

**A New Sixteenth Order Iterative Method for Solving Nonlinear Equations****Kedsadaphon Wongsim<sup>1</sup>, Napassanan Srisarakham<sup>2,\*</sup>**<sup>1</sup>*Department of Energy and Environment Engineering, Faculty of Engineering, Mahasarakham Rajabhat University, Mahasarakham 44000, Thailand*<sup>2</sup>*Department of Mathematics, Faculty of Science, Mahasarakham University, Mahasarakham 44150, Thailand**\*Corresponding author: Napassanan.sri@msu.ac.th*

**Abstract.** In this paper, we present a new iterative method for solving nonlinear equations. The method is a combination of the method of Chun and the method of Hu et.al. The new method requires six function evaluations and has the order of convergence sixteenth. Numerical experiments are made to demonstrate the convergence and validation of the iterative method.

**1. INTRODUCTION**

In this paper, we consider iterative methods to find a simple root  $\alpha$  of a nonlinear equation  $f(x) = 0$ , where  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$  for an open interval  $D$  is a scalar function. This problem is important for many nonlinear numerical problems. Newton's method is probably the most widely used algorithm for dealing with such problems, and it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (1.1)$$

which converges quadratically in some neighborhood of  $\alpha$

In recent years, researchers have made many modifications in this method to get higher order iterative methods. These methods are developed using various techniques by introducing some more steps to Newton's method. In this way, not only the convergence order but efficiency index of the method may also be increased. The well-known King's family of methods [1] is an example of this class of methods, which is given as

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$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (1.2)$$

$$x_{n+1} = y_n - \left[ \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \right] \frac{f(y_n)}{f'(x_n)}, \quad (1.3)$$

where  $\beta \in \mathbb{R}$ . Chun [3] presented a fourth-order family of new variants of King's fourth-order family of methods, defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2\beta f^2(y_n)f'(x_n)} \frac{f(y_n)}{f'(x_n)}, \quad (1.4)$$

where  $\beta \in \mathbb{R}$ . Hu et.al [4] presented a fifth-order method which is the variant of double newton's method, defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \left[ 1 + \left( \frac{f(y_n)}{f(x_n)} \right)^2 \right] \frac{f(y_n)}{f'(y_n)}, \quad (1.5)$$

In this paper, we present the new method which is formed from a combination of (1.4) and (1.5). We also incorporate Hermite interpolation to approximate the first derivative in the method. The discussion of the new method and its convergence and analysis are carried out in Section 2. Then, in Section 3 we perform numerical simulations using some test functions, and compare the new method with some other methods.

## 2. THE METHOD AND ANALYSIS OF CONVERGENCE

Consider the iteration method of the form

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \left[ 1 + \left( \frac{f(y_n)}{f(x_n)} \right)^2 \right] \frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \left[ 1 + \frac{f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)}{f(z_n)} \right]^2 \frac{f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)}{f'(z_n)}. \quad (2.1)$$

We use the method of Hu et.al (1.5) in the first and second steps of a three-step cycle, and in the third step we apply Chun's method (1.4) in the case  $\beta = \frac{1}{2}$ . We can easily prove that the method (2.1) is twentyth-order convergent and it requires seven evaluations of the function and its first derivative. The method (2.1) has an efficiency index of [5]  $20^{\frac{1}{7}} \approx 1.534$ . To derive a method with a higher efficiency index, we reduce the number of evaluation functions by approximate  $f'(y_n)$  using a Hermite interpolation.

To approximate  $f'(y_n)$ , we construct a Hermite interpolation polynomial,  $H_2(x)$ , that meets the interpolation conditions  $H_2(x_n) = f(x_n)$ ,  $H_2(y_n) = f(y_n)$ ,  $H'_2(x_n) = f'(x_n)$ . We have

$$H'_2(x_n) = \frac{2(x - x_n)}{(y_n - x_n)^2} (f(y_n) - f(x_n)) - \frac{2x - x_n - y_n}{y_n - x_n} f'(x_n), \quad (2.2)$$

an approximation of  $f'(y_n)$  is then obtained:

$$f'(y_n) \approx H'_2(x_n) = 2 \frac{(f(y_n) - f(x_n))}{y_n - x_n} - f'(x_n) = 2f[x_n, y_n] - f'(x_n), \quad (2.3)$$

where  $f[x_n, y_n] = \frac{f(y_n) - f(x_n)}{(y_n - x_n)}$ . Therefore, a new method is derived as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \left[ 1 + \left( \frac{f(y_n)}{f'(x_n)} \right)^2 \right] \frac{f(y_n)}{2f[x_n, y_n] - f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} - \left[ 1 + \frac{f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)}{f(z_n)} \right]^2 \frac{f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)}{f'(z_n)}. \end{aligned} \quad (2.4)$$

We prove the following convergence theorem for the method (2.4).

**Theorem 2.1.** Assume that the function  $f$  is sufficiently differentiable and  $f$  has a simple zero  $\alpha \in D$ . If the initial point  $x_0$  is sufficiently close to  $\alpha$ , then the method defined by (2.4) converges to  $\alpha$  with sixteenth-order and the error function is  $e_{n+1} = c_2^5 c_3^4 (4c_2^2 - c_3) e_n^{16} + O(e_n^{17})$  where  $e_n = x_n - \alpha$  and  $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$ ,  $k = 2, 3, 4, \dots$

*Proof.* Let  $e_n = x_n - \alpha$  and  $c_k = \frac{f^{(k)}(\alpha)}{k! f'(\alpha)}$ ,  $k = 2, 3, 4, \dots$  Using a Taylor expansion of  $f(x_n)$  and  $f'(x_n)$  about  $\alpha$  and taking into account  $f(\alpha) = 0$ , we obtain

$$f(x_n) = f'(\alpha) \left[ e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6) \right], \quad (2.5)$$

$$f'(x_n) = f'(\alpha) \left[ 1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6) \right], \quad (2.6)$$

from  $x_n = e_n + \alpha$ , (2.5) and (2.6), we obtain

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= \alpha + c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + (-7c_2 c_3 - 4c_2^3 + 3c_4) e_n^4 + O(e_n^5). \end{aligned} \quad (2.7)$$

Expanding  $f(y_n)$  about  $\alpha$  and from (2.7), we obtain

$$f(y_n) = f'(\alpha) \left[ c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + (-7c_2 c_3 - 5c_2^3 + 3c_4) e_n^4 + O(e_n^5) \right]. \quad (2.8)$$

From (2.5) and (2.8), we obtain

$$f[x_n, y_n] = f'(\alpha) \left[ 1 + c_2 e_n + (c_2^2 - c_3) e_n^2 + (3c_2 c_3 + c_4 - 2c_2^3) e_n^3 + O(e_n^4) \right], \quad (2.9)$$

and it follows that

$$\frac{f(y_n)}{2f[x_n, y_n] - f'(x)} = c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 + 3(c_2^2 - 2c_2 c_3 + c_4) e_n^4 + O(e_n^5). \quad (2.10)$$

From (2.5), (2.6), (2.8) and (2.9), we obtain

$$\begin{aligned} z_n &= y_n - \left[ 1 + \left( \frac{f(y_n)}{f(x_n)} \right)^2 \right] \frac{f(y_n)}{2f[x_n, y_n] - f'(x)} \\ &= \alpha - c_2 c_3 e_n^4 + (4c_2^2 + 2c_3 c_2^2 - 2c_2 c_4 - 2c_3^2) e_n^5 + O(e_n^6). \end{aligned} \quad (2.11)$$

Expanding  $f(z_n)$  and  $f'(z_n)$  about  $\alpha$  and from (2.11), we obtain

$$f(z_n) = f'(\alpha) \left[ -c_2 c_3 e_n^4 + (-2c_2 c_4 + 4c_2^4 + 2c_3 c_2^2 - 2c_3^2) e_n^5 + O(e_n^6) \right]. \quad (2.12)$$

$$f'(z_n) = f'(\alpha) \left[ 1 - 2c_3 c_2^2 e_n^4 + 4c_2 (-c_2 c_4 + 2c_2^4 + c_3 c_2^2 - c_3^2) e_n^5 + O(e_n^6) \right]. \quad (2.13)$$

From (2.11), (2.12) and (2.13), we obtain

$$z_n - \frac{f(z_n)}{f'(z_n)} = \alpha + c_2^3 c_3^2 e_n^8 + (4c_2^3 c_3 c_4 - 8c_3 c_2^6 - 4c_3^2 c_2^4 + 4c_2^2 c_3^3) e_n^9 + O(e_n^{10}). \quad (2.14)$$

Expanding  $f \left[ z_n - \frac{f(z_n)}{f'(z_n)} \right]$  about  $\alpha$  and from (2.14), we obtain

$$f \left[ z_n - \frac{f(z_n)}{f'(z_n)} \right] = f'(\alpha) \left[ c_2^3 c_3^2 e_n^8 + (4c_2^3 c_3 c_4 - 8c_3 c_2^6 - 4c_3^2 c_2^4 + 4c_2^2 c_3^3) e_n^9 + O(e_n^{10}) \right]. \quad (2.15)$$

From (2.14), (2.12), (2.13) and (2.15), we obtain

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \left[ 1 + \frac{f \left( z_n - \frac{f(z_n)}{f'(z_n)} \right)}{f(z_n)} \right]^2 \frac{f \left( z_n - \frac{f(z_n)}{f'(z_n)} \right)}{f'(z_n)} = \alpha + c_2^5 c_3^4 (4c_2^2 - c_3) e_n^{16} + O(e_n^{17}). \quad (2.16)$$

Therefore  $x_{n+1} = e_{n+1} + \alpha$ , and from (2.16), we get

$$e_{n+1} = c_2^5 c_3^4 (4c_2^2 - c_3) e_n^{16} + O(e_n^{17}). \quad (2.17)$$

□

This finishes the proof of Theorem 1.1.

In terms of computational cost, the developed method requires evaluations of four functions and two first derivatives per iteration. The new method has the efficiency index of  $16^{\frac{1}{6}} \approx 1.587$  which is better than the method (2.1),  $20^{\frac{1}{7}} \approx 1.534$ .

### 3. NUMERICAL EXAMPLES

In this section, we compare our new method of sixteenth-order defined in (2.4), with some famous equation solvers. For the sake of comparison, we consider the optimal sixteenth-order convergent method (28) (ZF) given by Zafar et.al [6] and the sixteenth-order convergent method (CW) given by Comemuang et.al [7] and the optimal sixteenth-order convergent methods (JRP) and (FSH) given by Sharma et al. [8] and Soleymani et al. [9], respectively. All the computations are done using software Maple 2000 with tolerance  $\epsilon = 10^{-1000}$  and 4,000 digits precision. The

stopping criterion is  $|f(x_n)| < \epsilon$ . The following functions are used for numerical testing which are the same in [6].

| Functions test                                      | $x_0$ | Exact zero ( $\alpha$ ) |
|-----------------------------------------------------|-------|-------------------------|
| $f_1(x) = \sin(x) - \frac{x}{100}$                  | -0.9  | 0.0000000000000000      |
| $f_2(x) = e^{\sin(x)} - 1 - \frac{x}{5}$            | 1.0   | 0.0000000000000000      |
| $f_3(x) = e^{-x} + \cos(x)$                         | 0.5   | 1.7461395304080130      |
| $f_4(x) = x^3 + 4x^2 - 15$                          | 3.0   | 1.6319808055660635      |
| $f_5(x) = 10xe^{-x^2} - 1$                          | 0.0   | 1.6796306104284499      |
| $f_6(x) = \sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3$ | 2.0   | 2.3319676558839640      |

**Table 1.** Numerical comparison of  $|f(x_n)|$ , number of iterations  $n = 3$ .

|                      |                         |                          |                           |
|----------------------|-------------------------|--------------------------|---------------------------|
| $f_1(x), x_0 = -0.9$ | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | $0.234 \times 10^{-5}$  | $0.205 \times 10^{-105}$ | $0.133 \times 10^{-2624}$ |
| FSH                  | $0.149 \times 10^{-5}$  | $0.286 \times 10^{-124}$ | $0.248 \times 10^{-2727}$ |
| ZF                   | $0.138 \times 10^{-5}$  | $0.561 \times 10^{-130}$ | $0.349 \times 10^{-2742}$ |
| CW                   | 0.000107                | $0.631 \times 10^{-76}$  | $0.402 \times 10^{-1376}$ |
| 2.4                  | $0.745 \times 10^{-19}$ | $0.271 \times 10^{-485}$ | 0                         |
| $f_2(x), x_0 = 1.0$  | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | $0.815 \times 10^{-9}$  | $0.143 \times 10^{-146}$ | $0.124 \times 10^{-2350}$ |
| FSH                  | $0.375 \times 10^{-8}$  | $0.299 \times 10^{-126}$ | $0.139 \times 10^{-1899}$ |
| ZF                   | $0.239 \times 10^{-8}$  | $0.610 \times 10^{-137}$ | $0.192 \times 10^{-2194}$ |
| CW                   | 0.000296                | $0.986 \times 10^{-22}$  | $0.134 \times 10^{-132}$  |
| 2.4                  | $0.153 \times 10^{-6}$  | $0.145 \times 10^{-134}$ | $0.529 \times 10^{-2695}$ |
| $f_3(x), x_0 = 0.5$  | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | $0.922 \times 10^{-9}$  | $0.111 \times 10^{-152}$ | $0.734 \times 10^{-2456}$ |
| FSH                  | $0.335 \times 10^{-8}$  | $0.701 \times 10^{-145}$ | $0.942 \times 10^{-2332}$ |
| ZF                   | $0.143 \times 10^{-8}$  | $0.850 \times 10^{-151}$ | $0.204 \times 10^{-2426}$ |
| CW                   | 0.0000550               | $0.687 \times 10^{-30}$  | $0.259 \times 10^{-185}$  |
| 2.4                  | $0.113 \times 10^{-8}$  | $0.246 \times 10^{-152}$ | $0.525 \times 10^{-2451}$ |
| $f_4(x), x_0 = 3.0$  | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | 0.0000130               | $0.455 \times 10^{-103}$ | $0.214 \times 10^{-1678}$ |
| FSH                  | 0.000231                | $0.471 \times 10^{-81}$  | $0.410 \times 10^{-1324}$ |
| ZF                   | 0.000231                | $0.471 \times 10^{-81}$  | $0.410 \times 10^{-1324}$ |
| CW                   | 0.000241                | $0.345 \times 10^{-32}$  | $0.296 \times 10^{-205}$  |
| 2.4                  | $0.494 \times 10^{-5}$  | $0.826 \times 10^{-112}$ | $0.309 \times 10^{-1820}$ |
| $f_5(x), x_0 = 0.0$  | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | $0.186 \times 10^{-19}$ | $0.128 \times 10^{-332}$ | 0                         |
| FSH                  | $0.101 \times 10^{-19}$ | $0.230 \times 10^{-337}$ | 0                         |
| ZF                   | $0.955 \times 10^{-20}$ | $0.884 \times 10^{-338}$ | 0                         |
| CW                   | $0.154 \times 10^{-9}$  | $0.222 \times 10^{-67}$  | $0.196 \times 10^{-414}$  |
| 2.4                  | $0.496 \times 10^{-19}$ | $0.744 \times 10^{-326}$ | 0                         |
| $f_6(x), x_0 = 2.0$  | $ f(x_1) $              | $ f(x_2) $               | $ f(x_3) $                |
| JRP                  | $0.263 \times 10^{-18}$ | $0.273 \times 10^{-313}$ | 0                         |
| FSH                  | $0.346 \times 10^{-18}$ | $0.549 \times 10^{-311}$ | $0.2 \times 10^{-3998}$   |
| ZF                   | $0.138 \times 10^{-18}$ | $0.119 \times 10^{-317}$ | 0                         |
| CW                   | 0.0618                  | $0.394 \times 10^{-14}$  | $0.263 \times 10^{-93}$   |
| 2.4                  | $0.549 \times 10^{-19}$ | $0.390 \times 10^{-92}$  | $0.2 \times 10^{-3998}$   |

Table 1 shows the absolute value of function  $|f(x_n)|$  of each methods. The computational results in Table 1 show that in the most cases, our method (2.4) requires less value of function  $|f(x_n)|$  than the other methods of the same order.

#### 4. CONCLUSION

This paper present the modification of three step iteration method to solve nonlinear equations  $f(x) = 0$ . The new iterative method is formed from a combination of Hu's method and Chun's method. To reduce the number of evaluation functions, some derivatives in this method are estimated by Hermite interpolation. This method requires four functions and two first derivative evaluations per iteration. We have that the order convergence of this method is sixteen. Some examples show that the new method presented in the paper performs better than or similary to some other methods.

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**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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