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Strongly Convergent Inertial Krasnosel'skiĭ–Mann and Ishikawa-Type Schemes for Fixed Point and Monotone Inclusion Problems with Applications to Image Restoration

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Abstract. We propose an inertial Krasnosel'skiĭ–Mann and Ishikawa-type iterative process with step-size control for finding fixed points of nonexpansive mappings in Hilbert spaces. Without relying on viscosity-type techniques and under mild assumptions on the control parameters, we establish strong convergence of the scheme. The method is also utilized for solving monotone inclusion problems and extended to image restoration applications. Numerical tests on several blurring operators confirm that the algorithm attains higher signal-to-noise ratio (SNR), delivers superior restoration quality compared with existing approaches.

1. Introduction

In what follows, we use the notations $\mathbb{N}$ for the natural numbers, $\mathbb{R}$ for the real numbers, and $\mathbb{R}^n$ for the n-dimensional Euclidean space with $n \in \mathbb{N}$. The symbol I refers to the identity operator. Throughout this work, H denotes a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

We next recall some standard classes of operators. Consider a mapping $T : H \to H$:

(i) T is called *Lipschitzian* if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

(ii) T is called *strictly pseudo-contractive* if there exists $\kappa \in (-\infty, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \delta\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \tag{1.1}$$

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Several well-known operator classes arise as special instances of (1.1):

When $\delta = -1$, T is *firmly nonexpansive*, equivalently,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \qquad \forall\, x, y \in H.$$

For $\delta = 0$, T becomes a *nonexpansive mapping*, that is,

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall\, x, y \in H.$$

If $\delta = 1$, then T reduces to a *pseudo-contractive mapping*; see [1] for further details.

Recall that if C is a nonempty closed convex subset of a Hilbert space H, then for every $u \in H$ there exists a unique element $\hat{x} \in C$ such that

$$\|u - \hat{x}\| = \inf_{x \in C} \|u - x\|.$$

The operator $P_C : H \to C$, defined by $P_C(u) = \hat{x}$ is known as the *metric projection* of H onto C; see [2].

Fixed Point Problem. For a mapping $T : H \to H$, the fixed point problem is to determine $x \in H$ such that

$$x = Tx. \tag{1.2}$$

The collection of such points is denoted by $F(T) := \{x \in H : Tx = x\}$. This formulation underlies a wide range of iterative methods aimed at solving optimization tasks as well as monotone inclusion problems, and it provides the theoretical foundation for the algorithmic scheme proposed in this paper. The study of fixed points of nonexpansive operators has long been a central topic in fixed point theory, with numerous applications in signal and image processing, including image restoration and recovery (see, e.g., [3–5]).

However, employing a straightforward iterative method like the Picard iteration [6], defined by

$$u_{k+1} = Tu_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \quad u_0 \in H \text{ arbitrary},$$

may fail to converge to a fixed point of (1.2). For instance, consider the mapping $T : [0,1] \to [0,1]$ given by $Tu = 1 - u$ for all $u \in [0,1]$. It is easy to verify that T is nonexpansive with $F(T) = \{\frac{1}{2}\}$. Starting with $u_0 = \frac{2}{5}$ yields $u_1 = 1 - u_0 = \frac{3}{5}$, $u_2 = 1 - u_1 = \frac{2}{5}$, and so on, producing the alternating sequence $(u_k)_{k \geq 0} = \left(\frac{2}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \ldots\right)$, which does not converge to the desired fixed point $\frac{1}{2}$.

To address such limitations, Krasnosel'skiĭ and Mann independently proposed what is now known as the *Krasnosel'skiĭ–Mann algorithm* [7], a widely used scheme for solving (1.2):

$$u_{k+1} = (1 - \alpha_k)u_k + \alpha_k Tu_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \tag{1.3}$$

where $(\alpha_k)_{k \geq 0} \subset [0,1]$ is an appropriately chosen control sequence. Reich [8] proved that if T is nonexpansive with a nonempty fixed point set and $(\alpha_k)_{k \geq 0}$ satisfies certain standard conditions, then the sequence generated by (1.3) converges *weakly* to a fixed point of T.

Building upon this direction, Ishikawa [9] proposed a two-step iterative scheme for Lipschitzian pseudo-contractive mappings in Hilbert spaces, which is now referred to as the *Ishikawa iteration*:

$$\begin{cases} w_k = (1-\alpha_k)u_k + \alpha_k T u_k, \\ u_{k+1} = (1-\beta_k)u_k + \beta_k T w_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.4}$$

where $(\alpha_k)_{k\geq 0}$ and $(\beta_k)_{k\geq 0}$ are sequences in $[0,1]$ subject to suitable constraints. Under appropriate assumptions on C, $(\alpha_k)_{k\geq 0}$, and $(\beta_k)_{k\geq 0}$, Ishikawa established the strong convergence of (1.4) to a fixed point of T.

Subsequently, Halpern [10] introduced another iterative method for solving the fixed point problem (1.2) associated with a nonexpansive mapping T. Unlike the Krasnosel'skiĭ–Mann scheme, Halpern's approach incorporates a reference element $u \in C$ that remains fixed throughout the iteration, yielding the following process:

$$\begin{cases} w, u_0 \in C, \\ u_{k+1} = (1-\alpha_k)w + \alpha_k T u_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.5}$$

where $(\alpha_k)_{k\geq 0} \subset [0,1]$. Halpern proved that, under suitable conditions, the sequence generated by (1.5) converges strongly to a fixed point of T. Moudafi [11] introduced an enhanced iterative procedure guaranteeing strong convergence, which later became known as the viscosity approximation method. This method was established by merging Halpern's iteration with the theory of contraction mappings. Since its introduction, the viscosity approximation approach has been extensively studied and generalized in many directions by various authors; see, for example, [12–17] for further developments and applications. In 2009, Yao et al. [18] proposed a modified Krasnosel'skiĭ-Mann iteration for nonexpansive mappings by introducing appropriate step-size parameters. They proved that the generated sequence converges strongly to a fixed point of a nonexpansive mapping in Hilbert spaces. Their scheme is given by

$$\begin{cases} w_k = (1-\alpha_k)\, u_k, \\ u_{k+1} = (1-\beta_k)\, w_k + \beta_k T w_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.6}$$

where $u_0 \in H$ and the sequences $(\alpha_k)_{k\geq 0}, (\beta_k)_{k\geq 0}$ lie in $[0,1]$. A further advancement was made in 2019 by Bot et al. [19], who refined (1.3) to obtain strong convergence to a fixed point of a nonexpansive mapping. Their method is formulated as

$$u_{k+1} = (1-\lambda_k)\rho_k u_k + \lambda_k T \rho_k u_k, \quad \forall k \in \mathbb{N} \cup \{0\}, \tag{1.7}$$

when $(\lambda_k)_{k\geq 0}, (\rho_k)_{k\geq 0} \subset (0,1]$. Under suitable assumptions on these sequences, they established that the iteration converges strongly to the fixed point $\hat{x}$ of T closest to the origin, i.e., $\hat{x} = P_{F(T)}(0)$.

Earlier, in 1964, Polyak [20] introduced several acceleration techniques to enhance the convergence speed of iterative schemes. These include the use of variable relaxation parameters and inertial extrapolation terms of the form $\theta_k(u_k - u_{k-1})$, where the sequence $(\theta_k)_{k\geq 0}$ satisfies certain conditions. Since then, inertial-type strategies have attracted considerable attention and have been extensively studied; see [21–27] for comprehensive references. In 2019, Shehu [28] introduced an algorithm combining inertial terms, Halpern's method, and error perturbations to approximate fixed points of nonexpansive mappings. Subsequently, Kitkuan et al. [29] applied inertial extrapolation techniques to the viscosity approximation method in order to solve certain monotone inclusion problems, with applications to image restoration. Along similar lines, Artsawang and Ungchittrakool [30] proposed and analyzed an inertial Mann-type iterative scheme, motivated by the work of Bot et al. [19], for approximating fixed points of nonexpansive mappings. Their method was further applied to monotone inclusion and image restoration problems. The iterative procedure can be written as

$$(\textbf{AU2020})\begin{cases} u_0, u_1 \in H, \\ w_k = u_k + \theta_k(u_k - u_{k-1}), \\ u_{k+1} = \rho_k w_k + \alpha_k(T\rho_k w_k - \rho_k w_k) + \varepsilon_k, \quad \forall k \in \mathbb{N}, \end{cases}$$

where $(\theta_k)_{k\geq 1}, (\alpha_k)_{k\geq 1}, (\rho_k)_{k\geq 1}$ are sequences chosen from $[0,1]$ which satisfy certain desirable properties.

The main contribution of this paper is to propose a new iterative algorithm for approximating fixed points of nonexpansive mappings in Hilbert spaces. The construction of our method is motivated by several classical and modern iterative schemes, in particular, the Ishikawa iteration and the modified Krasnosel'skiĭ-Mann iteration, together with inertial extrapolation techniques. By combining these ideas, we develop a unified framework which not only generalizes a number of existing algorithms but also improves their convergence behavior. Under suitable conditions on the involved control sequences, we prove that the sequence generated by our algorithm converges strongly to the fixed point $\hat{x} \in F(T)$ closest to the origin, namely, $\hat{x} = P_{F(T)}(0)$.

The remainder of this paper is organized as follows. Section 2 collects some basic definitions, lemmas, and preliminary results needed in the sequel. In Section 3, we introduce the proposed algorithm and provide a detailed proof of its strong convergence. In section 4, we apply the algorithm to monotone inclusion problems. Section 5 deals with applications to image restoration problems and provides numerical experiments illustrating the efficiency of the method. Finally, Section 6 concludes the paper with further remarks and possible research directions.

2. Preliminaries

In this part, we compile several auxiliary results in the setting of real Hilbert spaces, which will serve as key ingredients for establishing the main theorem in the subsequent section.

Lemma 2.1 ([31,32]). *Let H denote a real Hilbert space. The identities and inequalities below are valid for all $x, y \in H$ and $t \in \mathbb{R}$:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle x + y, y\rangle, \quad \forall x, y \in H;$
(2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad \forall t \in \mathbb{R}$ *and* $x, y \in H.$

Lemma 2.2 ([33, Lemma 2.5], [34, Lemma 3.1]). *Consider sequences* $(c_k)_{k\geq 0}, (\varepsilon_k)_{k\geq 0} \subseteq [0, +\infty)$, $(\mu_k)_{k\geq 0} \subseteq [0, 1]$ *and* $(\lambda_k)_{k\geq 0} \subseteq \mathbb{R}$ *satisfying*

$$c_{k+1} \leq (1 - \mu_k)c_k + \mu_k\lambda_k + \varepsilon_k, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

If $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$, *then the following assertions hold:*

(1) *If* $\mu_k\lambda_k \leq \sigma\mu_k$ *(where* $\sigma \geq 0$*), then* $(c_k)_{k\geq 0}$ *is bounded.*

(2) *If* $\sum_{k=0}^{\infty} \mu_k = +\infty$ *and* $\limsup_{k\to\infty} \lambda_k \leq 0$, *then* $\lim_{k\to\infty} c_k = 0$.

Proposition 2.1 ([35, Theorem 1.]). *Let* $T : H \to H$ *be a nonexpansive operator with a nonempty fixed point set* $F(T)$. *Then* $F(T)$ *forms a closed and convex subset of* H.

In this work, the notation "$\to$ " will denote strong convergence, while "$\rightharpoonup$ " will indicate weak convergence.

Lemma 2.3 (Demi-closed principle [2]). *Assume* $T : H \to H$ *is nonexpansive and let* $(u_k)_{k\geq 0} \subseteq H$. *The operator* $I - T$ *is demi-closed at the origin; that is, whenever* $u_k \rightharpoonup u \in H$ *and simultaneously* $\|u_k - Tu_k\| \to 0$ *as* $k \to \infty$, *it must hold that* u *is a fixed point of* T, *i.e.,* $u \in F(T)$.

We now recall certain properties of the metric projection, which will play an important role in establishing the main theorem of the next section. These can be formulated as follows:

Lemma 2.4 ([2]). *Let* $C \subset H$ *be a closed convex set with* $C \neq \emptyset$. *For each* $u \in H$ *and* $\hat{x} \in C$,

$$\hat{x} = P_C(u) \text{ if and only if } \langle u - \hat{x}, v - \hat{x} \rangle \leq 0, \quad \forall v \in C.$$

3. Main Results

In this section, we introduce a new iterative algorithm which incorporates inertial terms, the Ishikawa-type averaging process, and a Krasnosel'skiĭ-Mann relaxation step with error perturbations. The proposed method extends and unifies several existing schemes in the literature, and it will be shown to converge strongly to the nearest fixed point of a nonexpansive mapping. The formal description of the method is given as follows.

Algorithm 3.1 A strongly convergent inertial–Ishikawa KM-type algorithm

Initialization: Given real sequences $(\theta_k)_{k\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(s_k)_{k\geq 1}, (t_k)_{k\geq 1} \subseteq [0, 1)$, $(\rho_k)_{k\geq 1} \subseteq [0, 1]$, and an error sequence $(\varepsilon_k)_{k\geq 1} \subseteq H$. Choose arbitrary initial points $u_0, u_1 \in H$.

Iterative Steps: For a current iterate $u_k, u_{k-1} \in H$, repeat the following step:

Step 1. $w_k := u_k + \theta_k(u_k - u_{k-1})$,

Step 2. $v_k := (1 - s_k)w_k + s_kTw_k$,

Step 3. $u_{k+1} := (1 - t_k)\rho_kv_k + t_kT\rho_kv_k + \varepsilon_k$.

Update $k := k + 1$ and return to Step 1.

In order to establish the strong convergence of Algorithm 3.1, it is necessary to impose certain conditions on the control sequences and the perturbation terms. These conditions are presented below.

Assumption 1. *Let $(a_k)_{k\geq 0}, (b_k)_{k\geq 0} \subseteq [0,1)$, $(\rho_k)_{k\geq 0} \subseteq [0,1]$, and $(\varepsilon_k)_{k\geq 0} \subseteq H$ be sequences satisfying the following conditions:*

(1) $\sum\limits_{n=0}^{\infty} 1 - s_k < +\infty$.

(2) $\limsup\limits_{k\to\infty}\ t_k < 1$ *and* $\sum\limits_{k=1}^{\infty} |t_k - t_{k-1}| < +\infty$.

(3) $\lim\limits_{k\to\infty} \rho_k = 1$, $\sum\limits_{k=0}^{\infty} (1-\rho_k) = +\infty$, *and* $\sum\limits_{k=1}^{\infty} |\rho_k - \rho_{k-1}| < +\infty$.

(4) $\sum\limits_{k=0}^{\infty} \|\varepsilon_k\| < +\infty$.

We can create some examples of simple sequences that satisfy the Assumption 1 as follows:

Remark 3.1. *As a concrete example, take $v \in H$ and define $s_k = 1 - \frac{1}{2^k}$, $t_k = \frac{1}{3} + \frac{1}{k+1}$, $\rho_k = 1 - \frac{1}{k+2}$, and $\varepsilon_k = \frac{v}{5^k}$ for all $k \geq 0$. It is straightforward to check that the sequences above satisfy Assumption 1.*

Lemma 3.1. *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $(u_k)_{k\geq 0}$ be generated by Algorithm 3.1. Suppose $(\theta_k)_{k\geq 1} \subset [0,\theta]$ with $\theta \in (0,1)$ such that $\sum\limits_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$. and assume Assumption 1 holds. Then the sequence $(u_k)_{k\geq 0}$ is bounded.*

Proof. Let $k \in \mathbb{N}$ and $u \in F(T)$. Then we consider

$$\|w_k - u\| = \left\|u_k + \theta_k(u_k - u_{k-1}) - u\right\| \leq \|u_k - u\| + \theta_k \|u_k - u_{k-1}\|. \tag{3.1}$$

By using (3.1), we get that

$$\begin{aligned}
\|v_k - u\| &= \left\|(1-s_k)w_k + s_k T w_k - u\right\| \\
&= \left\|(1-s_k)(w_k - u) + s_k(Tw_k - u)\right\| \\
&\leq (1-s_k)\|w_k - u\| + s_k\|Tw_k - u\| \\
&\leq \|w_k - u\|
\end{aligned} \tag{3.2}$$

From (3.2), we deduce that

$$\begin{aligned}
\left\|\rho_k v_k - u\right\| &= \|\rho_k(v_k - u) + (\rho_k - 1)u\| \\
&\leq \rho_k \|v_k - u\| + (1-\rho_k)\|u\| \\
&\leq \rho_k \|u_k - u\| + \rho_k\theta_k\|u_k - u_{k-1}\| + (1-\rho_k)\|u\| \\
&\leq \rho_k \|u_k - u\| + \theta_k\|u_k - u_{k-1}\| + (1-\rho_k)\|u\|.
\end{aligned} \tag{3.3}$$

Using (3.3) for connecting, we will have

$$\begin{aligned}\left\|u_{k+1}-u\right\| &= \|(1-t_k)\rho_k v_k + t_k T\rho_k v_k + \varepsilon_k - u\| \\ &= \|(1-t_k)(\rho_k v_k - u) + t_k(T\rho_k v_k - u) + \varepsilon_k\| \\ &\leq (1-t_k)\|\rho_k v_k - u\| + t_k\|T\rho_k v_k - u\| + \|\varepsilon_k\| \\ &\leq \|\rho_k v_k - u\| + \|\varepsilon_k\| \\ &\leq \rho_k \|u_k - u\| + \theta_k\|u_k - u_{k-1}\| + (1-\rho_k)\|u\| + \|\varepsilon_k\|. \end{aligned} \tag{3.4}$$

Invoking Lemma 2.2 (1) in connection with (3.4) and choosing $\mu_k = 1-\rho_k$, $\|u_k - u\| = c_k$, $\|u\| = \lambda_k = \sigma$, and $\theta_k \|u_k - u_{k-1}\| + \|\varepsilon_k\| = \varepsilon_k$, it follows that $(u_k)_{k\geq 0}$ remains bounded. □

Lemma 3.2. *Let $T : H \to H$ be a nonexpansive mapping with nonempty fixed point set $F(T)$, and suppose $(u_k)_{k\geq 0}$ is generated by Algorithm 3.1. Assume that $(\theta_k)_{k\geq 1} \subseteq [0,\theta]$ for some $\theta \in [0,1)$ satisfies $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$, together with Assumption 1. Then $\left\|u_{k+1} - u_k\right\| \to 0$ as $k \to \infty$.*

Proof. To begin with, note that

$$\begin{aligned}\|w_k - w_{k-1}\| &= \left\|u_k - u_{k-1} + \theta_k(u_k - u_{k-1}) - \theta_{k-1}(u_{k-1} - u_{k-2})\right\| \\ &\leq \|u_k - u_{k-1}\| + \theta_k \|u_k - u_{k-1}\| + \theta_{k-1}\|u_{k-1} - u_{k-2}\|. \end{aligned} \tag{3.5}$$

Using (3.5), we get that

$$\begin{aligned}\|v_k - v_{k-1}\| &= \|(1-s_k)w_k + s_k T w_k - (1-s_{k-1})w_{k-1} - s_{k-1}Tw_{k-1}\| \\ &= \|(1-s_k)(w_k - w_{k-1}) + s_k(Tw_k - Tw_{k-1}) - (s_k - s_{k-1})(w_{k-1} - Tw_{k-1})\| \\ &\leq \|w_k - w_{k-1}\| + |s_k - s_{k-1}|\|w_{k-1} - Tw_{k-1}\| \\ &\leq \|w_k - w_{k-1}\| + |s_k - s_{k-1}| M_1, \end{aligned} \tag{3.6}$$

where $M_1 := \sup\{\|w_{k-1} - Tw_{k-1}\| : k \in \mathbb{N}\}$. Combining (3.6) with the inequality below, we deduce that

$$\begin{aligned}\left\|\rho_k v_k - \rho_{k-1}v_{k-1}\right\| &= \left\|\rho_k(v_k - v_{k-1}) + (\rho_k - \rho_{k-1})v_{k-1}\right\| \\ &\leq \rho_k \|v_k - v_{k-1}\| + \left|\rho_k - \rho_{k-1}\right| \|v_{k-1}\| \\ &\leq \rho_k \|w_k - w_{k-1}\| + |s_k - s_{k-1}| M_1 + \left|\rho_k - \rho_{k-1}\right| \|v_{k-1}\| \\ &\leq \rho_k \|u_k - u_{k-1}\| + \theta_k \|u_k - u_{k-1}\| + \theta_{k-1}\|u_{k-1} - u_{k-2}\| \\ &\quad + |s_k - s_{k-1}| M_1 + \left|\rho_k - \rho_{k-1}\right| M_2, \end{aligned} \tag{3.7}$$

where $M_2 := \sup\{\|v_{k-1}\| : k \in \mathbb{N}\}$. From (3.7), it follows that:

$$\begin{aligned}\left\|u_{k+1} - u_k\right\| &= \left\|(1-t_k)\rho_k v_k + t_k T\rho_k v_k + \varepsilon_k - ((1-t_{k-1})\rho_{k-1}v_{k-1} + t_{k-1}T\rho_{k-1}v_{k-1} + \varepsilon_{k-1})\right\| \\ &= \left\|(1-t_k)(\rho_k v_k - \rho_{k-1}v_{k-1}) - (t_k - t_{k-1})\rho_{k-1}v_{k-1} + t_k(T\rho_k v_k - T\rho_{k-1}v_{k-1})\right. \\ &\quad \left. + (t_k - t_{k-1})T\rho_{k-1}v_{k-1} + (\varepsilon_k - \varepsilon_{k-1})\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-t_k)\left\|\rho_k v_k - \rho_{k-1}v_{k-1}\right\| + |t_k - t_{k-1}|\left\|\rho_{k-1}v_{k-1}\right\| + t_k\left\|T\rho_k v_k - T\rho_{k-1}v_{k-1}\right\| \\
&\quad + |t_k - t_{k-1}|\left\|T\rho_{k-1}v_{k-1}\right\| + \|\varepsilon_k - \varepsilon_{k-1}\| \\
&\leq \left\|\rho_k v_k - \rho_{k-1}v_{k-1}\right\| + |t_k - t_{k-1}|M_3 + \|\varepsilon_k - \varepsilon_{k-1}\| \\
&\leq \rho_k\|u_k - u_{k-1}\| + \theta_k\|u_k - u_{k-1}\| + \theta_{k-1}\|u_{k-1} - u_{k-2}\| + |s_k - s_{k-1}|M_1 \\
&\quad + \left|\rho_k - \rho_{k-1}\right|M_2 + |t_k - t_{k-1}|M_3 + \|\varepsilon_k - \varepsilon_{k-1}\| \\
&= (1-(1-\rho_k))\|u_k - u_{k-1}\| + \gamma_k, \qquad (3.8)
\end{aligned}
$$

where $M_3 := \sup\left\{\left\|\rho_{k-1}v_{k-1}\right\| + \left\|T\rho_{k-1}v_{k-1}\right\| : k \in \mathbb{N}\right\}$ and

$$
\begin{aligned}
\gamma_k := {}& \theta_k\|u_k - u_{k-1}\| + \theta_{k-1}\|u_{k-1} - u_{k-2}\| + |s_k - s_{k-1}|M_1 + \left|\rho_k - \rho_{k-1}\right|M_2 \\
&+ |t_k - t_{k-1}|M_3 + \|\varepsilon_k - \varepsilon_{k-1}\|.
\end{aligned}
$$

By applying Lemma 2.2 (2) together with Assumption 1 in relation (3.8), we deduce that $\left\|u_{k+1} - u_k\right\| \to 0$ as $k \to \infty$. □

Theorem 3.1. *Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $(u_k)_{k\geq 0}$ be the sequence generated by Algorithm 3.1. Suppose $(\theta_k)_{k\geq 1} \subseteq [0,\theta]$ with $\theta \in [0,1)$ satisfies $\sum_{k=1}^{\infty}\theta_k\|u_k - u_{k-1}\| < +\infty$, and assume Assumption 1 holds. Then $(u_k)_{k\geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.*

Proof. From Lemma 3.1, we have $(u_k)_{k\geq 0}$ is bounded. Since $F(T) \neq \emptyset$, $w_k = u_k + \theta_k(u_k - u_{k-1})$ and $v_k = (1-s_k)w_k + s_k T w_k$, so $(w_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ are both bounded. Let $\hat{x} := P_{F(T)}(0)$. It follows that $\hat{x} \in F(T)$. Using Lemma 2.1 (1), we obtain

$$
\begin{aligned}
\|w_k - \hat{x}\|^2 &= \left\|u_k - \hat{x} + \theta_k(u_k - u_{k-1})\right\|^2 \\
&\leq \|u_k - \hat{x}\|^2 + 2\theta_k\langle w_k - \hat{x}, u_k - u_{k-1}\rangle \\
&\leq \|u_k - \hat{x}\|^2 + \theta_k\|u_k - u_{k-1}\|L_1, \qquad (3.9)
\end{aligned}
$$

where $L_1 := \sup\{2\|w_k - \hat{x}\| : k \in \mathbb{N}\}$. Therefore, applying (3.9), we deduce that

$$
\begin{aligned}
\|v_k - \hat{x}\|^2 &= \left\|(1-s_k)w_k + s_k T w_k - \hat{x}\right\|^2 \\
&\leq (1-s_k)\|w_k - \hat{x}\|^2 + s_k\|Tw_k - \hat{x}\|^2 + (1-s_k)s_k\|w_k - Tw_k\|^2 \\
&\leq \|w_k - \hat{x}\|^2 + (1-s_k)\|w_k - Tw_k\|^2 \\
&\leq \|u_k - \hat{x}\|^2 + \theta_k\|u_k - u_{k-1}\|L_1 + (1-s_k)M_1, \qquad (3.10)
\end{aligned}
$$

where $M_1 := \sup\left\{\|w_k - Tw_k\|^2 : k \in \mathbb{N}\right\}$. Relation (3.10) implies that

$$
\begin{aligned}
\left\|\rho_k v_k - \hat{x}\right\|^2 &= \left\|\rho_k(v_k - \hat{x}) + (\rho_k - 1)\hat{x}\right\|^2 \\
&= \rho_k^2\|v_k - \hat{x}\|^2 + 2\rho_k(1-\rho_k)\langle -\hat{x}, v_k - \hat{x}\rangle + (1-\rho_k)^2\|\hat{x}\|^2 \\
&\leq \rho_k\left(\|u_k - \hat{x}\|^2 + \theta_k\|u_k - u_{k-1}\|L_1 + (1-s_k)M_1\right) \\
&\quad + (1-\rho_k)\left(2\rho_k\langle -\hat{x}, v_k - \hat{x}\rangle + (1-\rho_k)\|\hat{x}\|^2\right)
\end{aligned}
$$

$$\leq (1-(1-\rho_k))\|u_k-\hat{x}\|^2+(1-\rho_k)\left(2\rho_k\langle -\hat{x}, v_k-\hat{x}\rangle+(1-\rho_k)\|\hat{x}\|^2\right)$$
$$+\theta_k\|u_k-u_{k-1}\|L_1+(1-s_k)M_1. \quad (3.11)$$

Relation (3.11) yields

$$\begin{aligned}
\left\|u_{k+1}-\hat{x}\right\|^2 &= \left\|\rho_k v_k+t_k(T\rho_k v_k-\rho_k v_k)+\varepsilon_k-\hat{x}\right\|^2\\
&= \left\|(1-t_k)(\rho_k v_k-\hat{x})+t_k(T\rho_k v_k-\hat{x})+\varepsilon_k\right\|^2\\
&\leq \left\|(1-t_k)(\rho_k v_k-\hat{x})+t_k(T\rho_k v_k-\hat{x})\right\|^2+2\langle u_{k+1}-\hat{x},\varepsilon_k\rangle\\
&\leq (1-t_k)\left\|\rho_k v_k-\hat{x}\right\|^2+t_k\left\|T\rho_k v_k-\hat{x}\right\|^2+2\langle u_{k+1}-\hat{x},\varepsilon_k\rangle\\
&\leq \left\|\rho_k v_k-\hat{x}\right\|^2+2\langle u_{k+1}-\hat{x},\varepsilon_k\rangle\\
&\leq (1-(1-\rho_k))\|u_k-\hat{x}\|^2+(1-\rho_k)\left(2\rho_k\langle -\hat{x}, v_k-\hat{x}\rangle+(1-\rho_k)\|\hat{x}\|^2\right)\\
&\quad+\theta_k\|u_k-u_{k-1}\|L_1+(1-s_k)M_1+\|\varepsilon_k\|L_2, \qquad (3.12)
\end{aligned}$$

where $L_2 := \sup\left\{2\left\|u_{k+1}-\hat{x}\right\| : k\in\mathbb{N}\right\}$. Our next step is to verify that $\left\|T\rho_k v_k-\rho_k v_k\right\|\to 0$ as $k\to\infty$. We observe that

$$\begin{aligned}
\left\|Tu_{k+1}-v_k\right\| &= \left\|Tu_{k+1}-(1-s_k)w_k-s_kTw_k\right\|\\
&= \left\|s_k(Tu_{k+1}-Tw_k)+(1-s_k)(Tu_{k+1}-w_k)\right\|\\
&\leq s_k\|Tu_{k+1}-Tw_k\|+(1-s_k)\|Tu_{k+1}-w_k\|\\
&\leq \|u_{k+1}-w_k\|+(1-s_k)\|Tu_{k+1}-w_k\|\\
&= \left\|u_{k+1}-u_k-\theta_k(u_k-u_{k-1})\right\|+(1-s_k)\|Tu_{k+1}-w_k\|\\
&\leq \left\|u_{k+1}-u_k\right\|+\theta_k\|u_k-u_{k-1}\|+(1-s_k)M_2, \qquad (3.13)
\end{aligned}$$

where $M_2 := \sup\{\|Tu_{k+1}-w_k\| : k\in\mathbb{N}\}$. By using (3.13), we get that

$$\begin{aligned}
\left\|T\rho_k v_k-\rho_k v_k\right\| &= \left\|T\rho_k v_k-Tu_{k+1}+Tu_{k+1}-\rho_k v_k\right\|\\
&\leq \left\|T\rho_k v_k-Tu_{k+1}\right\|+\left\|Tu_{k+1}-\rho_k v_k\right\|\\
&\leq \left\|\rho_k v_k-u_{k+1}\right\|+\left\|(1-\rho_k)Tu_{k+1}+\rho_k(Tu_{k+1}-v_k)\right\|\\
&\leq \left\|\rho_k v_k-((1-t_k)\rho_k v_k+t_k(T\rho_k v_k)+\varepsilon_k)\right\|\\
&\quad+(1-\rho_k)\left\|Tu_{k+1}\right\|+\rho_k\left\|Tu_{k+1}-v_k\right\|\\
&\leq t_k\left\|T\rho_k v_k-\rho_k v_k\right\|+\|\varepsilon_k\|+(1-\rho_k)L_3\\
&\quad+\left\|u_{k+1}-u_k\right\|+\theta_k\|u_k-u_{k-1}\|+(1-s_k)M_2, \qquad (3.14)
\end{aligned}$$

where $L_3 := \sup\left\{\left\|Tu_{k+1}\right\| : k\in\mathbb{N}\right\}$. It follows from (3.14), Assumption 1 and Lemma 3.2 that

$$\begin{aligned}
\left\|T\rho_k v_k-\rho_k v_k\right\| &\leq \frac{1}{(1-t_k)}\left(\|\varepsilon_k\|+(1-\rho_k)L_3+\left\|u_{k+1}-u_k\right\|+\theta_k\|u_k-u_{k-1}\|+(1-s_k)M_2\right)\\
&\to 0 \text{ as } k\to\infty. \qquad (3.15)
\end{aligned}$$

By (3.15), one obtains

$$\lim_{k\to\infty} \left\|T\rho_k v_k - \rho_k v_k\right\| = 0. \tag{3.16}$$

We proceed to show that the sequence $(u_k)_{k\geq 0}$ converges strongly to $\hat{x}$ which it is enough to show that

$$\limsup_{k\to\infty} \langle -\hat{x}, v_k - \hat{x}\rangle \leq 0. \tag{3.17}$$

Suppose, for contradiction, that (3.17) does not hold. Hence, there exists a real number $r > 0$ and a subsequence $(v_{k_m})_{m\geq 1} \subseteq (v_k)_{k\geq 1}$ such that

$$\langle -\hat{x}, v_{k_m} - \hat{x}\rangle \geq r > 0, \quad \forall m \geq 1.$$

The fact that $(v_{k_m})_{m\geq 1}$ is bounded ensures the existence of a subsequence $(v_{k_{m_l}})_{l\geq 1}$ of $(v_{k_m})_{m\geq 1}$ such that $v_{k_{m_l}} \rightharpoonup z \in H$ as $l \to \infty$. Therefore,

$$0 < r \leq \lim_{l\to\infty} \langle -\hat{x}, v_{k_{m_l}} - \hat{x}\rangle = \langle -\hat{x}, z - \hat{x}\rangle. \tag{3.18}$$

Given that $\lim_{k\to\infty} \rho_k = 1$, we deduce

$$\rho_{k_{m_l}} v_{k_{m_l}} \rightharpoonup z \text{ as } k \to \infty. \tag{3.19}$$

From (3.16), (3.19) and Lemma 2.3, one obtains $z \in F(T)$. The combination of Proposition 2.1 and Lemma 2.4 ensures that the inequality $\langle -\hat{x}, z - \hat{x}\rangle = \langle 0 - \hat{x}, z - \hat{x}\rangle \leq 0$ is valid which causes a contradiction with (3.18). This contradiction establishes that (3.17) is valid. And then, Assumption 1 (4.1) ensures that

$$\limsup_{k\to\infty} \left(2\rho_k \langle -\hat{x}, v_k - \hat{x}\rangle + (1-\rho_k)\|\hat{x}\|^2\right) \leq 0.$$

In the last step, applying (3.12) and Lemma 2.2 (2), we deduce $\lim_{k\to\infty} u_k = \hat{x}$. This completes the proof. □

Remark 3.2. *Let $\{\xi_k\}_{k\geq 1} \subseteq [0, +\infty)$ be the sequence such that $\sum_{k=1}^{\infty} \xi_k < +\infty$. Then, we define*

$$\tilde{\theta}_k = \begin{cases} \min\left\{\theta, \frac{\xi_k}{\|u_k - u_{k-1}\|}\right\}, & \text{if } u_k \neq u_{k-1}, \\ \theta, & \text{otherwise}, \end{cases}$$

where $(u_k)_{k\geq 0}$ and θ are specified by Theorem 3.1. In addition, when $(\theta_k)_{k\geq 1}$ is selected $[0, \tilde{\theta}_k]$ for all $k \in \mathbb{N}$ it follows that $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$.

4. Algorithmic Applications to Monotone Inclusions

This section is devoted to applying Algorithm 3.1 to the problem of finding a zero of certain monotone inclusion problems involving three operators in the setting of real Hilbert spaces.

Let $\Omega : H \to 2^H$ be a set-valued operator, where 2^H denotes the collection of all subsets of H. The set of zeros of Ω is defined as $\text{zer}(\Omega) := \{z \in H : 0 \in \Omega z\}$, while the graph of Ω is given by $\text{G}(\Omega) := \{(u, v) \in H \times H : v \in \Omega u\}$. We recall the following standard notions for a set-valued operator Ω:

(A) Ω is said to be *monotone* if

$$\langle x - y, \tilde{x} - \tilde{y} \rangle \geq 0, \quad \forall (x, \tilde{x}), (y, \tilde{y}) \in G(\Omega).$$

(B) Ω is called γ*-strongly monotone*, with $\gamma > 0$, if

$$\langle x - y, \tilde{x} - \tilde{y} \rangle \geq \gamma \|x - y\|^2, \quad \forall (x, \tilde{x}), (y, \tilde{y}) \in G(\Omega).$$

(C) Ω is λ*-cocoercive* (or equivalently, λ*-inverse strongly monotone*), with $\lambda > 0$, if

$$\langle x - y, \tilde{x} - \tilde{y} \rangle \geq \lambda \|\tilde{x} - \tilde{y}\|^2, \quad \forall (x, \tilde{x}), (y, \tilde{y}) \in G(\Omega).$$

(D) Ω is said to be *maximal monotone* if it is monotone and its graph cannot be properly contained in the graph of another monotone operator. In other words, if $\Psi : H \to 2^H$ is monotone and $G(\Omega) \subseteq G(\Psi)$, then necessarily $G(\Omega) = G(\Psi)$.

In the particular case when $\Omega : H \to H$ is single-valued, the above conditions simplify to:

(a) Ω is *monotone* if

$$\langle x - y, \Omega x - \Omega y \rangle \geq 0, \quad \forall x, y \in H.$$

(b) Ω is γ*-strongly monotone*, with $\gamma > 0$, if

$$\langle x - y, \Omega x - \Omega y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in H.$$

(c) Ω is λ*-cocoercive*, with $\lambda > 0$, if

$$\langle x - y, \Omega x - \Omega y \rangle \geq \lambda \|\Omega x - \Omega y\|^2, \quad \forall x, y \in H.$$

Recall that for a set-valued operator $\Omega : H \to 2^H$, the mapping $J_\Omega := (I + \Omega)^{-1} : H \to 2^H$ is called the *resolvent* of Ω. It is a classical result that if $\Omega : H \to 2^H$ is maximal monotone and $\eta > 0$, then the resolvent $J_{\eta\Omega}$ is single-valued and firmly nonexpansive.

In this section, we are concerned with the following monotone inclusion problem involving three operators:

$$\text{find } x \in H \text{ such that } 0 \in \Psi x + \Omega x + \Phi x, \tag{4.1}$$

where $\Psi, \Omega : H \to 2^H$ are maximal monotone operators and $\Phi : H \to H$ is a λ-cocoercive operator with some $\lambda > 0$.

To solve problem (4.1) via Algorithm 3.1, we require several auxiliary tools. One of the key ingredients is the following result of Davis and Yin.

Proposition 4.1 ([36]). *Let $F_1, F_2 : H \to H$ be two firmly nonexpansive operators and let $\Phi : H \to H$ be a λ-cocoercive operator with $\lambda > 0$. For any $\eta \in (0, 2\lambda)$, define*

$$T := F_1 \circ (2F_2 - I - \eta \Phi \circ F_2) + I - F_2.$$

Then T is τ-averaged with constant $\tau := \frac{2\lambda}{4\lambda - \eta} < 1$. In particular, for all $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1 - \tau}{\tau} \|(I - T)x - (I - T)y\|^2.$$

The set of solutions to the three-operator monotone inclusion problem can be represented in terms of fixed points of the operator T given in Proposition 4.1. More precisely, the following characterization holds.

Lemma 4.1 ([36, Lemma 2.2]). *Let $\Psi, \Omega : H \to 2^H$ be maximal monotone operators and $\Phi : H \to H$ be an operator. Suppose that* $\operatorname{zer}(\Psi + \Omega + \Phi) \neq \emptyset$. *Then*

$$\operatorname{zer}(\Psi + \Omega + \Phi) = J_{\eta\Omega}(F(T)),$$

where $T = J_{\eta\Psi} \circ \Big(2J_{\eta\Omega} - I - \eta\Phi \circ J_{\eta\Omega}\Big) + \Big(I - J_{\eta\Omega}\Big)$ and $\eta > 0$.

In particular, when T is defined as above, Step 2 of Algorithm 3.1 can be expressed as

$$\begin{aligned} v_k &= (1-s_k)w_k + s_k T w_k \\ &= (1-s_k)w_k + s_k\Big(J_{\eta\Psi} \circ \Big(2J_{\eta\Omega} - I - \eta\Phi \circ J_{\eta\Omega}\Big) + (I - J_{\eta\Omega})\Big)w_k. \end{aligned} \tag{4.2}$$

Moreover, for Step 3 we observe that

$$\begin{aligned} T(\rho_k v_k) - \rho_k v_k &= \Big(J_{\eta\Psi} \circ (2J_{\eta\Omega} - I - \eta\Phi \circ J_{\eta\Omega}) + (I - J_{\eta\Omega})\Big)(\rho_k v_k) - \rho_k v_k \\ &= J_{\eta\Psi}\Big(2J_{\eta\Omega}(\rho_k v_k) - \rho_k v_k - \eta\Phi \circ J_{\eta\Omega}(\rho_k v_k)\Big) + \rho_k v_k - J_{\eta\Omega}(\rho_k v_k) - \rho_k v_k \\ &= J_{\eta\Psi}\Big(2J_{\eta\Omega}(\rho_k v_k) - \rho_k v_k - \eta\Phi \circ J_{\eta\Omega}(\rho_k v_k)\Big) - J_{\eta\Omega}(\rho_k v_k). \end{aligned} \tag{4.3}$$

Therefore, by applying Algorithm 3.1 to the three-operator monotone inclusion problem (4.1), we obtain the following iterative scheme.

Algorithm 4.1 Algorithm for solving the three-operator monotone inclusion problem

Initialization: Given real sequences $(\theta_k)_{k\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(s_k)_{k\geq 1}, (t_k)_{k\geq 1} \subseteq [0, 1)$, $(\rho_k)_{k\geq 1} \subseteq [0, 1]$, and an error sequence $(\varepsilon_k)_{k\geq 1} \subseteq H$. Fix $\eta \in (0, 2\lambda)$.

Iterative Steps: For a current iterate $u_k, u_{k-1} \in H$, repeat the following step:

Step 1. $w_k := u_k + \theta_k(u_k - u_{k-1})$,

Step 2. $v_k := (1-s_k)w_k + s_k\Big(J_{\eta\Psi}\Big(2J_{\eta\Omega}(w_k) - w_k - \eta\Phi(J_{\eta\Omega}(w_k))\Big) + w_k - J_{\eta\Omega}(w_k)\Big)$,

Step 3. $u_{k+1} := \rho_k v_k + t_k\Big(J_{\eta\Psi}\Big(2J_{\eta\Omega}(\rho_k v_k) - \rho_k v_k - \eta\Phi(J_{\eta\Omega}(\rho_k v_k))\Big) - J_{\eta\Omega}(\rho_k v_k)\Big) + \varepsilon_k$.

Update $k := k + 1$ and return to Step 1.

Theorem 4.1. *Let $\Psi, \Omega : H \to 2^H$ be maximal monotone operators and $\Phi : H \to H$ be λ-cocoercive with $\lambda > 0$. Assume that* $\operatorname{zer}(\Psi + \Omega + \Phi)$ *is nonempty. Let $(\theta_k)_{k\geq 1}$ be a sequence in $[0, \theta]$ with $\theta \in [0, 1)$ and $\eta \in (0, 2\lambda)$. Let $(u_k)_{k\geq 0}$, $(w_k)_{k\geq 1}$, and $(v_k)_{k\geq 1}$ be generated by Algorithm 4.1. Assume that $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$ and the Assumption 1 hold. Then the following assertions are valid:*

(1) *$(u_k)_{k\geq 0}$, $(w_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ converge strongly to $\hat{x} := P_{F(T)}(0)$, where* $T := J_{\eta\Psi} \circ \Big(2J_{\eta\Omega} - I - \eta\Phi \circ J_{\eta\Omega}\Big) + \Big(I - J_{\eta\Omega}\Big)$.

(2) $\left(J_{\eta\Omega}(w_k)\right)_{k\geq 1}$ *and* $\left(J_{\eta\Omega}(\rho_k v_k)\right)_{k\geq 1}$ *converge strongly to* $J_{\eta\Omega}(\hat{x}) \in \operatorname{zer}(\Psi+\Omega+\Phi)$.

Proof. (1) From Proposition 4.1, the operator T is nonexpansive. Applying Theorem 3.1, we deduce that $(u_k)_{k\geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$. Since $w_k = u_k + \theta_k(u_k - u_{k-1})$ and $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$, it follows that $w_k \to \hat{x}$. Furthermore, because $s_k \to 1$ and T is continuous, we conclude that $v_k = (1-s_k)w_k + s_k T w_k \longrightarrow T\hat{x} = \hat{x}$.

(2) From part (1) we know $w_k \to \hat{x}$ and $v_k \to \hat{x}$. Since $\rho_k \to 1$, this gives $\rho_k v_k \to \hat{x}$. By continuity of the resolvent $J_{\eta\Omega}$, we obtain $J_{\eta\Omega}(w_k) \to J_{\eta\Omega}(\hat{x})$ and $J_{\eta\Omega}(\rho_k v_k) \to J_{\eta\Omega}(\hat{x})$. Finally, Lemma 4.1 ensures that $J_{\eta\Omega}(\hat{x}) \in J_{\eta\Omega}(F(T)) = \operatorname{zer}(\Psi+\Omega+\Phi)$. □

If we set $\Omega \equiv 0$ in Theorem 4.1, then the resolvent reduces to

$$J_{\eta\Omega}(x) = (I+\eta\Omega)^{-1}(x) = (I+0)^{-1}(x) = I(x), \qquad \forall x \in H.$$

In this case, the operator T becomes

$$\begin{aligned} T &= J_{\eta\Psi} \circ \left(2J_{\eta\Omega} - I - \eta\Phi \circ J_{\eta\Omega}\right) + (I - J_{\eta\Omega}) \\ &= J_{\eta\Psi} \circ (2I - I - \eta\Phi \circ I) + (I - I) \\ &= J_{\eta\Psi} \circ (I - \eta\Phi). \end{aligned}$$

By Lemma 4.1, it follows that $\operatorname{zer}(\Psi+\Phi) = F(T)$. This leads to the following corollary.

Corollary 4.1. *Let* $\Psi : H \to 2^H$ *be a maximal monotone operator and let* $\Phi : H \to H$ *be* λ*-cocoercive with* $\lambda > 0$. *Suppose that* $\operatorname{zer}(\Psi+\Phi) \neq \emptyset$. *Let* $(u_k)_{k\geq 0}$ *be generated by the following scheme:*

Algorithm 4.2 Algorithm for the two-operator inclusion problem

Initialization: Given real sequences $(\theta_k)_{k\geq 1} \subseteq [0,\theta]$ with $\theta \in [0,1)$, $(s_k)_{k\geq 1}, (t_k)_{k\geq 1} \subseteq [0,1)$, $(\rho_k)_{k\geq 1} \subseteq [0,1]$, and an error sequence $(\varepsilon_k)_{k\geq 1} \subseteq H$. Fix $\eta \in (0, 2\lambda)$.

Iterative Steps: For a current iterate $u_k, u_{k-1} \in H$, repeat the following step:

Step 1. $w_k := u_k + \theta_k(u_k - u_{k-1})$,

Step 2. $v_k := (1-s_k)w_k + s_k J_{\eta\Psi}\Big(w_k - \eta\Phi w_k\Big)$,

Step 3. $u_{k+1} := (1-t_k)\rho_k v_k + t_k J_{\eta\Psi}\Big(\rho_k v_k - \eta\Phi(\rho_k v_k)\Big) + \varepsilon_k$.

Update $k := k+1$ and return to Step 1.

Assume that $\sum_{k=1}^{\infty} \theta_k \|u_k - u_{k-1}\| < +\infty$, *and that Assumption 1 holds. Then* $(u_k)_{k\geq 0}$ *converges strongly to the projection* $P_{\operatorname{zer}(\Psi+\Phi)}(0)$.

5. Applications to Image Restoration and Computational Experiments

In this section, we demonstrate how the proposed iterative method can be applied to image restoration tasks, where the aim is to recover high-quality images from degraded observations that suffer from blur, noise, or other distortions. A standard model for the degradation process is given by

$$y = Bx + w, \tag{5.1}$$

where x represents the original image, B is a blur operator, and w denotes additive noise. The recovery problem is often formulated as a regularized least-squares optimization model of the form

$$\min_{x} \left\{ \frac{1}{2} \|Bx - y\|_2^2 + \tau \Psi(x) \right\}, \tag{5.2}$$

where $\tau > 0$ is a regularization parameter and $\Psi(\cdot)$ is a regularization function. A widely used choice for Ψ is the ℓ_1 norm, which serves as a sparsity-promoting penalty and is well known in the context of Tikhonov-type regularization [37]. With this choice, problem (5.2) becomes

$$\min_{x \in \mathbb{R}^k} \left\{ \frac{1}{2} \|Bx - y\|_2^2 + \tau \|x\|_1 \right\}, \tag{5.3}$$

where y denotes the observed degraded image and B is a bounded linear operator. It is worth noting that (5.3) can be cast as a particular instance of the two-operator monotone inclusion problem (4.1) with the specifications: $\Psi = \partial f(\cdot)$, $\Omega \equiv 0$, $\Phi = \nabla L(\cdot)$, where $f(x) = \|x\|_1$, $L(x) = \frac{1}{2}\|Bx - y\|_2^2$, and the regularization parameter is chosen as $\tau = 0.001$. Under this setting, the cocoercive operator becomes $\Phi(x) = \nabla L(x) = B^*(Bx - y)$, where B^* denotes the adjoint (transpose) of B. To conduct the experiments, a set of images was selected and corrupted by different blurring operators. The restoration procedure was then carried out using Algorithm 4.2, corresponding to Corollary 4.1, with the following control parameters: $s_k = 1 - \frac{1}{(10k+1)^2}$, $t_k = 0.97 + \frac{1}{(k+100)^2}$, $\rho_k = 1 - \frac{1}{10k+1}$, $\varepsilon_k = 0$, and the inertial parameter (θ_k) defined by

$$\theta_k = \begin{cases} \min\left\{ \frac{70k-9}{100k}, \frac{1}{(k+1)^2\|u_k - u_{k-1}\|} \right\}, & \text{if } u_k \neq u_{k-1}, \\ \frac{70k-9}{100k}, & \text{otherwise.} \end{cases} \tag{5.4}$$

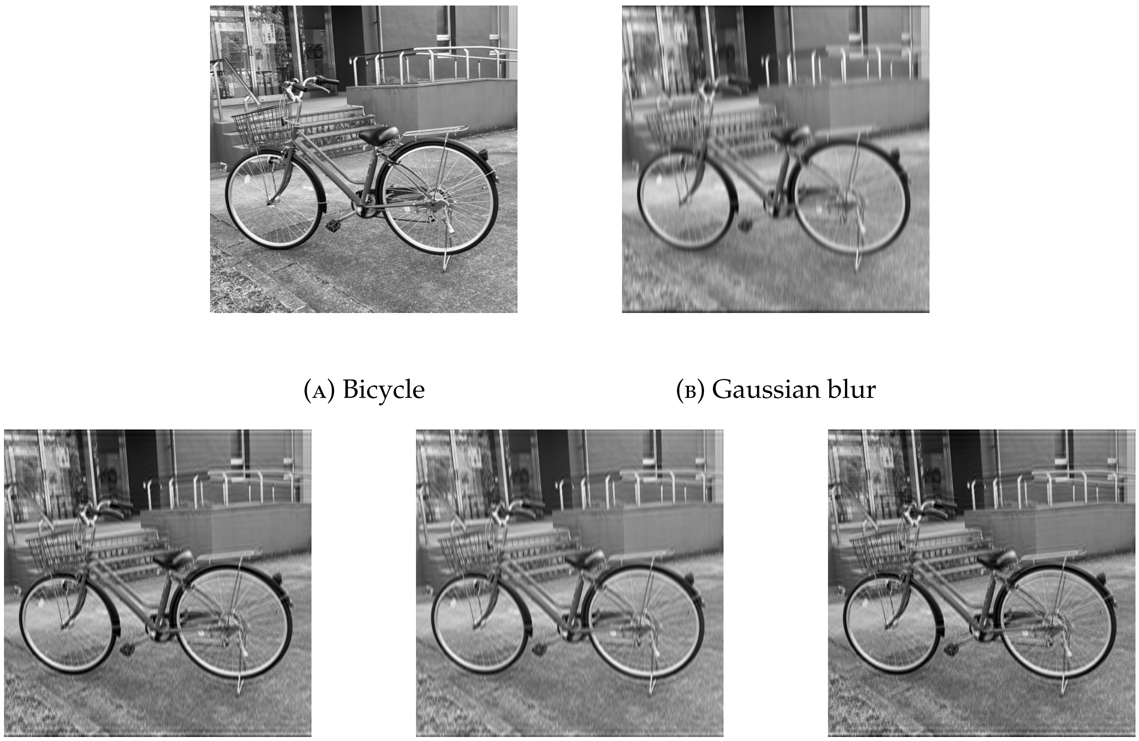
Finally, to assess the effectiveness of the proposed scheme, we perform a comparative study with two existing approaches: the inertial Mann-type iteration introduced by Artsawang and Ungchittrakool (abbreviated as AU2020) in [30, Corollary 2], and the iterative procedure described in [38, Algorithm (4.1)] (referred to as Akutsah et al. Alg. 2023).

For the algorithm proposed by Artsawang and Ungchittrakool (AU2020), the parameters are specified as $\alpha_k = 0.97 + \frac{1}{(k+100)^2}$, $\rho_k = 1 - \frac{1}{100k+1}$, $\lambda_k = 0.7$. In the case of the scheme introduced by Akutsah et al. (2023), we adopt the parameter choice $\alpha_k = \beta_k = B_k = 0.1 + \frac{1}{(10k)^2}$. To assess the fidelity of the reconstructed images, we measure the signal-to-noise ratio (**SNR**), defined as

$$\mathbf{SNR}(k) = 20 \log_{10} \frac{\|x\|_2^2}{\|x - u_k\|_2^2},$$

where x represents the original image and u_k denotes the approximation obtained after k iterations.

All numerical experiments were implemented in Matlab 9.19 (R2022b) and executed on a MacBook Pro (14-inch, 2021) equipped with an Apple M1 Pro processor and 16 GB of memory. The results obtained under the above parameter settings are illustrated in the following figures.



(A) Bicycle (B) Gaussian blur

(C) AU2020 (D) Akutsah et al.Alg.2023 (E) Algorithm 3

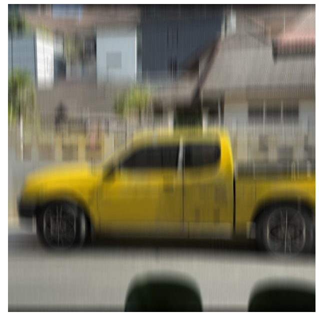
Figure 1. Image restoration results on the Bicycle test image. Subfigure (A) shows the original image, while (B) displays the degraded version corrupted by Gaussian blur. Subfigures (C), (D), and (E) present the reconstructions obtained using AU2020, Akutsah et al. Algorithm 2023, and the proposed Algorithm 4.2, respectively.

(A) Motorcycle (B) Average blur

(C) AU2020 (D) Akutsah et al.Alg.2023 (E) Algorithm 3

Figure 2. Image restoration results on the Motorcycle test image. Subfigure (A) shows the original image, while (B) displays the degraded version corrupted by average blur. Subfigures (C), (D), and (E) present the reconstructions obtained using AU2020, Akutsah et al. Algorithm 2023, and the proposed Algorithm 4.2, respectively.

(A) Pickup truck



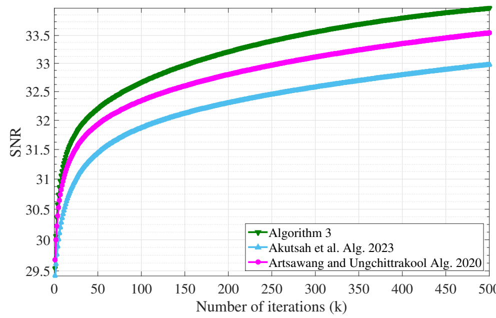
(B) Motion blur

(C) AU2020

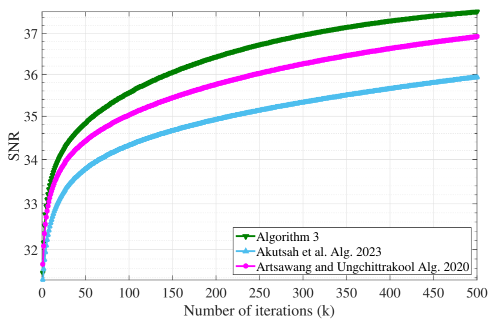
(D) Akutsah et al.Alg.2023

(E) Algorithm 3

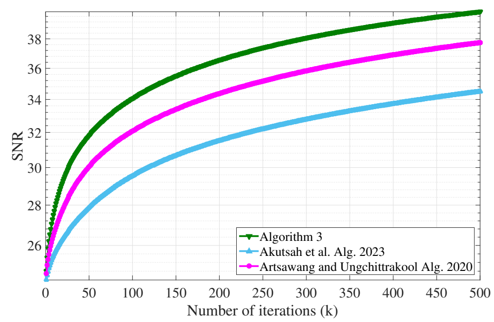
Figure 3. Image restoration results on the Pickup truck test image. Subfigure (A) shows the original image, while (B) displays the degraded version corrupted by motion blur. Subfigures (C), (D), and (E) present the reconstructions obtained using AU2020, Akutsah et al. Algorithm 2023, and the proposed Algorithm 4.2, respectively.



(A)



(B)



(C)

Figure 4. Signal-to-noise ratio (SNR) comparison across iterations for three benchmark algorithms: AU2020, Akutsah et al. Alg. 2023, and the proposed Algorithm 4.2. Subfigures (A), (B), and (C) correspond to the test images Bicycle, Motorcycle, and Pickup truck, respectively.

Figures 1–3 present the outcomes of the proposed method compared with the AU2020 and Akutsah et al. Alg. 2023 algorithms under different degradation settings. In Figure 1 (Bicycle), the original image and its Gaussian-blurred version are displayed in subfigures (a) and (b), while the restored images obtained from AU2020, Akutsah et al. Alg. 2023, and Algorithm 4.2 are shown in (c), (d), and (e), respectively. It can be observed that Algorithm 4.2 provides sharper structural details of the bicycle, particularly along the wheel spokes and frame, compared to the other two methods.

Figure 2 (Motorcycle) illustrates the case of Average blur. While all three restoration methods are capable of reducing blur, Algorithm 4.2 consistently achieves higher visual clarity, preserving fine details of the motorcycle body and background elements more effectively than AU2020 and Akutsah et al. Alg. 2023.

In Figure 3 (Pickup truck), the images degraded by Motion blur are considered. The reconstruction obtained from Algorithm 4.2 demonstrates superior restoration of edge features, especially around the vehicle's contours and horizontal patterns, which appear noticeably sharper than in the results of the comparison methods.

Figure 4 further validates these observations by reporting the signal-to-noise ratio (SNR) performance across 500 iterations. In all three test images, the SNR values of Algorithm 4.2 dominate those of AU2020 and Akutsah et al. Alg. 2023, indicating higher reconstruction accuracy.

Table 1. Signal-to-noise ratio (SNR) values obtained at different iterations n for three test images (Bicycle, Motorcycle, and Pickup truck). The performance of AU2020, Akutsah et al. Alg. 2023, and the proposed Algorithm 4.2 is compared.

k	Bicycle			Motorcycle			Pickup truck		
	AU2020	Akutsah2023	Alg. 3	AU2020	Akutsah2023	Alg. 3	AU2020	Akutsah2023	Alg. 3
1	29.6787	29.4194	29.5414	31.6844	31.3545	31.5117	24.7017	24.4017	24.8135
20	31.3909	30.8607	31.6437	33.7237	33.1011	34.0474	27.8139	26.3347	29.2731
50	31.9315	31.4457	32.2002	34.4268	33.7878	34.8360	30.0526	27.8802	31.8144
100	32.3417	31.8720	32.6608	35.0309	34.3238	35.5568	32.0809	29.5517	34.0522
200	32.8018	32.3033	33.2077	35.7577	34.9212	36.4231	34.3713	31.5191	36.5418
500	33.5486	32.9776	33.9982	36.9241	35.9316	37.5448	37.7407	34.5007	39.9385

6. Conclusion

We have introduced and analyzed an inertial Krasnosel'skiĭ–Mann and Ishikawa-type iterative scheme with step-size parameters for nonexpansive mappings, formulated as Algorithm 3.1. It was proved under mild control conditions that Algorithm 3.1 converges strongly to a fixed point of the underlying nonexpansive operator, namely the nearest point $\hat{x} = P_{F(T)}(0)$ to the origin (see Theorem 3.1). To further demonstrate the usefulness of the proposed method, Algorithm 4.1, derived as a direct application of Algorithm 3.1, was employed to solve a monotone inclusion problem involving three operators (4.1) (see Theorem 4.1). Moreover, we showed that the image restoration problem (5.3) can be effectively handled by Algorithm 4.2, which is obtained as a refinement of Algorithm 4.1 (see Corollary 4.1). Numerical experiments conducted on different blurred images confirmed the effectiveness of our approach. In particular, the proposed scheme produced higher signal-to-noise ratio (SNR) values when compared to the algorithms of Artsawang and Ungchittrakool Alg. 2020 and Akutsah et al. Alg. 2023. These results clearly validate that our method provides superior performance and constitutes a more efficient iterative tool for both theoretical and practical applications.

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