0\\0.0010&1.9980&0.0010&0&0.1000&-0.2000&0.1000&0\\0&0.0010&19.9980&0.0010&0&0.1000&-0.2000&0.1000\\0&0&0.0020&99.9980&0&0&0.2000&-0.2000\\1.0000&0&0&0&0&0&0&0\\0&1.0000&0&0&0&0&0&0\\0&0&1.0000&0&0&0&0&0\\0&0&0&1.0000&0&0&0&0\end{bmatrix}.$$

Figure 5 displays the spectral features of matrix M_1, including the computation of spectrum, singular values, structured singular values, and pseudo-spectrum. In Figure 6, we plot the eigenmode corresponding to the eigenvalues. The top plot in the figure shows an envelope which is produced by plotting the absolute value of an eigenmode minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations.

In Figure 6, we plot the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The pseudomode displays the right singular vector that corresponds to the least singular value in the matrix (zI_8-M_3).

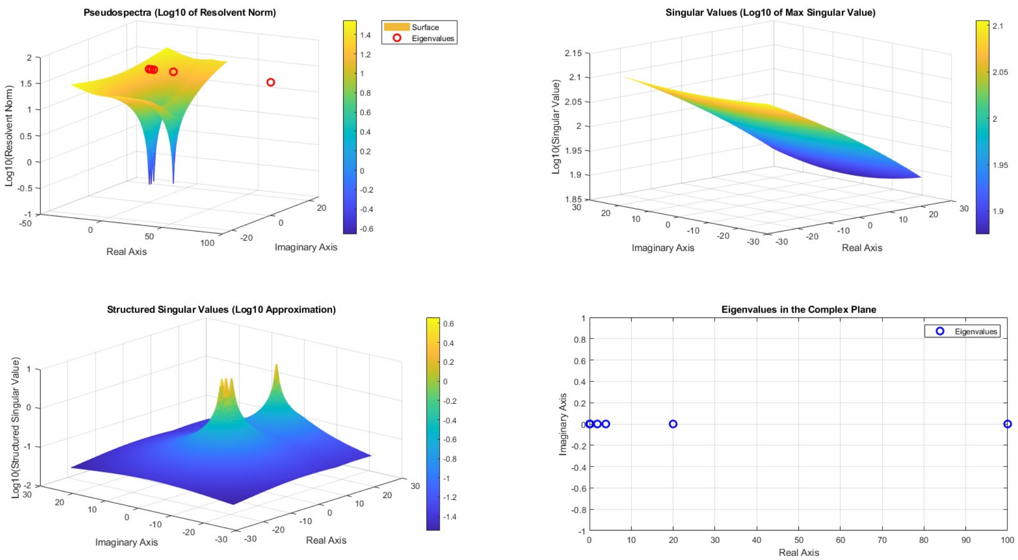

FIGURE 5. Spectral properties of matrix M_3 in Example-3.

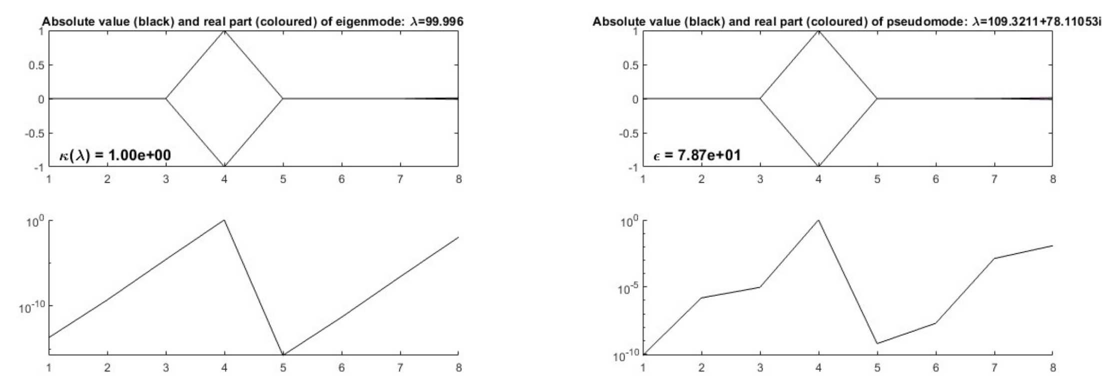

FIGURE 6. Eigenmode (left) and inverse of resolvent norm (right) of matrix M_3 in Example-3

Example 4. We consider 1000, 200 and 300 dimensional stiffness, mass, and damping matrices which are generated by MATLAB command **rand**. The spectral properties like the computation of spectrum, singular values, structured singular values, and pseudo-spectrum of 100, 200 and 300 dimensional Haar matrices are presented in Figure 7.

In Figures 8-10, we plot the eigenmode corresponding to the eigenvalues. The top plot in each figure shows an envelope which is produced by plotting the absolute value of an eigenmode and minus the absolute value. The real part is shown with a cyan line. The plot at the bottom level in each figure show absolute value of eigenmode being ploted at a log scale. Further it shows that how quickly an eigenmode is decaying with the time. The condition number computed for an eigenvalue is shown in the top plot. The large condition number means that eigenvalue is sensitive to perturbations. Further, we present the plot of the value of inverse of the resolvent norm. We show real part of pseudomode in magenta. The right singular vector corresponding to the smallest singular value corresponding to matrix $zI - M$, is shown in pseudomde.

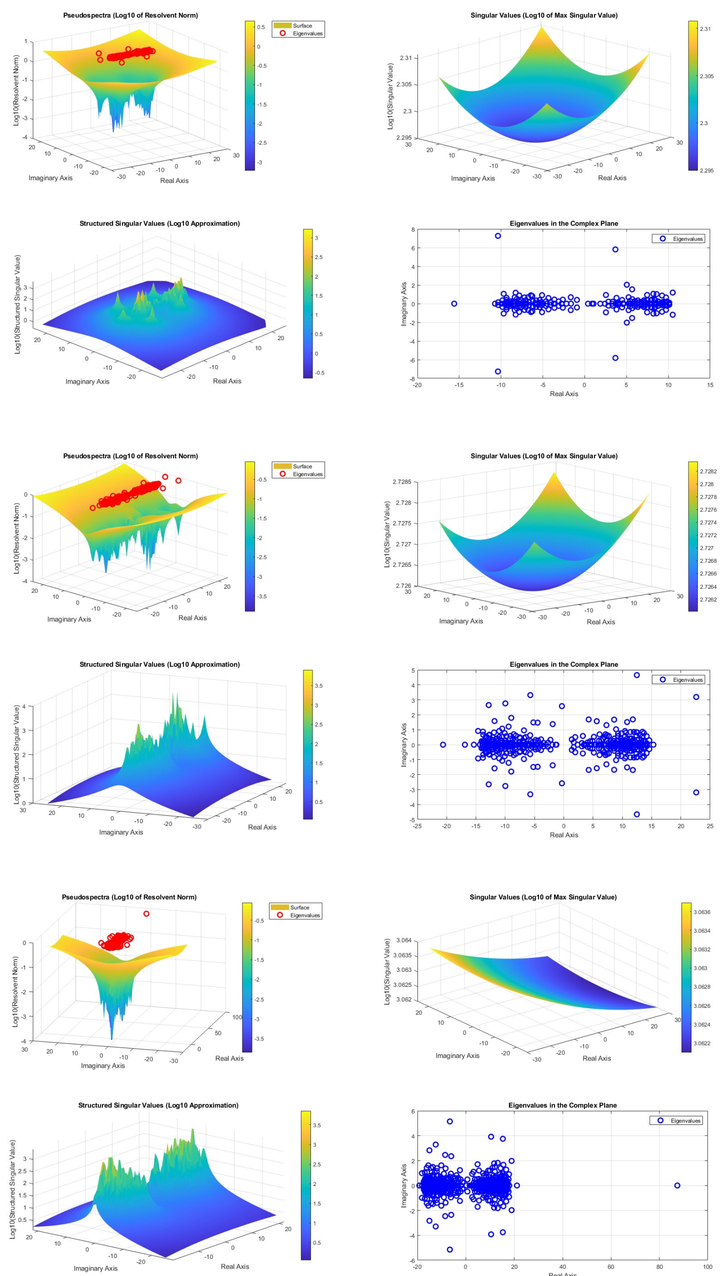

Figure 7. Spectral properties of 100, 200 and 300 dimensional stiffness, mass, and damping matrices in Example-4.

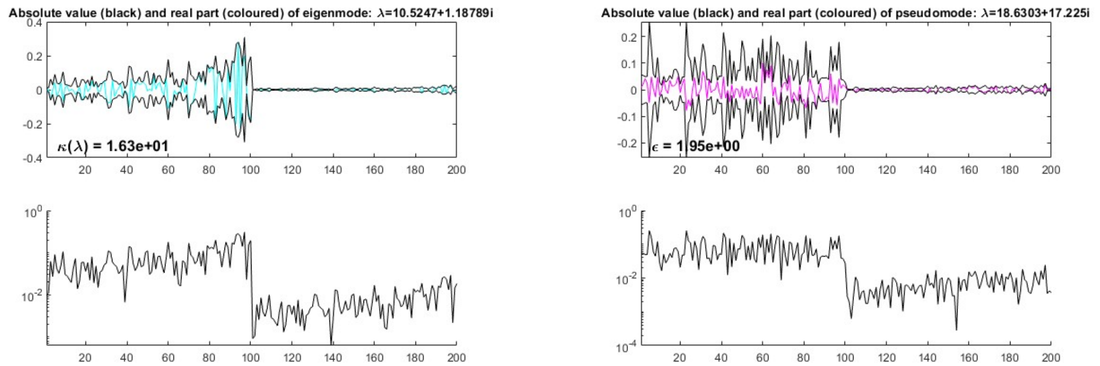

Figure 8. Eigenmode (left) and inverse of resolvent norm (right) of 100 dimensional stiffness, mass, and damping matrices in Example-4

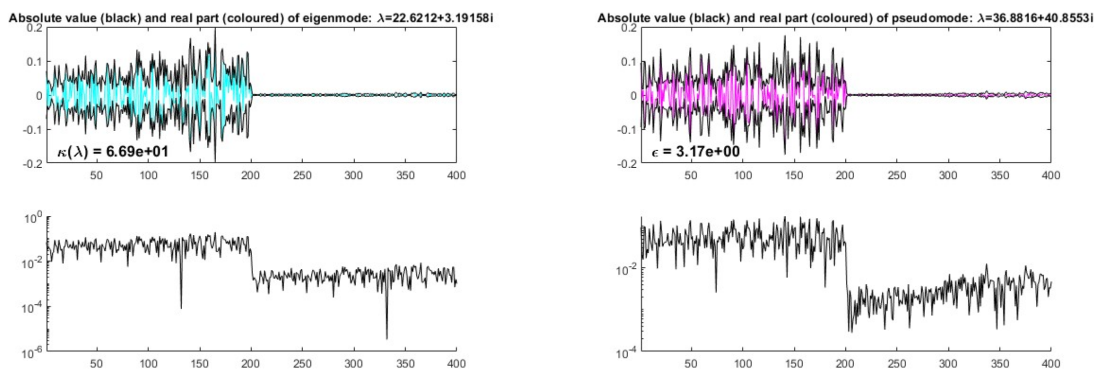

Figure 9. Eigenmode (left) and inverse of resolvent norm (right) of 200 dimensional stiffness, mass, and damping matrices in Example-4

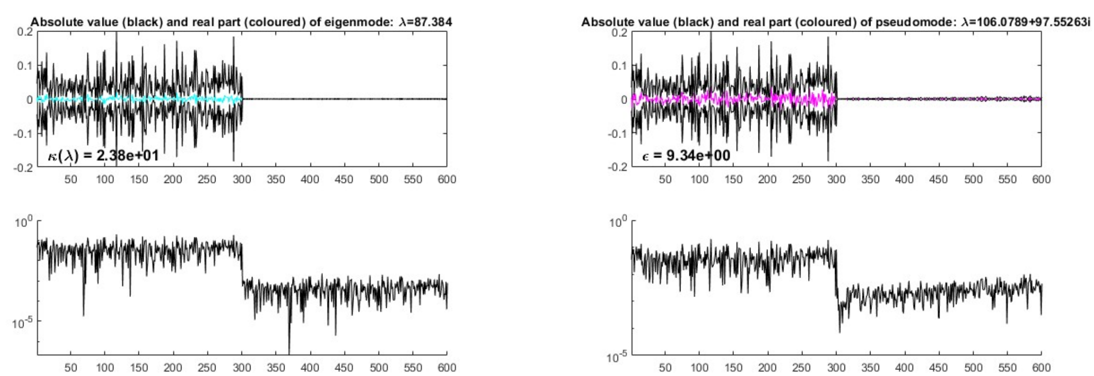

Figure 10. Eigenmode (left) and inverse of resolvent norm (right) of 300 dimensional stiffness, mass, and damping matrices in Example-4

6. Conclusion

This work establishes novel theoretical contributions concerning D-stability, speed-dependent additive D-stability, and coordinate-dependent additive D-stability. Our analysis primarily focuses on linear dynamical systems characterized by n degrees of freedom, represented in the form

$$A\frac{d^2\mu(\tau)}{d\tau^2} + B\frac{d\mu(\tau)}{d\tau} + C\mu(\tau) = 0,\ \tau \in \mathbb{R},\ \tau > 0,$$

with $A, B, C \in \mathcal{M}^{n\times n}$, are the stiffness, mass, and damping matrices, $\mu(\tau) \in \mathbb{R}^n$, $\frac{d\mu(\tau)}{d\tau}$, the vector of generalized speed. The analytical findings concerning D-stability, speed-dependent additive D-stability, and coordinate-dependent additive D-stability emerge through a synthesis of principles from D-stability theory and μ-analysis. The gathering of several techniques from matrix analysis, system theory, and numerical linear algebra forms the basis of our suggested methodology. To evaluate the efficacy of the suggested methodology, numerical tests are provided on spectrum computation and behavior, structured singular values and pseudo-spectrum in three dimensional space for A, B and $C \in \mathcal{M}^{n\times n}$ to be regarded as Toeplitz, and transportation matrices.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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