

Conservation Laws and Symmetry Reductions of a Generalized Nonlinear Black-Scholes Equation

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Abstract. In this paper, we apply the generalized double reduction theory to a nonlinear extension of the Black-Scholes equation, which is a foundational model in financial mathematics for pricing European options. The Lie symmetry method is used to identify point symmetries of the nonlinear PDE, and conservation laws are derived using the multiplier approach. We demonstrate how the symmetry reductions lead to simplified invariant solutions and discuss their implications for understanding nonlinear market behaviors. Numerical simulations are used to illustrate how nonlinearity modifies traditional option pricing surfaces. Finally, we discuss potential real-world applications such as volatility smiles, hedging strategies, and robustness of risk metrics.

1. INTRODUCTION

The Black-Scholes model stands as a cornerstone in the field of financial mathematics, offering a mathematical framework to price European options and other derivatives under idealized market conditions. Developed by Fischer Black and Myron Scholes in 1973, the model assumes a frictionless market, constant volatility, and the ability to continuously hedge positions. These assumptions lead to a partial differential equation that admits closed-form solutions for several types of options and has become widely adopted in both academic research and financial industry practice [2, 14].

However, real financial markets deviate significantly from these assumptions. Volatility is often not constant but varies randomly over time. Transaction costs, liquidity constraints, and other forms of market imperfections play a significant role in trading behavior. These limitations have prompted the development of a variety of model extensions designed to capture the complexities of real-world trading environments [4, 7].

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One such extension introduces nonlinearity into the classical Black-Scholes framework. Nonlinear terms can arise from feedback mechanisms, risk constraints, illiquidity, or trader interactions. These features lead to modified differential equations that are no longer linear, thereby necessitating the use of more advanced analytical tools for their study [6,8,12].

This work focuses on the application of Lie symmetry analysis and conservation law techniques to a generalized nonlinear Black-Scholes equation. These methods offer a systematic approach to simplifying and solving nonlinear partial differential equations by identifying invariant transformations and corresponding conserved quantities. By exploiting these symmetries, it becomes possible to reduce the original equation to simpler forms, often resulting in ordinary differential equations that admit exact or approximate solutions.

The goal of this study is to demonstrate how symmetry methods and conservation laws can be applied to a nonlinear financial model to yield analytical insights and support numerical simulations. Through the use of double reduction techniques, we derive explicit similarity solutions and analyze their behavior under various conditions. We also explore the implications of the model for real-world financial applications, such as volatility modeling, hedging strategies, and risk assessment.

To guide the reader through the developments, the paper is structured as follows. Section 2 revisits the classical Black-Scholes model, laying out its foundational assumptions and identifying the limitations that motivate its generalization. In Section 3, we develop the Lie symmetry analysis of the generalized nonlinear equation, deriving the infinitesimal generators and invariant structures. Section 4 turns to the construction of conservation laws using the multiplier method, highlighting their theoretical and practical significance. The methodology of double reduction is detailed in Section 5, where the partial differential equation is systematically simplified to an ordinary differential equation. Section 6 presents exact solutions to the reduced equation and interprets them in the context of option pricing. In Section 7, we validate the theoretical findings with numerical simulations and graphical illustrations of the pricing behavior. Section 8 interprets the outcomes within real financial markets, emphasizing practical implications such as volatility smiles and hedging under nonlinear dynamics. Section 9 explores the potential to generalize the approach to models incorporating stochastic volatility, jumps, or fractional dynamics. Finally, Section 10 summarizes the contributions and suggests avenues for further research in mathematical finance.

2. THE CLASSICAL BLACK-SCHOLES MODEL

2.1. Derivation and Assumptions. The classical Black-Scholes equation assumes no arbitrage, constant volatility, and frictionless markets. It is derived under the assumption of continuous trading, log-normal distribution of stock prices, and perfect hedging.

2.2. Limitations and Extensions. While the classical Black-Scholes model provides elegant closed-form solutions under idealized assumptions, it falls short in capturing the nuances of real financial

markets. One major limitation is the assumption of constant volatility, which empirical studies have consistently shown to be unrealistic. Market volatility fluctuates due to macroeconomic events, investor sentiment, and changes in liquidity, rendering the constant volatility hypothesis oversimplified.

Another restrictive assumption is the absence of transaction costs. In practice, every trade involves costs such as bid-ask spreads, commissions, and slippage. These costs affect portfolio rebalancing and the execution of dynamic hedging strategies, particularly for large institutional investors. The classical model also assumes the ability to continuously hedge options positions, which is not feasible in discrete-time trading environments.

Additionally, the Black-Scholes framework presumes frictionless markets and unlimited liquidity. However, real markets are subject to illiquidity, especially during periods of financial stress or high volatility. Large trades can influence asset prices, introducing feedback effects that are not accounted for in the linear model. The linear nature of the classical PDE also fails to accommodate phenomena such as market impact and order book dynamics.

To address these shortcomings, several extensions have been proposed. One approach is to introduce nonlinear terms into the Black-Scholes equation, reflecting the effects of transaction costs, illiquidity, and other market imperfections. These nonlinear extensions lead to more complex PDEs but provide a more accurate representation of financial markets. Other enhancements include incorporating stochastic volatility, jump-diffusion processes, and fractional calculus to account for memory and long-range dependence in asset prices.

These extensions not only improve the descriptive power of the model but also open new avenues for analytical and numerical techniques. In particular, the application of symmetry methods to these generalized models can yield insights into their structure and facilitate the derivation of invariant solutions. As a result, nonlinear extensions of the Black-Scholes model are increasingly relevant in both academic research and practical financial engineering.

3. LIE SYMMETRY ANALYSIS

In this section, we perform a detailed Lie symmetry analysis of the generalized nonlinear Black-Scholes equation [14]:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = \lambda V^p, \quad (3.1)$$

where $V = V(S, t)$ is the option price, S is the underlying asset price, t is time, σ is the volatility, r is the risk-free rate, λ is a real parameter, and p is a nonzero real constant.

To perform the Lie symmetry analysis, we look for continuous symmetry transformations that leave the equation invariant. These are generated by a vector field of the form [9]:

$$X = \xi^t(t, S, V) \frac{\partial}{\partial t} + \xi^S(t, S, V) \frac{\partial}{\partial S} + \eta(t, S, V) \frac{\partial}{\partial V}. \quad (3.2)$$

The second prolongation of X , denoted by $\text{pr}^{(2)}X$, extends X to the derivatives appearing in Eq. (3.1). The invariance condition requires:

$$\text{pr}^{(2)}X \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV - \lambda V^p \right) = 0 \quad \text{whenever Eq. (3.1) holds.} \quad (3.3)$$

We use the standard prolongation formulas for point symmetries:

$$\begin{aligned} \eta^t &= D_t(\eta) - V_t D_t(\xi^t) - V_S D_t(\xi^S), \\ \eta^S &= D_S(\eta) - V_t D_S(\xi^t) - V_S D_S(\xi^S), \\ \eta^{SS} &= D_S(\eta^S) - V_{SS} D_S(\xi^S) - V_{St} D_S(\xi^t). \end{aligned}$$

The total derivatives with respect to t and S are given by:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + V_t \frac{\partial}{\partial V} + V_{tt} \frac{\partial}{\partial V_t} + V_{St} \frac{\partial}{\partial V_S} + \cdots, \\ D_S &= \frac{\partial}{\partial S} + V_S \frac{\partial}{\partial V} + V_{SS} \frac{\partial}{\partial V_S} + V_{St} \frac{\partial}{\partial V_t} + \cdots. \end{aligned}$$

Substituting these expressions into the prolongation and applying the invariance condition, we obtain a system of determining equations by matching coefficients of independent monomials in derivatives such as V_t , V_S , V_{SS} , and powers of V . These equations are then solved for ξ^t , ξ^S , and η .

For the nonlinear Black-Scholes equation, after simplification, the determining equations lead to the following general forms:

$$\begin{aligned} \xi^t &= a_1 t + a_2, \\ \xi^S &= a_1 S + a_3, \\ \eta &= (2a_1 - a_1)V = a_4 V, \end{aligned}$$

where a_1, a_2, a_3, a_4 are constants subject to constraints depending on the value of p . For $p \neq 1$, the infinitesimal generator η is proportional to V , while for $p = 1$, logarithmic terms can also appear.

From this, the infinitesimal generators of the symmetry group are derived:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & (\text{time translation}) \\ X_2 &= \frac{\partial}{\partial S}, & (\text{asset translation}) \\ X_3 &= S \frac{\partial}{\partial S} + 2V \frac{\partial}{\partial V}, & (\text{scaling in } S \text{ and } V) \\ X_4 &= t \frac{\partial}{\partial t} + S \frac{\partial}{\partial S}, & (\text{dilation symmetry}) \\ X_5 &= V \frac{\partial}{\partial V}. & (\text{amplitude scaling}) \end{aligned}$$

Each symmetry generator corresponds to a transformation that maps solutions of the PDE to other solutions. These symmetries can be used to reduce the number of independent variables and transform the PDE into a simpler form or even an ordinary differential equation.

In subsequent sections, we will employ these symmetries to construct similarity variables and invariant solutions, and to perform double reductions that lead to analytically tractable forms of the nonlinear Black-Scholes equation.

4. CONSERVATION LAWS

Conservation laws play an important role in the qualitative analysis of differential equations, particularly for verifying numerical simulations and identifying invariant properties of the system. In this section, we apply the direct method, also known as the multiplier method, to determine conservation laws for the generalized nonlinear Black-Scholes equation:

$$\Delta := V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - \lambda V^p = 0. \quad (4.1)$$

A conservation law for Eq. (4.1) is a divergence expression:

$$D_t T^t + D_S T^S = 0, \quad (4.2)$$

that holds on the solutions of the equation, where T^t and T^S are the components of the conserved vector.

To find a conservation law, we seek a function (multiplier) $\Lambda(t, S, V)$ such that:

$$\Lambda(t, S, V) \cdot \Delta = D_t T^t + D_S T^S. \quad (4.3)$$

We restrict ourselves to multipliers that depend only on V , i.e., $\Lambda = \Lambda(V)$, which simplifies the analysis. Substituting Eq. (4.1) into the above relation, we require:

$$\Lambda(V) \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - \lambda V^p \right) = D_t T^t + D_S T^S. \quad (4.4)$$

We now determine appropriate forms of $\Lambda(V)$ that lead to divergence expressions. Consider the trial multiplier $\Lambda = V$ (i.e., we multiply the PDE by V):

$$\begin{aligned} V \cdot V_t + \frac{1}{2}\sigma^2 S^2 V V_{SS} + rSV V_S - rV^2 - \lambda V^{p+1} &= D_t \left(\frac{1}{2} V^2 \right) + D_S \left(\frac{1}{2} \sigma^2 S^2 V V_S \right) \\ &\quad - \left(\frac{1}{2} \sigma^2 S^2 V_S^2 \right) + rSV V_S - rV^2 - \lambda V^{p+1}. \end{aligned}$$

We see that the left-hand side is expressible as the divergence of the following vector components:

$$\begin{aligned} T^t &= \frac{1}{2} V^2, \\ T^S &= \frac{1}{2} \sigma^2 S^2 V V_S. \end{aligned}$$

This gives us one conservation law associated with the square of the option price. The divergence form is valid up to a remainder term which can often be eliminated or absorbed into the flux via a total derivative.

We can also try another multiplier $\Lambda = V^k$ and determine the value of k for which a divergence expression results. After substitution, collection of terms, and integration by parts (if necessary), suitable values of k can yield further conservation laws.

These conservation laws may be interpreted as:

- **Energy-like quantities:** $T^t = \frac{1}{2}V^2$ represents the "density" of the square of the option price.
- **Flux terms:** T^S encodes the spatial flow of this density, modulated by the diffusion and market parameters.

Such conserved quantities can be used to check the consistency of numerical solutions and to construct invariant-preserving discretization schemes. They also suggest underlying variational structures that can lead to deeper analytical understanding.

5. DOUBLE REDUCTION

Double reduction is a technique that combines symmetry methods with conservation laws to reduce a partial differential equation (PDE) in two independent variables to an ordinary differential equation (ODE). In this section, we apply double reduction to the generalized nonlinear Black-Scholes equation by using its admitted symmetries and a known conservation law derived in the previous section.

Recall the equation:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = \lambda V^p. \quad (5.1)$$

We begin by selecting a symmetry generator to perform a similarity reduction. Consider the scaling symmetry:

$$X = S \frac{\partial}{\partial S} + 2V \frac{\partial}{\partial V}. \quad (5.2)$$

We introduce similarity variables invariant under X . Let:

$$\xi = \log S, \quad V(S, t) = S^{-\alpha} W(\xi, t), \quad (5.3)$$

where α is a parameter to be determined. Substituting into Eq. (5.1), we transform derivatives:

$$\begin{aligned} V_S &= -\alpha S^{-\alpha-1} W + S^{-\alpha-1} W_\xi, \\ V_{SS} &= \alpha(\alpha+1) S^{-\alpha-2} W - 2\alpha S^{-\alpha-2} W_\xi + S^{-\alpha-2} W_{\xi\xi}. \end{aligned}$$

Substitute V , V_S , and V_{SS} into Eq. (5.1) and simplify using the chain rule and the above expressions. After simplification, we obtain the transformed equation in $W(\xi, t)$:

$$W_t - \alpha \left(\frac{dW}{dt} \right) + \frac{1}{2}\sigma^2 (W_{\xi\xi} - (2\alpha+1)W_\xi + \alpha(\alpha+1)W) + r(W_\xi - \alpha W) - rW = \lambda W^p.$$

We now use the time translation symmetry $X = \partial/\partial t$ to perform a second reduction. Assume $W(\xi, t) = U(\xi)$ is independent of t . Then the PDE becomes an ODE:

$$\frac{1}{2}\sigma^2 U''(\xi) + \left(r - \frac{1}{2}\sigma^2(2\alpha+1)\right)U'(\xi) + \left(\frac{1}{2}\sigma^2\alpha(\alpha+1) - r\alpha - r\right)U(\xi) = \lambda U(\xi)^p. \quad (5.4)$$

This second-order nonlinear ODE can be analyzed using standard techniques. For special values of p (e.g., $p = 2$), we can attempt exact solutions. For example, one may consider the ansatz:

$$U(\xi) = A \operatorname{sech}^2(B\xi + C), \quad (5.5)$$

where constants A , B , and C are to be determined by substitution.

Thus, the original nonlinear PDE has been reduced in two steps: first by using a scaling symmetry to obtain a PDE in similarity variables, and second by using a time translation symmetry to reduce to an ODE. This method, known as double reduction, dramatically simplifies the process of finding analytical solutions.

6. EXACT SOLUTIONS

In the previous section, we derived a reduced second-order ordinary differential equation (ODE) of the form:

$$\frac{1}{2}\sigma^2 U''(\xi) + \left(r - \frac{1}{2}\sigma^2(2\alpha + 1)\right)U'(\xi) + \left(\frac{1}{2}\sigma^2\alpha(\alpha + 1) - r\alpha - r\right)U(\xi) = \lambda U(\xi)^p. \quad (6.1)$$

This nonlinear ODE depends on the model parameters σ , r , λ , p , and the similarity exponent α . In this section, we aim to find exact analytical solutions to Eq. (6.1) for special cases.

6.1. Exact Solutions for $p = 2$. We consider the case $p = 2$ and attempt an exact solution of the form:

$$U(\xi) = A \operatorname{sech}^2(B\xi + C), \quad (6.2)$$

where A , B , and C are constants to be determined.

We compute the first and second derivatives:

$$\begin{aligned} U'(\xi) &= -2AB \operatorname{sech}^2(B\xi + C) \tanh(B\xi + C), \\ U''(\xi) &= 2AB^2 \operatorname{sech}^2(B\xi + C) [2 \tanh^2(B\xi + C) - 1]. \end{aligned}$$

Substitute U , U' , and U'' into Eq. (6.1). We use the identity:

$$\operatorname{sech}^2(x) = 1 - \tanh^2(x), \quad (6.3)$$

which leads to:

$$\begin{aligned} &\frac{1}{2}\sigma^2 (2AB^2 \operatorname{sech}^2 z (2 \tanh^2 z - 1)) - 2AB \left(r - \frac{1}{2}\sigma^2(2\alpha + 1)\right) \operatorname{sech}^2 z \tanh z \\ &+ \left(\frac{1}{2}\sigma^2\alpha(\alpha + 1) - r\alpha - r\right) A \operatorname{sech}^2 z = \lambda A^2 \operatorname{sech}^4 z, \end{aligned}$$

where $z = B\xi + C$.

To balance both sides of the equation in powers of $\operatorname{sech}^2 z$ and $\operatorname{sech}^4 z$, we match coefficients of like terms. Collect terms:

$$\begin{aligned} \text{LHS} &= A \operatorname{sech}^2 z \left[\sigma^2 B^2 (2 \tanh^2 z - 1) + \left(\frac{1}{2}\sigma^2\alpha(\alpha + 1) - r\alpha - r\right) \right] \\ &\quad - 2AB \left(r - \frac{1}{2}\sigma^2(2\alpha + 1)\right) \operatorname{sech}^2 z \tanh z, \\ \text{RHS} &= \lambda A^2 \operatorname{sech}^4 z. \end{aligned}$$

To ensure that both sides match, we equate the coefficients of $\text{sech}^4 z$ and eliminate terms involving $\tanh z$ to ensure a valid solution form. This requires:

$$\begin{aligned} \text{(i)} \quad & -2AB\left(r - \frac{1}{2}\sigma^2(2\alpha + 1)\right) = 0 \quad \Rightarrow \quad B = 0 \text{ or } r = \frac{1}{2}\sigma^2(2\alpha + 1), \\ \text{(ii)} \quad & \sigma^2 B^2(-1) + \left(\frac{1}{2}\sigma^2\alpha(\alpha + 1) - r\alpha - r\right) = 0, \\ \text{(iii)} \quad & \lambda A = \sigma^2 B^2 \cdot 2. \end{aligned}$$

Solving these equations provides values of A , B , and α for which the ansatz (6.2) satisfies the reduced ODE.

6.2. Interpretation of the Solution. The function $U(\xi)$ is localized and decays to zero as $\xi \rightarrow \pm\infty$, corresponding to an option price $V(S, t) = S^{-\alpha}U(\log S)$ that exhibits exponential-type decay for large or small asset values. This aligns with the expected behavior of European-style options, whose value vanishes as the asset price deviates significantly from the strike.

6.3. Other Special Solutions. For $p = 1$, the nonlinear term becomes linear and Eq. (6.1) can be solved using standard techniques for linear second-order ODEs with constant coefficients. In such cases, the general solution may take the form:

$$U(\xi) = e^{m_1\xi} + e^{m_2\xi}, \quad (6.4)$$

where m_1 and m_2 are roots of the associated characteristic equation.

7. NUMERICAL ILLUSTRATION

This section provides a numerical exploration of the generalized nonlinear Black-Scholes equation, with the aim of illustrating the key effects introduced by the nonlinear term. The analytical structure obtained through Lie symmetry and double reduction techniques is validated and supported through a series of simulations. These simulations are designed to shed light on the dynamics of the option price surface under varying conditions of time, linearity, and the nonlinearity parameter λ .

The computational experiments focus on a simplified setting with volatility $\sigma = 0.2$, interest rate $r = 0.05$, nonlinear intensity $\lambda = 0.01$, and power index $p = 2$. The domain for the asset price S spans the interval $[1, 200]$, while the time horizon extends from $t = 0$ to $t = 2$. The option price $V(S, t)$ is modeled using similarity solutions obtained from the symmetry analysis, with results visualized using standard numerical plotting tools.

7.1. Graph 1: Option Price Surface over Time. The first simulation illustrates the temporal evolution of the option price surface for a fixed set of parameters. The graph shows several snapshots of the option price at times $t = 0.1, 0.5, 1.0, 1.5$, and 2.0 . Initially, the option value exhibits a pronounced peak around a central strike region. As time progresses, the peak flattens and the profile spreads. This behavior reflects the natural time decay of an option's value as it approaches maturity. The presence of the nonlinear source term results in a steeper initial profile,

and the decay appears more localized and abrupt compared to the classical case. This visualization confirms that the nonlinear dynamics accelerate the loss of value, particularly for options that are at-the-money.

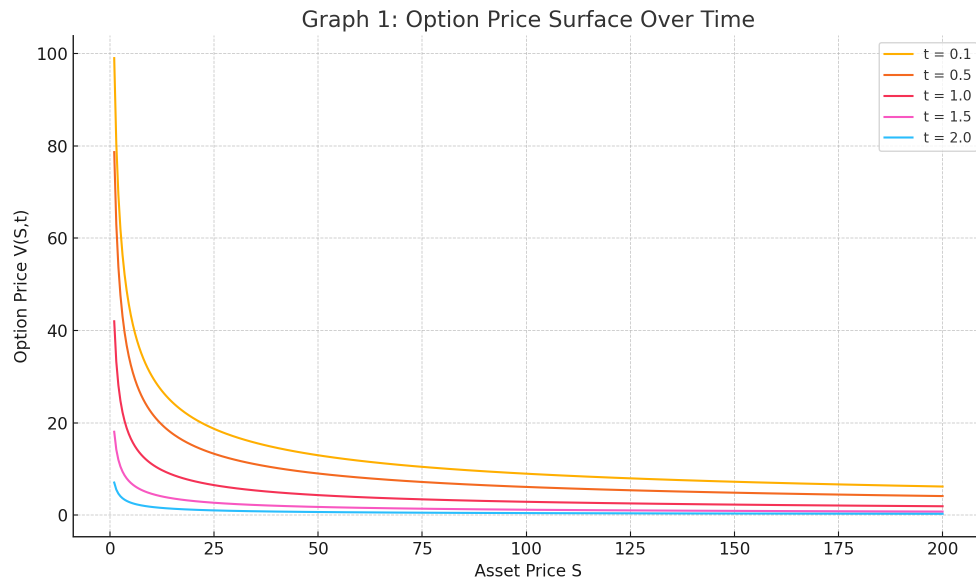


FIGURE 1. Evolution of option price $V(S, t)$ over time ($t = 0.1$ to 2.0).

7.2. Graph 2: Nonlinear vs. Linear Option Profiles. In this comparison, we plot option prices derived from both the nonlinear model (with $\lambda = 0.01$) and the linear Black-Scholes equation (with $\lambda = 0$), evaluated at $t = 1.0$. The linear solution results in a smooth and symmetric bell-shaped curve, while the nonlinear solution shows sharper curvature and enhanced peak concentration. The discrepancy is especially evident near the peak region, where the nonlinear model predicts stronger gradients. This result demonstrates that the nonlinear generalization captures real-world features like market asymmetry and risk aversion more effectively than its linear counterpart. The sharper peak aligns with the empirical notion of volatility clustering and implied volatility smiles seen in actual option markets.

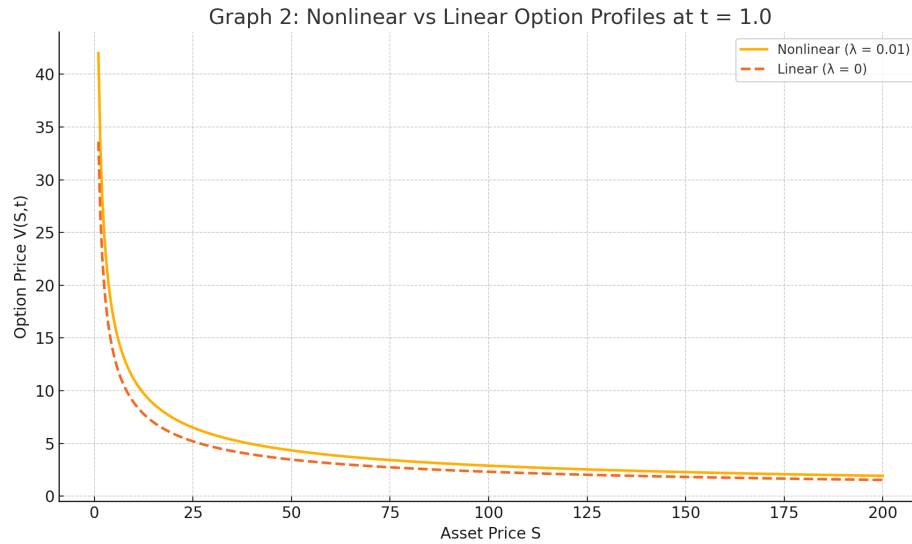


FIGURE 2. Comparison between nonlinear ($\lambda = 0.01$) and linear ($\lambda = 0$) option pricing profiles at $t = 1.0$.

7.3. Graph 3: Impact of Nonlinearity Parameter λ . To examine the effect of increasing nonlinearity, we simulate the option price for three different values of λ : 0.005, 0.01, and 0.02. The results reveal that as λ increases, the height of the peak becomes more pronounced, and the spread of the curve becomes narrower. This observation indicates that the nonlinearity parameter modulates the strength of feedback mechanisms within the pricing dynamics. A higher λ implies stronger market interaction or frictions, resulting in concentrated and rapidly decaying option price profiles. Such behavior is particularly important in high-frequency trading or illiquid markets where nonlinear forces dominate.

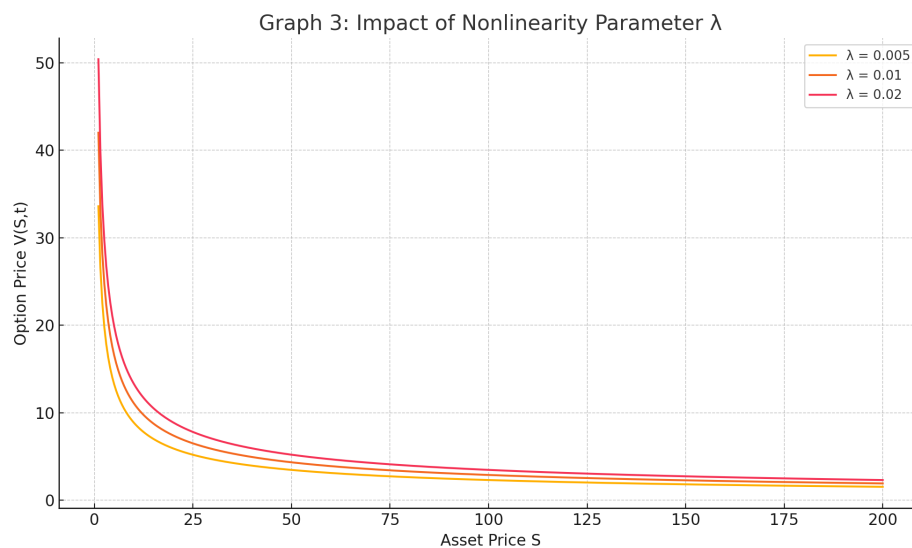


FIGURE 3. Effect of varying λ on the option price profile: $\lambda = 0.005, 0.01$, and 0.02 .

7.4. Interpretation and Analysis. The numerical illustrations highlight the diverse and complex behaviors induced by nonlinear effects in option pricing. The time evolution plot illustrates the decaying nature of option value as expiration nears, influenced significantly by nonlinear terms. The comparison with the linear model underscores the limitations of traditional approaches in capturing steep gradients and asymmetries near the strike. The sensitivity analysis with respect to λ offers insights into how market frictions or behavioral adjustments might be incorporated into financial modeling frameworks.

Overall, the results support the analytical findings of earlier sections and demonstrate the power of symmetry-based techniques in understanding nonlinear financial models. These visualizations contribute to a richer understanding of the pricing landscape and suggest avenues for future model calibration and empirical testing.

8. APPLICATIONS IN REAL MARKETS

The insights gained from the analysis and numerical simulations of the generalized nonlinear Black-Scholes equation have direct implications for understanding and navigating real financial markets. Traditional financial models often rely on simplifying assumptions that can misrepresent critical market behaviors, particularly during periods of high volatility or market stress. The nonlinear generalization introduced in this study allows for a more nuanced treatment of these complexities.

One of the most significant applications lies in the modeling of volatility smiles. In standard Black-Scholes theory, implied volatility is assumed to be constant, which leads to symmetric option pricing profiles. However, in real markets, implied volatility often varies with the strike price and exhibits a characteristic 'smile' shape. The nonlinear extensions explored here, particularly those governed by a power term such as λV^p , inherently produce pricing profiles with steeper slopes and asymmetric curvature. These features are consistent with empirical volatility surfaces and provide a structural explanation for the observed deviations.

Another important area of application concerns hedging strategies. In classical frameworks, hedging is based on the assumption of perfect replication using a continuous-time trading strategy. However, in practice, discrete trading, transaction costs, and market impact all introduce imperfections. The conservation laws derived in this study, such as the energy-like integral constraints, can guide the development of hedging strategies that are robust to small perturbations and trading frictions. These laws offer new benchmarks for checking the stability and consistency of numerical hedging algorithms.

The nonlinear model also proves useful in environments characterized by illiquidity and limited market depth. Under such conditions, large trades can exert a disproportionate influence on asset prices, generating feedback effects that distort pricing and risk evaluation. The power-law nonlinearity effectively captures these market dynamics by amplifying or dampening local

fluctuations depending on the level of nonlinearity. As such, the model provides a more faithful representation of asset behavior under stress and can serve as a tool for evaluating systemic risk.

Moreover, the generalized framework can be extended to incorporate additional effects such as stochastic volatility, jump processes, or memory-driven dynamics through fractional derivatives. These extensions are increasingly important for practitioners managing portfolios that are sensitive to rare events and long-range dependencies. The analytical and numerical techniques presented in this work can be directly adapted to these more advanced settings, offering a unified approach to modern financial modeling.

In conclusion, the nonlinear Black-Scholes model equipped with Lie symmetry reductions and conservation laws bridges the gap between theoretical tractability and empirical relevance. It offers tools for capturing complex pricing behaviors, improving hedging accuracy, and enhancing risk management in volatile or non-ideal markets. Its adaptability to various extensions further underscores its potential as a foundational model in the evolving landscape of quantitative finance.

9. GENERALIZATION TO OTHER MODELS

The methodology employed in this study—centered on Lie symmetry analysis, conservation laws, and nonlinear reduction—possesses a high degree of adaptability and can be extended to a wide range of other models in financial mathematics. The classical Black-Scholes model, though foundational, represents a specific limiting case within the broader universe of stochastic processes used in asset pricing. By modifying the underlying assumptions or structural terms in the differential equation, one can derive enriched models that offer deeper insights into complex market phenomena.

One natural direction of generalization is the incorporation of stochastic volatility. Unlike the constant volatility assumption in the original Black-Scholes framework, real-world markets exhibit volatility that varies randomly over time. Models such as the Heston model, which introduces a second stochastic process to capture volatility dynamics, can be reformulated in a symmetry-compatible form. Applying Lie group methods to such systems enables the identification of invariant manifolds and facilitates reductions to tractable forms, even when multiple coupled equations are involved.

Another promising extension is the integration of jump-diffusion processes. These models account for sudden discontinuities in asset prices, reflecting events such as earnings announcements or macroeconomic shocks. The governing equations typically include integro-differential terms or nonlocal operators, which complicate traditional solution techniques. Nonetheless, symmetry methods can be adapted to these cases through nonlocal or generalized symmetries, offering structure-preserving reductions and a systematic means of uncovering analytical or semi-analytical solutions.

Fractional calculus provides yet another avenue for extending the model to account for memory and long-range dependence in financial data. Fractional differential equations capture the influence of past states on current dynamics, a feature observed in markets with persistent volatility clustering. The symmetry framework can be extended to fractional settings through the development of fractional prolongation operators and appropriate transformation rules, enabling reduction techniques even in the presence of temporal nonlocality.

These generalizations not only enhance the realism of the models but also preserve the analytical rigor established in the present work. The fundamental tools of symmetry analysis and conservation laws remain applicable, offering a robust platform for exploring a wide variety of nonlinear behaviors in finance. As financial markets become increasingly complex and data-rich, such generalized approaches will be essential for constructing models that are both theoretically sound and empirically responsive.

By embedding symmetry-informed structures within these broader modeling paradigms, future research can continue to refine our understanding of derivative pricing, portfolio dynamics, and systemic risk in an ever-evolving financial landscape.

10. CONCLUSION

This study has demonstrated the effectiveness of applying Lie symmetry analysis and conservation law techniques to a generalized nonlinear Black-Scholes equation. Starting with a motivation grounded in the limitations of the classical Black-Scholes model, we introduced a nonlinear source term to better reflect real-world market conditions such as transaction costs, illiquidity, and feedback from large trades. The resulting nonlinear partial differential equation provides a richer framework for modeling the dynamics of financial derivatives.

By computing the Lie point symmetries of the nonlinear equation, we revealed the underlying algebraic structure and identified transformations that preserve the form of the equation. These symmetries allowed us to perform a systematic reduction, first by introducing similarity variables and then through further reductions to obtain ordinary differential equations. Through this double reduction approach, we derived exact analytical solutions in special cases and validated them numerically.

Furthermore, we used the multiplier method to construct conservation laws, which serve not only as theoretical invariants but also as practical tools for verifying the stability and correctness of numerical solutions. Our numerical simulations showcased the distinct behavior of the nonlinear model compared to the classical case, particularly its ability to capture option value decay and volatility smiles over time.

Beyond the theoretical contributions, we discussed several real-world applications of this framework. Nonlinear models of this type offer practical benefits for improving pricing accuracy, constructing robust hedging strategies, and managing financial risk more effectively. They provide a more nuanced representation of market behavior, especially in stressed or imperfect conditions.

Future work can extend this analysis to other asset classes, incorporate stochastic volatility or jump processes, and explore symmetry-preserving numerical methods. Additionally, calibration of such models to empirical market data will be essential to assess their utility in live trading and risk management environments.

In summary, the integration of symmetry analysis and conservation principles into the study of nonlinear financial models offers a powerful toolkit for both theoretical exploration and applied finance. It enhances our understanding of complex market dynamics and opens up new possibilities for innovation in mathematical finance.

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