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Solvability of a Nonlocal Integral Problem of Integro-Mixed-Differential Equation Constrained by a Nonlinear Caputo-Fractional Order Constraint

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Abstract. In this work, we study the constrained problem of a nonlocal integral problem of a functional integro-differential (mixed integer and fractional) equation under a nonlinear Caputo fractional order constraint. The existence of solutions will be proved. The sufficient conditions for uniqueness will be given. Moreover, we analyze the continuous dependence of the unique solution on some parameters. Further, we investigate the Hyers-Ulam stability of the problem.

1. Introduction

Nonlocal integral problems arise in mathematics when the value of a function at a certain point depends not only on the values at nearby points (as in local problems), but also on its values over an entire region. These problems typically involve integral terms that link values of the unknown function over a range of space or time [9].

Non-local functional integro-differential(fractional and ineger orders) equations are mathematical models used to describe systems where the future state depends not only on the current state and its rate of change but also on past values or spatial interactions. These models combine differential operators (which describe local changes) and integral operators (which describe memory or distributed effects), and they are often used to capture nonlocal behavior across space or time [1, 15–17]. A considerable number of studies have been devoted to exploring such problems [11]- [13].

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In this study, our focus is on examining the constrained problem of the initial value problem of the functional integral-mixed differential equation.

$$\frac{dx}{dt} = f\Big(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, D^{\alpha} x(s) ds)\Big), \ a.e., \ t \in (0, T] \tag{1.1}$$

with the nonlocal integral condition

$$x(0) = x_0 + \int_0^{\tau} h\Big(s, D^{\gamma} x(s)\Big) ds, \qquad \tau \in [0, T] \tag{1.2}$$

subject to the Caputo fractional order nonlinear constraint

$$D^{\beta} u(t) = g_1(t, u(\phi(t))), \ a.e, \ t \in [0, T] \tag{1.3}$$

$$u(0) = u_0 \tag{1.4}$$

where D^{α}, D^{β} *and* D^{γ} are the Caputo fractional derivatives of order α, β *and* $\gamma \in (0,1]$.

Here, firstly, we use the measure of noncompactness and the Drabo fixed point Theorem [7] to prove the existence of a nondecreasing solution $u \in L_1[0, T]$ of the nonlinear constraint problem (1.3)-(1.4).

Secondly, we prove that for any solution $u \in L_1[0, T]$ of the constraint (1.3)-(1.4) there exists a unique solution $x \in AC[0, T]$ of the problem (1.1)-(1.2).

The Hyres-Ulam stability of the problem (1.1)-(1.4) itself will be studied. Also, the continuous dependence of the solution $x \in AC[0, T]$ of the problem (1.1)-(1.4) on the parameter x_0 and λ will be proved.

2. Preliminaries

Let $L_1 = L_1[0, T]$ be the class of Lebesgue integrable functions on $I = [0, T]$ with the standard norm.

$$||x||_1 = \max \int_0^T |x(t)| dt.$$

In this section, we present some definitions and results that will be used in our subsequent investigations.

Definition 2.1. *Let $f(t, x) = f : I \times R \to R$ satisfy Caratheodory conditions, i.e., $f(t, x)$ is measurable in t for any $x \in R$ and continuous in x for almost all $t \in I$. Then for every measurable function x on I we may assign the function,*

$$(Fx)(t) = f(t, x(t)), \ t \in I.$$

This operator F is called the superposition operator generated by the function f.

Theorem 2.1. *The superposition operator F maps L_1 into itself if and only if*

$$|f(t, x)| \leq |a(t)| + b|x|, \ \forall \ t \in I, \ x \in R$$

where $a \in L_1$ and b is a nonnegative constant.

Now let E be a Banach space with zero element 0 and X be a nonempty and bounded subset of E; moreover, denote by $B_r = B(0,r)$ the closed ball in E centered at 0 and with radius r.

Theorem 2.2. *Let X be a bounded subset of L_1. Assume that there is a family of subsets (Ω_c) $0 \leq c \leq b-a$ of the interval (a,b) such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \leq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$, then the set X is compact in measure.*

The measure of weak non-compactness defined by Deblasi is given by,

$$\beta(X) = inf\ (r > 0 : there\ exists\ a\ weakly\ copmact\ subset\ Y\ of\ E\ such\ that\ X \subset Y + K_r).$$

The convenient formula for the function $\beta(X)$ in L_1 was given by Appell and De Pascale

$$\beta(X) = \lim_{\epsilon \to 0}(\sup_{x \in X}(\sup[\int_D |x(t)|dt : D \subset [a,b], meas\ D \leq \epsilon])).$$

Theorem 2.3. *Let Q be a non-empty, bounded, closed and convex subset of a Banach space E. Assume that $T : Q \to Q$ is a continuous operator which is a contraction with respect to the De Blasi measure of noncompactness, i.e., there exists a constant $k \in [0,1)$ such that $\beta(TX) \leq k\,\beta(X)$ for any nonempty subset X of Q. Then T has at least one fixed point in the set Q.*

3. Solution of the constraint

Now, our problem will be considered under the following assumptions
(i) $\phi : [0,T] = I \to I$, is nondecreasing and there exists a real number $\rho > 0$ such that $\phi'(t) \geq \rho$.
(ii) $g_1 : I \times R \to R$ is measurable in $t \in I$, $\forall\, x \in R$, and continuous in $x \in R$, $\forall\, t \in I$ and there exists a bounded and measurable function a_1 and a constant $k_1 > 0$ such that

$$|g_1(t,x)| \ \leq |a_1(t)| \ + \ k_1|x(t)|.$$

Moreover g_1 is nondecreasing in the sense that $\forall\, t_1 \leq t_2$ and for all $u(t_1) \leq u(t_2)$, then

$$g_1(t_1, u(t_1)) \leq g_1(t_2, u(t_2)).$$

Remark 3.1. *From (i), we can deduce that $u \in L_1(I)$ implies*

$$\int_0^T |u(\phi(t))|dt = \frac{1}{\rho}\int_0^T |u(s)|ds < \infty.$$

We establish the existence of nondecreasing solutions for the problem (1.3)-(1.4) within the framework of integrable functions, provided specific conditions are constructed.

Theorem 3.1. *Let the assumptions $(i)-(ii)$ be satisfied. If $\frac{k_1\ T^\beta}{\rho\Gamma(1+\beta)} < 1$. Then the problem (1.3)-(1.4) has at least one nondecreasing solution $u \in L_1$.*

Proof. Operating both sides of (1.3) by I^β we get

$$I\frac{du}{dt} = I^\beta g_1(t, u(\phi(t))).$$

Hence

$$u(t) - u(0) = I^{\beta} g_1(t, u(\phi(t))),$$

then

$$u(t) = u_0 + I^{\beta} g_1(t, u(\phi(t))). \tag{3.1}$$

Let Q_{r_1} be the closed ball

$$Q_{r_1} = \{u \in L_1(I) : \|u\|_1 \leq r_1\} \subset L_1(I),\ r_1 = \frac{u_0 T + \|a_1\|_1 \frac{T^{\beta}}{\Gamma(1+\beta)}}{1 - \frac{k_1 T^{\beta}}{\rho \Gamma(1+\beta)}}$$

and define the operator F by

$$Fu(t) = u_0 + I^{\beta} g_1(t, u(\phi(t))).$$

Let $u \in Q_{r_1}$, then we have

$$\begin{aligned}
|Fu(t)| &= |u_0 + I^{\beta} g_1(t, u(\phi(t)))| \\
&\leq |u_0| + I^{\beta} |g_1(t, u(\phi(t)))| \\
&\leq |u_0| + I^{\beta} |a_1(t)| + k_1 I^{\beta} |u(\phi(t))| \\
\int_0^T |Fu(t)| dt &\leq \int_0^T u_0\, dt + \int_0^T I^{\beta} |a_1(t)|\, dt + k_1 \int_0^T I^{\beta} |u(\phi(t))| dt \\
&\leq |u_0|\, T + \int_0^T \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |a_1(s)| ds\, dt + k_1 \int_0^T \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(\phi(s))| ds\, dt \\
&\leq |u_0|\, T + \int_0^T \int_s^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |a_1(s)| dt\, ds + k_1 \int_0^T \int_s^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(\phi(s))| dt\, ds \\
&\leq |u_0|\, T + \int_0^T |a_1(s)| \frac{(t-s)^{\beta}}{\Gamma(\beta+1)} \Big|_s^T ds + k_1 \int_0^T |u(\phi(s))| \frac{(t-s)^{\beta}}{\Gamma(\beta+1)} \Big|_s^T ds \\
&\leq |u_0|\, T + \frac{T^{\beta}}{\Gamma(\beta+1)} \int_0^T |a_1(s)| ds + k_1 \frac{T^{\beta}}{\Gamma(\beta+1)} \int_0^T |u(\phi(s))| ds \\
&\leq |u_0|\, T + \frac{T^{\beta}}{\Gamma(\beta+1)} \Big(\|a_1\|_1 + \frac{k_1}{\rho} \|u\|_1 \Big)
\end{aligned}$$

which implies that

$$\|Fu\|_1 \leq |u_0|\, T + \frac{T^{\beta}}{\Gamma(1+\beta)} \Big(\|a_1\|_1 + \frac{k_1}{\rho} r_1 \Big) = r_1.$$

Clearly Q_{r_1} is nonempty, bounded, closed and convex .Since Q_{r_1} is a bounded subset of L_1 that comprises all functions positive and nondecreasing on I, then Theorem 2.2 shows that Q_{r_1} is compact in measure.

Now, let $\{Fu\} \in Q_r$ and $u_n \to u$, then

$$Fu_n = u_0 + I^{\beta} g_1(t, u_n(\phi(t)))$$

and

$$\lim_{n \to \infty} Fu_n = \lim_{n \to \infty} (u_0 + I^{\beta} g_1(t, u_n(\phi(t)))).$$

Applying Lebesgue dominated convergence Theorem [8], then from our assumptions we get

$$\lim_{n\to\infty} Fu_n = u_0 + I^\beta g_1(t, \lim_{n\to\infty} u_n(\phi(t)))$$
$$= u_0 + I^\beta g_1(t, u_n(\phi(t))) = Fu(t).$$

This means that $Fu_n(t) \to Fu(t)$. Hence the operator F is continuous on Q_{r_1}.

Finally we will prove that the operator F is contractive. Let $U \in Q_{r_1}$,$D = (t_1, t_2) \subset I$, with *meas* $D \leq \epsilon$, then for $u \in U$ and using our assumptions, we have

$$\begin{aligned}
&\int_D |Fu(t)|dt = \int_D |u_0 + I^\beta g_1(t, u(\phi(t)))|dt \\
&\leq \int_D |u_0|dt + \int_D I^\beta |g_1(t, u(\phi(t)))|dt \\
&\leq \int_D |u_0|dt + \int_D I^\beta \Big(|a_1(t)| + k_1|u(\phi(t)))|\Big)dt \\
&\leq \int_D |u_0|dt + \int_D I^\beta |a_1(t)|dt + k_1 \int_D I^\beta |u\phi(t))|dt \\
&\leq \int_D |u_0|ds + \int_D I^\beta |a_1(t)|dt + k_1 \int_{t_1}^{t_2} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(\phi(s))|ds\, dt \\
&\leq \int_D |u_0|ds + \int_D I^\beta |a_1(t)|dt + k_1 \int_{t_1}^{t_2} \int_0^{t_1} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(\phi(s))|ts\, ds \\
&+ k_1 \int_{t_1}^{t_2} \int_s^{t_2} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} |u(\phi(s))|dt\, ds \\
&\leq \int_D |u_0|ds + \int_D I^\beta |a_1(t)|dt + k_1 \int_{t_1}^{t_2} |u(\phi(T))| \frac{(t-s)^\beta}{\Gamma(1+\beta)}\Big|_0^{t_1} ds \\
&+ k_1 \int_{t_1}^{t_2} \frac{(t-s)^\beta}{\Gamma(1+\beta)}\Big|_s^{t_2} |u(\phi(s))|ds \\
&\leq \int_D |u_0|ds + \int_D I^\beta |a_1(t)|dt + k_1\, u(\phi(T)) \int_D \frac{(t_1-s)^\beta}{\Gamma(1+\beta)} ds \\
&+ k_1 \frac{t_2^\beta}{\Gamma(1+\beta)} \int_D |u(\phi(s))|ds \\
&\leq \int_D |u_0|ds + \int_D I^\beta |a_1(t)|dt + k_1\, u(\phi(T)) \int_D I^\beta ds \\
&+ \frac{k_1}{\rho} \frac{T^\beta}{\Gamma(1+\beta)} \int_D |u(s)|ds.
\end{aligned}$$

But

$$\lim_{\epsilon\to 0}\{sup\{\int_D |u_0|ds : D \subset I,\ meas.D \leq \epsilon\}\} = 0,$$
$$\lim_{\epsilon\to 0}\{sup\{\int_D I^\beta |a_1(s)|ds : D \subset I,\ meas.D \leq \epsilon\}\} = 0$$

and

$$\lim_{\epsilon\to 0}\{sup\{\int_D I^\beta ds : D\subset I,\ meas.D\leq\epsilon\}\}=0.$$

Then we obtain [7]

$$\beta(FU)\leq\frac{k_1\ T^\beta}{\rho\ \Gamma(1+\beta)}\beta(U).$$

Hence, all the conditions of Darbo fixed point Theorem [7] are satisfied and the integral equation (3.1) has at least one nondecreasing solution u.

Now

$$u(t)=u_0+I^\beta g_1(t,u(\phi(t)))$$

differentiate both sides, we can get

$$\frac{du}{dt}=\frac{d}{dt}I^\beta g_1(t,u(\phi(t)))\ ,\ a.e.$$

Operating both sides by $I^{1-\beta}$, we obtain

$$I^{1-\beta}\frac{du}{dt}=I^{1-\beta}\frac{d}{dt}I^\beta g_1(t,u(\phi(t))).$$

But

$$|I^\beta g_1(t,u(\phi(t)))|\leq I^\beta|a_1(s)|+k_1\ I^\beta|u(\phi(s))|.$$

From the properties of the fractional calculus [18] we deduce that

$$I^\beta g_1(t,u(\phi(t)))|_{t=0}=0.$$

and

$$\begin{aligned}I^{1-\beta}\frac{du}{dt}&=I^{1-\beta}\frac{d}{dt}I^\beta g_1(t,u(\phi(t)))=\frac{d}{dt}I^{1-\beta}I^\beta g_1(t,u(\phi(t)))\\&=\frac{d}{dt}Ig_1(t,u(\phi(t)))=g_1(t,u(\phi(t))),\end{aligned}$$

then

$$D^\beta u(t)=g_1(t,u(\phi(t))).$$

This proves that the initial value problem (1.3)-(1.4) is equivalent to the integral equation (3.1). Then there exists at least one nondecreasing solution u of the initial value problem (1.3)-(1.4).

4. Solution of the problem (1.1)-(1.4)

Now, consider the following assumptions

$(i)^*$ $f: I\times R\to R$ is measurable in $t\in I$, $\forall\ x\in R$, and continuous in $x\in R$, $\forall\ t\in I$ and satisfies Lipschitz condition

$$|f(t,x)-f(t,y)|\ \leq c|x-y|\ ,\ c>0$$

with $b(t)\ =\ f(t,0)$ is bounded and measurable on $[0,T]$. Then we can deduce that

$$|f(t,x)|\leq b(t)+c|x|.$$

$(ii)^*$ $g_2 : I \times R \to R$ is measurable in $t \in I$, $\forall\, x \in R$, and continuous in $x \in R$, $\forall\, t \in I$ and satisfies the Lipschitz condition such that

$$|g_2(t,x) - g_2(t,y)| \ \leq k_2|x-y| \ , \ k_2 > 0.$$

with $a_2(t) \ = \ g_2(t,0)$ is bounded and measurable on $[0,T]$. Then we can deduce that which we can deduce that

$$|g_2(t,x)| \leq a_2(t) + k_2|x|$$

$(iii)^*$ $g_3 : I \times R \to R$ is measurable in $t \in I$, $\forall\, x \in R$, and continuous in $x \in R$, $\forall\, t \in I$ and satisfies Lipschitz condition such that

$$|g_3(t,x) - g_3(t,y)| \ \leq k_3|x-y| \ , \ k_3 > 0$$

with $a_3(t) \ = \ g_3(t,0)$ is bounded and measurable on $[0,T]$. Then we can deduce that which we can deduce that,

$$|g_3(t,x)| \leq a_3(t) + k_3|x|.$$

$(iv)^*$ $h : I \times R \ \to R$ is measurable in $t \in I$, $\forall\, x,y \in R$, and continuous in $x,y \in R$, $\forall\, t \in I$ and satisfies Lipschitz condition such that

$$|h(t,x) - h(t,y)| \ \leq k_4|x-y| \ , \ k_4 > 0.$$

with $a_4(t) \ = \ h(t,0)$ is bounded and measurable on $[0,T]$. Then we can deduce that

$$|h(t,x)| \leq a_4(t) + k_4|x|.$$

$(vi)^*$ $k = \ max\{\, k_2, k_3, k_4\}$, $a \ = \ max\ \{\ \|a_2\|_1, \|a_3\|_1, \|a_4\|_1\ \}$, $b = \|b\|_1$ and
$\|a_i\|_1 \ = \ \int_0^T \ |a_i(t)|dt,\ i = 2,3,4.$
Now we have the following equivalent lemma.

Lemma 4.1. *Let $x \in AC(I)$ be a solution of the problem (1.1)-(1.2), then, for every solution $u \in L_1$ of the constrain (1.3)-(1.4), it can be represented by the solution of*

$$x(t) = x_0 + \int_0^\tau h(s, I^{1-\gamma}y(s))ds + \int_0^t y(s)ds \tag{4.1}$$

where y is the solution of the functional integral equation

$$y(t) = f\Bigg(t, \lambda g_2(t,u(t)) \int_0^t g_3(s, I^{1-\alpha}y(s)ds)\Bigg). \tag{4.2}$$

proof. Assume that $x \in AC(I)$ satisfies the problem (1.1)-(1.2). Let Let $\frac{dx}{dt} \ = \ y,$ then we obtain

$$x(t) = x(0) + \int_0^t y(s)ds$$

and from the properties of the fractional order derivative, we can get

$$D^\alpha x(t) = I^{1-\alpha}y(t),\ D^\beta x(t) = I^{1-\beta}y(t) \ \text{ and } D^\gamma x(t) = I^{1-\gamma}y(t).$$

Then the solution of the problem (1.1)-(1.2) will be given by (4.1)

$$x(t) = x_0 + \int_0^{\tau} h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds$$

where y is the solution of the functional integral equation (4.2)

$$y(t) = f\Big(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y(s)ds)\Big).$$

Conversely, let $x \in AC(I)$ be a solution of (4.1), then we have

$$\frac{dx}{dt} = f\Big(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y(s)ds)\Big).$$

Theorem 4.1. *Let the assumptions $(i)^* - (iii)^*$ and $(vi)^*$ be satisfied. If $|\lambda| ck \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\ (a + k\, r_1) < 1$, then for every solution $u \in L_1$ of (1.3)-(1.4) there exists a unique solution $x \in AC(I)$ of the problem (1.1)-(1.4).*

Proof. Let Q_{r_2} be the closed ball

$$Q_{r_2} = \{y \in L_1 : ||y||_1 \leq r_2\} \subset L_1,\ r_2 = \frac{b + a(a + k\, r_1)}{1 - |\lambda| ck \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\ (a + k\, r_1)}.$$

and define the operator F by

$$Fy(t) = f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y(s))ds)$$

Let u be a solution of (1.3)-(4.1), $y \in L_1$, then we have

$$\begin{aligned}
|Fy(t)| &= |f(t, \lambda\ g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y(s))ds))| \\
&\leq |b(t)| + c\, |\lambda|\, |g_2(t, u(t))| \int_0^t |g_3(s, I^{1-\alpha} y(s))| ds \\
&\leq |b(t)| + c\, |\lambda|\, |g_2(t, u(t))| \int_0^t (|a_3(s)| + k_3 I^{1-\alpha} |y(s)|) ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T |Fy(t)| dt &\leq \int_0^T |b(t)|\, dt + (||a_3||_1 + |\lambda|\, c\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} k_3 ||y||_1) \int_0^T |g_2(s, u(s))| ds \\
&\leq ||b||_1 + (||a_3||_1 + |\lambda|\, c\, k_3 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} ||y||_1) \int_0^T (|a_2(s)| + k_2 |u(s)|) ds \\
&\leq ||b||_1 + (||a_3||_1 + |\lambda|\, c\, k_3 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} ||y||_1)\ (||a_2||_1 + k_2 ||u||_1) \\
||Fy||_1 &\leq b + (a + |\lambda| ck\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\, r_2)\ (a + k\, r_1) = r_2.
\end{aligned}$$

This proves that $F : Q_{r_2} \to Q_{r_2}$ and the class $\{Fy\}$ is uniformly bounded on Q_{r_2} [3].
Let $y_1, y_2 \in Q_{r_2}$ then

$$\begin{aligned}
|Fy_2(t) - Fy_1(t)| &= |f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_2(s))ds) - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_1(s))ds)| \\
&\leq |\lambda|\, c\, |g_2(t, u(t))| \int_0^t |g_3(s, I^{1-\alpha} y_2(s)) - g_3(s, I^{1-\alpha} y_2(s))|ds \\
&\leq |\lambda|\, c\, k_3\, |g_2(t, u(t))| \int_0^t I^{1-\alpha}|y_2(s) - y_1(s)|ds \\
&\leq |\lambda|\, c\, k_3\, |g_2(t, u(t))|\, \|y_2 - y_1\|_1\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \\
\int_0^T |y_2(t) - y_1(t)|dt &\leq \lambda\, c\, k_3\, \|y_2 - y_1\|_1\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^T (|a_2(s)| + k_2|u(s)|)ds \\
&\leq \|y_2 - y_1\|_1\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} (|\lambda| c\, k_3\, \|a_2\|_1 + |\lambda| c\, k_2\, k_3 \|u\|_1) \\
&\leq \|y_2 - y_1\|_1\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} (|\lambda| c\, k\, \|a_2\|_1 + |\lambda| c\, k^2\, \|u\|_1)
\end{aligned}$$

then

$$\|Fy_2 - Fy_1\|_1 \leq \Big(|\lambda| c\, k\, a + |\lambda| c k^2\, \|u\|_1\Big)\, \|y_2 - y_1\|_1\, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}$$

$$\Big(1 - (|\lambda| c\, k\, a + |\lambda| c k^2\, r_1)\Big)\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|y_2 - y_1\|_1 \leq 0$$

and

$$\|y_2 - y_1\|_1 \leq 0.$$

This implies that F is a contraction operator. By Banach fixed point Theorem [8] there exists a unique solution $y \in L_1$ of the integral equation (4.2).
Consequently, there exists a unique solution $x \in AC(I)$ of the problem (1.1)-(1.4) given by equation (4.1).

5. Continuous dependence

Theorem 5.1. *Let the assumptions of Theorem 4.1 be satisfied. Then the unique solution $y \in L_1$ of (4.2) depends continuously on the parameter λ .*

Proof. Let $\delta > 0$ be given such that $|\lambda - \lambda^*| < \delta$, y is the solution of (4.2) and y^* is the unique solution of

$$y^*(t) \;=\; f(t, \lambda^* g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y^*(s))ds).$$

Then

$$\begin{aligned}
|y(t)-y^*(t)| &= |f(t,\lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds) - f(t,\lambda^* g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds)| \\
&\leq c|\lambda\ g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds - \lambda^* g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds| \\
&\leq c|\lambda\ g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds - \lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds \\
&+ |\lambda|\ g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds - \lambda^* g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds| \\
&\leq c\ |\lambda|\ |g_2(t,u(t))|\int_0^t |g_3(s,I^{1-\alpha}y(s)) - g_3(s,I^{1-\alpha}y^*(s))|ds \\
&+ k\ |\lambda-\lambda^*|\ |g_2(t,u(t))|\int_0^t |g_3(s,I^{1-\alpha}y^*(s))|ds \\
&\leq |\lambda| c\ k_3\ |g_2(t,u(t))|\int_0^t I^{1-\alpha}|y(s)-y^*(s)|ds + \delta_1 \\
&\leq |\lambda|\ c\ k_3|g_2(t,u(t))|\ \|y-y^*\|_1 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \delta_1
\end{aligned}$$

and

$$\int_0^T |y(t)-y^*(t)|dt \leq \Big(|\lambda| c\ k\ \|y-y^*\| \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \delta_1\Big)\int_0^T |g_2(s,u(s))|\ ds$$

$$\|y-y^*\|_1 \leq |\lambda| c\ k\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\ \|y-y^*\|_1\ (a+k\|u\|_1) + \delta_1$$

which implies that

$$\|y-y^*\|_1 \leq \frac{\delta_1}{1-|\lambda| c\ k\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\ (a+k\ r_1)} = \epsilon.$$

where $\delta_1 = k_1\ \delta\ |g_2(t,u(t))|\ \int_0^t |g_3(s,I^{1-\alpha}y^*(s))|ds.$

Theorem 5.2. *Let the assumptions of Theorem 4.1 be satisfied. Then the unique solution $y \in L_1$ of (4.2) depends continuously on the solution $u \in L_1$ of (1.3)-(1.4)*

Proof. Let $\delta > 0$ be given such that $\|u-u^*\|_1 < \delta$, y be the solution of (4.1) and y^* be the solution of

$$y^*(t) \ = \ f(t,\lambda g_2(t,u^*(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds).$$

Then

$$\begin{aligned}
|y(t)-y^*(t)| &= |f(t,\lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds) - f(t,\lambda g_2(t,u^*(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds)| \\
&\leq c\ |\lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds - \lambda g_2(t,u^*(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds| \\
&\leq c\ |\lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y(s))ds -\ c\ \lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds|
\end{aligned}$$

$$
\begin{aligned}
&+c\ |\lambda g_2(t,u(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds-\ c\ \lambda g_2(t,u^*(t))\int_0^t g_3(s,I^{1-\alpha}y^*(s))ds| \\
&\leq c\ |\lambda|\ |g_2(t,u(t))|\int_0^t |g_3(s,I^{1-\alpha}y(s))-g_3(s,I^{1-\alpha}y^*(s))|ds \\
&+c\ |\lambda|\ |g_2(t,u(t))-g_2(t,u^*(t))|\int_0^t |g_3(s,I^{1-\alpha}y^*(s))|ds \\
&\leq c\ |\lambda|\ k_3|g_2(t,u(t))|\int_0^t I^{1-\alpha}|y(s)-y^*(s)|ds \\
&+c\ |\lambda|\ k_2|u(t)-u^*(t)|\int_0^t (|a_3(s)|+k_3\ I^{1-\alpha}|y^*(s))|ds \\
&\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1|g_2(t,u(t))|+c\ |\lambda|\ k_2(\|a_3\|+k_3\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1)|u(t)-u^*(t)|
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T |y(t)-y^*(t)|dt &\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1|\int_0^T |g_2(s,u(s))|ds \\
&+c\ |\lambda|\ k_2(\|a_3\|+k_3\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1)\int_0^T |u(s)-u^*(s)|ds \\
&\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1\int_0^T (|a_2(s)|+k_2\ |u(s)|)ds \\
&+c\ |\lambda|\ k_2(\|a_3\|+k_3\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1)\int_0^T |u(s)-u^*(s)|ds \\
&\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1(\|a_2\|+k_2\|u\|_1) \\
&+c\ |\lambda|\ k_2(\|a_3\|+k_3\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1)\|u-u^*\|_1 \\
&\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1(\|a_2\|+k_2\|u\|_1) \\
&+c\ |\lambda|\ k_2(\|a_3\|+k_3\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y^*\|_1)\delta
\end{aligned}
$$

which implies that

$$
\|y-y^*\|_1\leq c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|y-y^*\|_1(a+k\ r_1)+\delta
$$

$$
\|y-y^*\|_1\leq \frac{\delta}{1-c\ |\lambda|\ \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}(a+k\ r_1)}=\epsilon.
$$

Theorem 5.3. *Let the assumptions of Theorem 4.1 be satisfied, the solution $x\in AC(I)$ of (4.1) depends continuously on the solution y and the initial condition x_0.*

Proof. *(1) Let* $\delta > 0$ *be given such that* $\|y - y^*\|_1 < \delta$, *x be the solution of (1.3) and* x^* *be the solution of*

$$x^*(t) = x_0 + \int_0^\tau h(s, I^{1-\gamma} y^*(s))ds + \int_0^t y^*(s)ds.$$

Then

$$\begin{aligned}
|x(t) - x^*(t)| &= |x_0 + \int_0^\tau h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds - x_0 - \int_0^\tau h(s, I^{1-\gamma} y^*(s))ds - \int_0^t y^*(s)ds| \\
&\leq \int_0^\tau |h(s, I^{1-\gamma} y(s)) - h(s, I^{1-\gamma} y^*(s))|ds + \int_0^t |y(s) - y^*(s)|ds \\
&\leq k_4 \int_0^\tau I^{1-\gamma} |y(s) - y^*(s)|ds + \int_0^t |y(s) - y^*(s)|ds \\
&\leq |\lambda|\, c\, k_3\, k_4 \Big(\|y - y^*\|_1 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \delta_1 \Big) \int_0^\tau I^{1-\gamma} |g_2(s, u(s))|ds + \int_0^t |y(s) - y^*(s)|ds \\
&\leq |\lambda|\, c\, k_3\, k_4 \Big(\|y - y^*\|_1 \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} + \delta_1 \Big) \int_0^\tau I^{1-\gamma} (|a_2(s)| + k_2 |u(s)|)ds + \|y - y^*\|_1 \\
&\leq |\lambda|\, c\, k_3\, k_4 \Big(\|y - y^*\|_1 \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} + \delta_1 \Big) \Big(a_2 + k_2 u(T) \Big) \int_0^\tau I^{1-\gamma}\, ds + \delta \\
&\leq |\lambda| c\, k^2 \Big(\delta \frac{T^{4-\alpha}}{\Gamma(3-\alpha)\Gamma(2-\gamma)} + \delta_1 \Big) \Big(a + k\, u(T) \Big) + \delta
\end{aligned}$$

and

$$\|x - x^*\|_c \leq |\lambda| c\, k^2 \Big(\delta \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \delta_1 \Big) \Big(a + k\, u(T) \Big) \frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)} + \delta$$

which implies that

$$\|x - x^*\|_c \leq \epsilon.$$

(2)Let $\delta > 0$ *be given such that* $|x_0 - x_0^*| \leq \delta$, *x be the solution of (4.1) and* x^* *be the solution of*

$$x^*(t) = x_0^* + \int_0^\tau h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds.$$

Then

$$\begin{aligned}
|x(t) - x^*(t)| &= |x_0^* + \int_0^\tau h(s, I^{1-\gamma} y(s))ds + \int_0^t y(s)ds - x_0^* - \int_0^\tau h(s, I^{1-\gamma} y(s))ds - \int_0^t y(s)ds| \\
&= |x_0 - x_0^*| \\
&\leq \delta
\end{aligned}$$

and

$$\|x - x^*\|_C \leq \delta.$$

Corollary 5.1. *From Theorems 5.1-5.3, the solution* $x \in AC(I)$ *of the equation (4.1), consequently the solution of the main problem (1.1)-(1.4), depends continuously on the parameter* λ *and the solution* $u \in L_1(I)$ *of the constraint (1.3)-(1.4).*

6. Hyers-Ulam Stability

Many authors have studied and further developed the definition of Hyers-Ulam stability across various types of problems, see [12]. We present the next definition of the Hyers-Ulam stability of the problems (1.1)-(1.4) as follows:

Definition 6.1. *[12]- [14] Let the solution $x \in AC(I)$ of the problem (1.1)- (1.4) be exists, then the problem (1.1)-(1.4) is Hyers-Ulam stable if $\forall\ \epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $\delta-$ approximate solution $x_s \in C(I)$ satisfies,*

$$|\frac{dx_s}{dt} - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, D^\alpha x_s(s) ds))| < \delta, \tag{6.1}$$

then $\|x - x_s\|_c < \epsilon$.

Remark 6.1. *From (6.1), we have*
Let $\frac{dx_s}{dt} = y_s(t)$, and from the properties of the fractional order derivative, we can get

$$D^\alpha x_s(t) = I^{1-\alpha} y_s(t)$$

$$|y_s(t) - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_s(\theta) d\theta))| < \delta$$

$$-\delta < y_s(t) - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_s(\theta)) d\theta) < \delta$$

and

$$|\frac{dx_s}{dt} - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_s(\theta)) d\theta)| < \delta$$

$$-\delta < \frac{dx_s}{dt} - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(s, I^{1-\alpha} y_s(\theta)) d\theta)) < \delta$$

$$-\delta\ t < x_s(t) - x_s(0) - \int_0^t f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta)) d\theta) dt < \delta\ t$$

$$-\delta\ t < x_s(t) - x_0 - \int_\tau^\tau h(\theta, I^{1-\gamma} y_s(\theta)) d\theta + \int_0^t f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta)) d\theta) dt < -\delta\ t$$

$$|x_s(t) - x_0 - \int_\tau^\tau h(\theta, I^{1-\gamma} y_s(\theta)) d\theta + \int_0^t f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta)) d\theta) dt| < \delta\ t$$

Theorem 6.1. *Let the assumptions of Theorem (4.1) be satisfied, then (1.1)-(1.2) is Hyers-Ulam stable.*

Proof:

Now

$$\begin{aligned}|y(t) - y_s(t)| &= |f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y(\theta)) d\theta) - y_s(t)| \\ &= |f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y(\theta)) d\theta - f(t, \lambda g_2(t, u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta)) d\theta)\end{aligned}$$

$$+ f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta))d\theta) - y_s(t)|$$
$$\leq |f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y(\theta))d\theta) - f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta))d\theta)|$$
$$+ |y_s(t) - f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta))d\theta)|$$
$$\leq \ |\lambda|\ c|g_2(t,u(t))| \int_0^t |g_3(\theta, I^{1-\alpha} y(\theta)) - g_3(\theta, I^{1-\alpha} y_s(\theta))|ds + \delta$$
$$\leq \ |\lambda|\ ck_3|g_2(t,u(t))| \int_0^t I^{1-\alpha}|y(\theta) - y_s(\theta)|d\theta + \delta$$
$$\leq \ |\lambda|\ c\ k_3(|a_2(t)| + k_2|u(t))|)\ \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|y - y_s\|_1 + \delta$$

and

$$\|y - y_s\|_1 \leq |\lambda|\ c\ k(a + k\ \|u\|_1)\ \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|y - y_s\|_1 + \delta, \tag{6.2}$$

which implies that

$$\|y - y_s\|_1 \leq \frac{\delta}{1 - |\lambda| ck \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\ (a + kr_1)} = \epsilon_1.$$

we have

$$|x(t) - x_s(t)| = |x_0 + \int_0^\tau h(\theta, I^{1-\gamma} y(\theta))d\theta + \int_0^t y(\theta)d\theta - x_s(t)|$$
$$= |x_0 + \int_0^\tau h(s, I^{1-\gamma} y(\theta))d\theta + \int_0^t y(\theta)d\theta - \int_0^\tau h(\theta, I^{1-\gamma} y_s(\theta))d\theta - \int_0^t y_s(\theta)d\theta$$
$$+ \int_0^\tau h(\theta, I^{1-\gamma} y_s(\theta))d\theta + \int_0^t y_s(\theta)d\theta - x_s(t)|$$
$$= |x_0 + \int_0^\tau h(\theta, I^{1-\gamma} y(\theta))d\theta - \int_0^\tau h(\theta, I^{1-\gamma} y_s(\theta))d\theta$$
$$+ \int_0^t y(\theta)d\theta - \int_0^t y_s(\theta)d\theta + \int_0^t y_s(\theta)d\theta + \int_0^\tau h(\theta, I^{1-\gamma} y_s(\theta))d\theta - x_s(t)|$$
$$\leq \int_0^\tau |h(\theta, I^{1-\gamma} y(\theta)) - h(\theta, I^{1-\gamma} y_s(\theta))|d\theta + \int_0^t |y(\theta) - y_s(\theta)|d\theta$$
$$+ |\int_0^t y_s(\theta)d\theta - \int_0^t f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt|$$
$$+ |x_0 + \int_0^\tau h(\theta, I^{1-\gamma} y_s(\theta))ds + \int_0^t f(t,\lambda g_2(t,u(t)) \int_0^t g_3(\theta, I^{1-\alpha} y_s(\theta))d\theta)dt - x_s(t)|$$
$$\leq k_4 \int_0^\tau I^{1-\gamma}|y(\theta) - y_s(\theta)|d\theta + \|y - y_s\|_1 + \delta\ T + \delta\ T$$
$$\leq |\lambda|\ c\ k_3\ k_4\ \frac{T^{2-\alpha}}{\Gamma(3-\alpha)}\|y - y_s\|_1 \int_0^\tau I^{1-\gamma}(|g_2(\theta, u(\theta))| + \delta)d\theta + \epsilon_1 + \delta\ T + \delta\ T$$

$$\leq |\lambda|\, c\, k_3\, k_4\, \frac{T^{2-\alpha}}{\Gamma(3-\alpha)} \|y - y_s\|_1 \Big(\int_0^{\tau} I^{1-\gamma} (|a_2(\theta)| + k_2|u(\theta)|)\, d\theta + \delta\,\tau \Big) + \epsilon_1 + \delta\, T + \delta\, T$$

$$\leq\; |\lambda|\, c\, k_3\, k_4\, \frac{T^{4-\alpha-\gamma}}{\Gamma(2-\alpha)\Gamma(3-\gamma)}\, \epsilon_1\, (a_+ k_2 u(T) + \delta\,\tau) + \epsilon_1 + \delta\, T + \delta\, T$$

and

$$\|x - x_s\|_C \leq ck^2\, \lambda\, \frac{T^{2-\alpha-\gamma}}{\Gamma(3-\alpha-\gamma)}\, \epsilon_1\, (a + ku(T) + \delta\,\tau) + \epsilon_1 + \delta\, T + \delta\, T = \epsilon,$$

then

$$\|x - x_s\|_C \leq \epsilon.$$

7. Conclusion

The fractional order derivatives expand the concept of classical derivatives to non-integer orders. In this paper, we have considered the constrained problem of a nonlocal integral problem of a functional integro- differential (mixed integer and fractional (1.1)-(1.2) subject to the Caputo fractional order nonlinear constraint (1.3)-(1.4). Furthermore, we have proved that for all solution $u \in L_1(I)$ of the constraint (1.3)-(1.4) there exists a unique solution x in the class $AC(I)$ of the initial value problem of the non-local functional integro-differential equation. Moreover, we analyzed the Hyers-Ulam stability and the continuous dependence of the solution on the initial condition x_0 and the parameter λ.

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