Int. J. Anal. Appl. (2025), 23:300

International Journal of Analysis and Applications

Clique Numbers in Direct Sum Matrix Commuting Graphs: Structural Analysis and Optimal Bounds

Ahmed Al-Shujary[1,2], Manal Al-Labadi[3,*], Shuker Khalil[4], Eman Almuhur[5], Wasim Audeh[6]

[1]*Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Iraq*

[2]*Department of Accounting, College of Administration and Economics, Al-Farahidi University, Iraq*

[3]*Dept. of Mathematics, Faculty of Arts and Sciences, University of Petra, Amman, Jordan*

[4]*Dept. of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq*

[5]*Dept. of Mathematics, Faculty of Arts and Sciences, Applied Science Private University (ASU), Amman, Jordan*

[6]*Department of Mathematics, University of Petra, Jordan*

[*]*Corresponding author:* manal.allabadi@uop.edu.jo

Abstract. We investigate the clique numbers and structural properties of commuting graphs associated with direct sum matrix rings over finite commutative rings. For a finite commutative ring L with unity, we study the commuting graph $\Gamma(M(m \oplus m, L))$ whose vertex set consists of all non-central matrices in $M(m \oplus m, L)$, where two distinct vertices are adjacent if and only if they commute. Our main contributions establish fundamental lower bounds for the clique number $\omega\Gamma(M(m \oplus m, L)))$ across various ring structures. We prove that for any finite commutative ring R with unity and positive integer $m \geq 3$, the clique number satisfies $\omega(\Gamma(M(m, R))) \geq |R|^{2m} - |R|^2$. For rings isomorphic to Z_{p^r} where $r \geq 3$ is odd, we establish the improved bound $\omega(\Gamma(M(m, R))) \geq \max\{(p^r)^{2m} - p^{2r}, (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}\}$. When $r \geq 2$ is even, the bound becomes $\omega(\Gamma(M(m, R))) \geq \max\{(p^r)^{2m} - p^{2r}, (p^r)^{m^2-1}p^{2r} - p^{2r}\}$. Our approach combines sophisticated matrix-theoretic techniques with graph-theoretic analysis to construct explicit maximal cliques and derive optimal bounds. The results provide new insights into the intersection of algebraic graph theory and matrix ring theory, with potential applications in coding theory and combinatorial optimization.

1. Introduction

The study of algebraic graph theory has emerged as a fundamental bridge between discrete mathematics and abstract algebra, providing profound insights into the structural properties of algebraic objects through graph-theoretic methods. Within this rich mathematical landscape,

Received: Aug. 23, 2025.

2020 *Mathematics Subject Classification.* 05C69, 05C38, 16S50.

Key words and phrases. clique number; commuting graph; matrix ring; direct sum; finite rings.

https://doi.org/10.28924/2291-8639-23-2025-300

ISSN: 2291-8639

commuting graphs associated with matrix rings represent a particularly fascinating area of investigation, offering deep connections between linear algebra, ring theory, and combinatorial graph theory.

The foundational work in this field traces back to Beck's pioneering investigation of coloring properties of commutative rings, which established the theoretical framework for associating graphs with algebraic structures [21]. This seminal contribution was subsequently formalized and extended by Anderson and Livingston, who developed the systematic study of zero-divisor graphs and established many of the fundamental principles that continue to guide research in algebraic graph theory [1-17].

Building upon these foundational contributions, the study of commuting graphs has gained significant momentum, particularly through the influential work of Akbari and colleagues. Their investigations into the diameters of commuting graphs [19] and the structural properties of commuting graphs in matrix algebras [18-20] have provided essential insights into the graph-theoretic behavior of non-commutative algebraic structures. The non-commuting graph perspective, as explored by Abdollahi, Akbari, and Maimani [18], has further enriched our understanding of these fundamental constructions.

For a non-commutative ring R, the commuting graph $\Gamma(R)$ is defined with vertex set $R \setminus Z(R)$, where $Z(R)$ denotes the center of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. This construction naturally extends to matrix rings, where the non-commutativity arises from the inherent matrix structure rather than properties of the base ring. The investigation of such graphs in the context of matrix rings has revealed rich structural phenomena that continue to challenge and inspire researchers.

The present research focuses specifically on direct sum matrix constructions over finite commutative rings, a setting that provides both mathematical elegance and computational tractability. For matrices $T \in M(r_1, L)$ and $N \in M(r_2, L)$ over a ring L, the direct sum $T \oplus N$ forms a block diagonal matrix structure that preserves many algebraic properties while introducing fascinating combinatorial features. The collection $M(r_1 \oplus r_2, L)$ of all such direct sum matrices serves as the foundation for our graph-theoretic investigation.

Recent advances in understanding the structural properties of matrix rings have been complemented by sophisticated techniques from both matrix theory and graph theory. The work of Friedland on simultaneous similarity of matrices [22] and Gerstenhaber's investigations into dominance and varieties of commuting matrices [23] have provided crucial theoretical foundations. Additionally, the computational approaches developed by Giudici and Pope for analyzing diameters of commuting graphs in linear groups and matrix rings [24] have demonstrated the practical feasibility of systematic analysis in these complex algebraic structures.

The central focus of this investigation concerns the clique number $\omega(G)$ of commuting graphs, defined as the maximum size of a complete subgraph. This parameter captures fundamental information about the extent to which elements can simultaneously commute, providing insights into the algebraic structure through combinatorial means. Our analysis establishes several key theoretical contributions:

(1) For any finite commutative ring R with unity and positive integer $m \geq 3$, we prove the fundamental lower bound:

$$\omega(\Gamma(M(m \oplus m, R))) \geq |R|^{2m} - |R|^2$$

(2) For rings isomorphic to $\mathbb{Z}_{p^r}$ where $r \geq 3$ is odd, we establish the improved bound:

$$\omega(\Gamma(M(m \oplus m, R))) \geq \max\left\{(p^r)^{2m} - p^{2r}, (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}\right\}$$

(3) When $r \geq 2$ is even, the corresponding bound becomes:

$$\omega(\Gamma(M(m \oplus m, R))) \geq \max\left\{(p^r)^{2m} - p^{2r}, (p^r)^{m^2-1}p^{2r} - p^{2r}\right\}$$

These results represent significant advances in our understanding of clique structures in matrix commuting graphs, providing both theoretical insights and computational tools for further investigation. The methodology combines sophisticated matrix-theoretic analysis with graph-theoretic techniques, demonstrating the power of interdisciplinary mathematical approaches.

The broader significance of this research extends beyond pure mathematical interest. The techniques and results developed here have potential applications in coding theory, where commuting properties of matrices play crucial roles in error-correction algorithms. Similarly, the combinatorial optimization aspects of our clique analysis may inform computational approaches to problems in algebraic complexity theory and symbolic computation.

Furthermore, the systematic construction of maximal cliques presented in this work, particularly through the analysis of diagonal matrix families and specialized block structures, provides a template for investigating similar problems in related algebraic contexts. The explicit characterization of clique sizes offers both theoretical understanding and practical computational advantages for researchers working with matrix rings over finite fields and more general finite commutative rings.

This research contributes to the ongoing development of algebraic graph theory as a mature mathematical discipline, bridging fundamental questions in abstract algebra with sophisticated techniques from combinatorial optimization. The results establish new benchmarks for clique analysis in matrix commuting graphs while opening avenues for future investigation into related graph parameters and more general algebraic structures.

2. Mathematical Framework and Definitions

We establish the foundational concepts necessary for our analysis of direct sum commuting graphs. Throughout this work, all rings are assumed to be finite and commutative with unity.

Definition 2.1. *For matrices $T \in M(r_1, L)$ and $N \in M(r_2, L)$ over a ring L, the direct sum $T \oplus N$ is defined as the block diagonal matrix of T and N, both are square matrices*

$$T \oplus N = \begin{pmatrix} T_{r_1 \times r_1} & O_{r_1 \times r_2} \\ O_{r_2 \times r_1} & N_{r_2 \times r_2} \end{pmatrix},$$

yields a matrix in $M((r_1 + r_2) \times (r_1 + r_2), L)$, for simplicity denoted by $M(r_1 \oplus r_2, L)$, with a specific block structure that preserves many algebraic properties while introducing interesting combinatorial features. The collection $M(r_1 \oplus r_2, L)$ of all such direct sum matrices forms the foundation for our graph-theoretic investigation and provides a rich setting for studying domination properties.

Definition 2.2. *The commuting graph $\Gamma(M(m \oplus m, R))$ has vertex set consisting of all non-central matrices in $M(m \oplus m, R)$:*

$$V(\Gamma(M(m \oplus m, R))) = M(m \oplus m, R) \setminus Z(M(m \oplus m, R))$$

where $Z(M(m \oplus m, R)) = \{cI_m \oplus fI_m : c, f \in R\}$. Two distinct vertices A and B are adjacent if and only if $AB = BA$.

3. Preliminaries and Basic Properties

Throughout this paper, R denotes a finite commutative ring with unity 1_R. We denote by I_m the $m \times m$ identity matrix and by E_{ij} the matrix with 1 in position (i, j) and 0 elsewhere.

Definition 3.1. *The commuting graph $\Gamma(M(m \oplus m, R))$ of $M(m \oplus m, R)$ is the simple graph with vertex set*

$$V(\Gamma(M(m \oplus m, R))) = M(m \oplus m, R) \setminus Z(M(m \oplus m, R)) = M(m \oplus m, R) \setminus \{cI_m \oplus fI_m : c, f \in R\}$$

and edge set $E(\Gamma(M(m \oplus m, R))) = \{\{A, B\} : A, B \in V(\Gamma(M(m \oplus m, R))), A \neq B, AB = BA\}$.

Definition 3.2. *[24] Let $G = (V, E)$ be a simple undirected graph. A subset $S \subseteq V$ is called a **clique** if every pair of distinct vertices in S is adjacent, i.e., for all $u, v \in S$ with $u \neq v$, we have $\{u, v\} \in E$.*

Equivalently, S is a clique if and only if the induced subgraph $G[S]$ is a complete graph.

Definition 3.3. *[24] A clique S in a graph $G = (V, E)$ is called **maximal** if there exists no clique S' in G such that $S \subsetneq S'$. In other words, S is maximal if it cannot be extended by adding any additional vertex while preserving the clique property.*

Formally, S is a maximal clique if:

(1) *S is a clique, and*
(2) *For every $v \in V \setminus S$, the set $S \cup \{v\}$ is not a clique.*

Definition 3.4. *[24] The **clique number** of a graph G, denoted $\omega(G)$, is the size of the largest clique in G:*

$$\omega(G) = \max\{|S| : S \text{ is a maximal clique in } G\}.$$

Lemma 3.1. *The center of $M(m \oplus m, R)$ is $Z(M(m \oplus m, R)) = \{cI_m \oplus fI_m : c, f \in R\}$, and $|Z(M(m \oplus m, R))| = |R|^2$.*

Proof. Since R is commutative, any scalar matrix $cI_m \oplus fI_m$ commutes with all matrices in $M(m \oplus m, R)$. Conversely, if $A \oplus B = (a_{ij})_{i,j=1}^m \oplus (b_{i,j})_{i,j=1}^m \in Z(M(m \oplus m, R))$, then $A \oplus B$ commutes with all elementary matrices $E_{kl} \oplus E_{kl}$. The condition $(A \oplus B)(E_{12} \oplus E_{12}) = (E_{12} \oplus E_{12})(A \oplus B)$ implies $a_{ij} = 0, b_{ij} = 0$ for $i \neq j$ and $a_{11} = a_{22}, b_{11} = b_{22}$. Similarly, $(A \oplus B)(E_{23} \oplus E_{23}) = (E_{23} \oplus E_{23})(A \oplus B)$ implies $a_{22} = a_{33}, b_{22} = b_{33}$. Continuing this process shows $A \oplus B = a_{11}I_m \oplus b_{11}I_m$ for some $a_{11}, b_{11} \in R$. □

Proposition 3.1. *The commuting graph $\Gamma(M(m \oplus m, R))$ has $|R|^{4m^2} - |R|^2$ vertices.*

Proof. This follows immediately from $|M(m \oplus m, R)| = |R|^{4m^2}$ and Lemma 3.1. □

4. Structure of Maximal Cliques

We now establish our principal results on the clique number of matrix commuting graphs.

Lemma 4.1. *Let $D(m \oplus m, R) = \{diag(d_1, d_2, \ldots, d_m) \oplus diag(x_1, x_2, \ldots, x_m) : d_i, x_i \in R\} \setminus Z(M(m \oplus m, R))$ be the set of non-central diagonal matrices. Then $D(m \oplus m, R)$ is a maximal cliques in $\Gamma(M(m \oplus m, R))$.*

Proof. Any two diagonal matrices commute since

$$\begin{aligned}
& (diag(a_1, \ldots, a_m) \oplus diag(x_1, x_2, \ldots, x_m))(diag(b_1, \ldots, b_m) \oplus diag(y_1, y_2, \ldots, y_m)) \\
= \; & (diag(a_1b_1, \ldots, a_mb_m) \oplus diag(x_1y_1, \ldots, x_my_m)) \\
= \; & (diag(b_1a_1, \ldots, b_ma_m) \oplus diag(y_1x_1, \ldots, y_mx_m)) \\
& \text{(since } a_i, b_i, x_i, y_i \in R, R \text{ is a commutative ring)} \\
= \; & (diag(b_1, \ldots, b_m) \oplus diag(y_1, \ldots, y_m))(diag(a_1, \ldots, a_m) \oplus diag(x_1, \ldots, x_m)).
\end{aligned}$$

Any matrix commute with all diagonal matrices must be a diagonal matrix, therefore $D(m \oplus m, R) = \{diag(d_1, d_2, \ldots, d_m) \oplus diag(x_1, x_2, \ldots, x_m) : d_i, x_i \in R\} \setminus Z(M(m \oplus m, R))$ is a maximal cliques in $\Gamma(M(m \oplus m, R))$. □

Theorem 4.1. *For any finite commutative ring R with unity and positive integer $m \geq 3$,*

$$\omega(\Gamma(M(m, R))) \geq |R|^{2m} - |R|^2. \tag{4.1}$$

Proof. By Lemma 4.1, the diagonal matrices form a maximal clique. The number of diagonal matrices is $|R|^{2m}$, and the number of scalar matrices (which are central) is $|R|$. Therefore, $|D(m \oplus m, R)| = |R|^{2m} - |R|^2$, giving the desired lower bound. □

Now, we give other examples of maximal cliques in $\Gamma(M(m \oplus m, Z_{p^r}))$ when $r \geq 3$ is odd. We start with the following remark.

Remark 4.1. *Any matrix* $A \oplus B = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$ *commutes with the matrix* $(A + c_1 I) \oplus (B + c_2 I) = \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m}+c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m}+c_2 \end{pmatrix}$.

Let $S_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$ and $O_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{t-1}{2}\}$. Let L be the set of all matrices $A \oplus B$ of the form

$$\begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2} & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix},$$

such that $x_{1,1}, \ldots, x_{m-1,m-1}, y_{1,1}, \ldots, y_{m-1,m-1} \in O_1$, $c_1, c_2 \in Z_{p^r}$, and $x_{i,j}, y_{i,j} \in S_1$, $i \neq j$.

Lemma 4.2. *Suppose that r is an odd number, L and S_1 are defined as above. Then L induces a maximal clique in $\Gamma(M(m \oplus m, Z_{p^r}))$.*

Proof. Let X and Y be any two matrices in the set L. Then

$$X = \begin{pmatrix} a_{1,1}+c_1 & a_{1,2} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{2,1} & a_{2,2}+c_1 & \cdots & a_{2,m-1} & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & \vdots & \cdots & a_{m-1,m-1}+c_1 & a_{m-1,m} \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} f_{1,1}+c_2 & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ f_{2,1} & f_{2,2}+c_2 & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1,1} & \vdots & \cdots & f_{m-1,m-1}+c_2 & f_{m-1,m} \\ f_{m,1} & f_{m,2} & \cdots & f_{m,m-1} & c_2 \end{pmatrix},$$

$a_{i,j}, f_{i,j} \in O_1$, $a_{i,i}, f_{i,i} \in S_1$, $i \neq j$, $c_1, c_2 \in Z_{p^r}$. One can write $X = (J + (c_1 I \oplus c_2 I) + G)$, where

$$G = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & 0 & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & f_{1,2} & \cdots & f_{1,m-1} & f_{1,m} \\ f_{2,1} & 0 & \cdots & f_{2,m-1} & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,m-1} & 0 \end{pmatrix}$$

is a matrix with all elements in S_1, and the matrix

$$J = \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix} \oplus \begin{pmatrix} f_{1,1} & 0 & \cdots & 0 \\ 0 & f_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists of elements that belong to O_1. Similarly

$$Y = \begin{pmatrix} b_{1,1}+c_3 & b_{1,2} & \cdots & b_{1,m-1} & b_{1,m} \\ b_{2,1} & b_{2,2}+c_3 & \cdots & b_{2,m-1} & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{m-1,1} & b_{m-1,2} & \cdots & b_{m-1,m-1}+c_2 & b_{m-1,m} \\ b_{m,1} & b_{m,2} & \cdots & b_{m,m-1} & c_3 \end{pmatrix} \oplus \begin{pmatrix} l_{1,1}+c_4 & l_{1,2} & \cdots & l_{1,m-1} & l_{1,m} \\ l_{2,1} & l_{2,2}+c_4 & \cdots & l_{2,m-1} & l_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & \cdots & l_{m-1,m-1}+c_4 & l_{m-1,m} \\ l_{m,1} & l_{m,2} & \cdots & l_{m,m-1} & c_4 \end{pmatrix}$$

and $Y = (J^* + (c_3 I \oplus c_4 I) + G^*)$, where

$$G^* = \begin{pmatrix} 0 & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & 0 & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & l_{1,2} & \cdots & l_{1,m} \\ l_{2,1} & 0 & \cdots & l_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ l_{m,1} & l_{m,2} & \cdots & 0 \end{pmatrix}$$

is a matrix with all elements in S_1

$$J^* = \begin{pmatrix} b_{1,1} & 0 & \cdots & 0 \\ 0 & b_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{m,m} \end{pmatrix} \oplus \begin{pmatrix} l_{1,1} & 0 & \cdots & 0 \\ 0 & l_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_{m,m} \end{pmatrix}$$

is a diagonal matrix where the main diagonal consists elements belonging to O_1. Then $XY = (J + (c_1 I \oplus c_2 I) + G)(J^* + (c_3 I \oplus c_4 I) + G^*) = JJ^* + c_2 J + 0 + c_1 J^* + c_1 c_2 I + c_1 G^* + 0 + c_2 G + 0 = J^* J + c_1 J^* + 0 + c_2 J + c_2 c_1 I + c_2 G + 0 + c_1 G^* + 0 = (J^* + c_2 I + G^*)(J + c_1 I + G) = YX$. Hence X and Y commute and the induced subgraph on L is complete.

Let $A \oplus B = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix}$; $x_{i,j}, y_{i,j} \in Z_{p^r}$ be any matrix that commutes with all elements in L. The matrix $A \oplus B$ commute with $p^{\frac{r-1}{2}} E_{1,1} \oplus p^{\frac{r-1}{2}} E_{1,1}$ and hence $(A p^{\frac{r-1}{2}} E_{1,1}) \oplus (B p^{\frac{r-1}{2}} E_{1,1}) = (p^{\frac{r-1}{2}} E_{1,1} A) \oplus (p^{\frac{r-1}{2}} E_{1,1} B)$. From that we get $x_{1,2} p^{\frac{r-1}{2}} = 0 = \cdots = x_{1,m} p^{\frac{r-1}{2}} = 0$, $x_{2,1} p^{\frac{r-1}{2}} = 0 = \cdots = x_{m,1} p^{\frac{r-1}{2}} = 0$ and $y_{1,2} p^{\frac{r-1}{2}} = 0 = \cdots = y_{1,m} p^{\frac{r-1}{2}} = 0$, $y_{2,1} p^{\frac{r-1}{2}} = 0 = \cdots = y_{m,1} p^{\frac{r-1}{2}} = 0$. Similarly $A \oplus B$ commutes with all the matrices $p^{\frac{r-1}{2}} E_{2,2} \oplus p^{\frac{r-1}{2}} E_{2,2} \cdots p^{\frac{r-1}{2}} E_{m,m} \oplus p^{\frac{r-1}{2}} E_{m,m}$. From that we get $x_{1,m} p^{\frac{r-1}{2}} = 0 = \cdots = x_{m-1,m} p^{\frac{r-1}{2}}$, $x_{m,1} p^{\frac{r-1}{2}} = 0 = \cdots = x_{m,m-1} p^{\frac{r-1}{2}}$. So, $x_{l,j} = (c_{l,j} + id_{l,j}) p^e$, $p^e \in \{0, p^{\frac{r+1}{2}}, \ldots, p^{r-1}\}$, $l \neq j$ $x_{1,m} p^{\frac{r-1}{2}} = 0 = \cdots = x_{m-1,m} p^{\frac{r-1}{2}}$, and $y_{m,1} p^{\frac{r-1}{2}} = 0 = \cdots = y_{m,m-1} p^{\frac{r-1}{2}}$. So, $y_{l,j} = (c_{l,j} + id_{l,j}) p^e$, $p^e \in \{0, p^{\frac{r+1}{2}}, \ldots, p^{r-1}\}$, $l \neq j$. So

$$A \oplus B = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix};$$

where $x_{1,1}, x_{1,2}, \ldots, x_{m,m}, y_{1,1}, y_{1,2}, \ldots, y_{m,m} \in Z_{p^r}$, $x_{i,j}, y_{i,j} \in S_1$ $\forall i \neq j$. Now, we need to show that $A \oplus B \in L$. Since $A \oplus B$ commutes with $p^{\frac{r+1}{2}} E_{1,2} \oplus p^{\frac{r+1}{2}} E_{1,2}, \ldots, p^{\frac{r+1}{2}} E_{m-1,m} \oplus p^{\frac{r+1}{2}} E_{m-1,m}$, we have $(Ap^{\frac{r+1}{2}} E_{1,2}) \oplus (Bp^{\frac{r+1}{2}} E_{1,2}) = (p^{\frac{r+1}{2}} E_{1,2} A) \oplus (p^{\frac{r+1}{2}} E_{1,2} B), \ldots, (Ap^{\frac{r+1}{2}} E_{m-1,m}) \oplus (Bp^{\frac{r+1}{2}} E_{m-1,m}) = (p^{\frac{r+1}{2}} E_{m-1,m} A) \oplus (p^{\frac{r+1}{2}} E_{m-1,m} B)$. So,

$$(Ap^{\frac{r+1}{2}} E_{1,2}) \oplus (Bp^{\frac{r+1}{2}} E_{1,2}) = \begin{pmatrix} 0 & x_{1,1}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & y_{1,1}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{1,2} A) \oplus (p^{\frac{r+1}{2}} E_{1,2} B) = \begin{pmatrix} 0 & x_{2,2}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & y_{2,2}p^{\frac{r+1}{2}} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

So, we get $x_{1,1}p^{\frac{r+1}{2}} = x_{2,2}p^{\frac{r+1}{2}}$ and hence $(x_{1,1} - x_{2,2})p^{\frac{r+1}{2}} = 0$, $y_{1,1}p^{\frac{r+1}{2}} = y_{2,2}p^{\frac{r+1}{2}}$ and hence $(y_{1,1} - y_{2,2})p^{\frac{r+1}{2}} = 0$. From that we get $y_{1,1} = y_{2,2} + v_1 p^{\frac{r-1}{2}}$; $v_1 \in Z_{p^r}$. Similarly since

$$(Ap^{\frac{r+1}{2}} E_{2,3}) \oplus (Bp^{\frac{r+1}{2}} E_{2,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{2,2}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{2,2}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{2,3} A) \oplus (p^{\frac{r+1}{2}} E_{2,3} B) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_{3,3}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & y_{3,3}p^{\frac{r+1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \vdots & \vdots \end{pmatrix},$$

we get $x_{2,2} = x_{3,3} + u_2 p^{\frac{r-1}{2}}$; $u_2 \in Z_{p^r}$. So $x_{1,1} = x_{2,2} + u_1 p^{\frac{r-1}{2}}$ then $x_{1,1} = x_{3,3} + u_1^{(1)} p^{\frac{r-1}{2}}$, and $y_{2,2} = y_{3,3} + v_2 p^{\frac{r-1}{2}}$; $v_2 \in Z_{p^r}$. So $y_{1,1} = y_{2,2} + v_1 p^{\frac{r-1}{2}}$ then $y_{1,1} = y_{3,3} + v_1^{(1)} p^{\frac{r-1}{2}}$.

By the same technique we get

$$(Ap^{\frac{r+1}{2}} E_{m-1,m}) \oplus (Bp^{\frac{r+1}{2}} E_{m-1,m}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} x_{m-1,m-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} y_{m-1,m-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= (p^{\frac{r+1}{2}} E_{m-1,m} A) \oplus (p^{\frac{r+1}{2}} E_{m-1,m} B) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} x_{m,m} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \cdots & p^{\frac{r+1}{2}} y_{m,m} \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

then $x_{m-1,m-1} = x_{m,m} + u_{m-1}p^{\frac{r-1}{2}}$; $u_{m-1} \in Z_{p^r}$. So $x_{1,1} = x_{m,m} + u^{(m)}_{m-1}p^{\frac{r-1}{2}}$; $u^{(m)}_{m-1} \in Z_{p^r}$ and $y_{m-1,m-1} = y_{m,m} + v_{m-1}p^{\frac{r-1}{2}}$; $v_{m-1} \in Z_{p^r}$. So $y_{1,1} = y_{m,m} + v^{(m)}_{m-1}p^{\frac{r-1}{2}}$; $v^{(m)}_{m-1} \in Z_{p^r}$.

Then

$$A \oplus B = \begin{pmatrix} d_{1,1}+x_{m,m} & x_{1,2} & \cdots & x_{1,m} \\ x_{2,1} & d_{2,2}+x_{m,m} & \cdots & x_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m} \end{pmatrix} \oplus \begin{pmatrix} f_{1,1}+y_{m,m} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & f_{2,2}+y_{m,m} & \cdots & y_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m} \end{pmatrix};$$

$d_{1,1}, d_{1,2}, \ldots, d_{m-1,m-1}, f_{1,1}, f_{1,2}, \ldots, f_{m-1,m-1} \in O_1$, $x_{m,m}, y_{m,m} \in Z_{p^r}$ and $x_{i,j}, y_{i,j} \in S_1$ $\forall i \neq j$. Then $A \oplus B$ is a matrix in L. So, the set L is a maximal clique of $\Gamma(M(m \oplus m, Z_{p^r}))$. □

Observe that $|D(m \oplus m, Z_{p^r})| = (p^r)^{2m} - p^{2r}$, since D contains diagonal matrices except scalar matrices. We have

$$L = \left\{ \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix} : x_{i,i}, y_{i,i} \in O_1, x_{i,j}, y_{i,j} \in S_1; i \neq j, c_1, c_2 \in Z_{p^r} \right\}.$$

Where $S_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r+1}{2}\}$. And $O_1 = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r-1}{2}\}$. Observe that $|S_1| = p^{r-1}$, $|O_1| = p^{r+1}$. Hence

$$|L| = (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}. \tag{4.2}$$

Now, we give an example of a maximal clique of the graph $\Gamma(M(m \oplus m, Z_{p^r}))$ where r is an even integer.

Let $S = \{x_j p^t, y_j p^t : x_j, y_j \in Z_{p^r} \text{ and } t \geq \frac{r}{2}\}$. Let L be the set of all matrices $A \oplus B$ of the form

$$A \oplus B = \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix},$$

such that $x_{i,j}, y_{i,j} \in S$, for all $i, j, c_1, c_2 \in Z_{p^r}$.

Lemma 4.3. *Let r be an even number, L and S are defined as above. Then L is a maximal clique in $\Gamma(M(m \oplus m, Z_{p^r}))$.*

Proof. The proof is similar to that one of Lemma 4.2. □

Now, we look at the size of the set L.

We have

$$L = \left\{ \begin{pmatrix} x_{1,1}+c_1 & x_{1,2} & \cdots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2}+c_1 & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2}+c_1 & \cdots & x_{m-1,m-1}+c_1 & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,m-1} & c_1 \end{pmatrix} \oplus \begin{pmatrix} y_{1,1}+c_2 & y_{1,2} & \cdots & y_{1,m-1} & y_{1,m} \\ y_{2,1} & y_{2,2}+c_2 & \cdots & y_{2,m-1} & y_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_{m-1,1} & y_{m-1,2}+c_2 & \cdots & y_{m-1,m-1}+c_2 & y_{m-1,m} \\ y_{m,1} & y_{m,2} & \cdots & y_{m,m-1} & c_2 \end{pmatrix} : x_{i,i}, y_{i,i} \in O_1, x_{i,j}, y_{i,j} \in S_1; i \neq j, c_1, c_2 \in Z_{p^r} \right\}.$$

Where $S = \{x_j p^t,\ y_j p^t : x_j, y_j \in \mathbb{Z}_{p^r}$ and $t \geq \frac{r}{2}\}$. Observe that $|S| = p^r$. Hence

$$|L| = (p^r)^{m^2-1} p^{2r} - p^{2r} \tag{4.3}$$

.

Corollary 4.1. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 3$ is an odd number,*

$$\omega(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^{r-1})^{m^2-m}(p^{r+1})^{m-1}p^{2r} - p^{2r}\}. \tag{4.4}$$

Proof. Using inequality (4.1) and equation (4.2), then we conclude the result. □

Corollary 4.2. *For any finite commutative ring R isomorphic to Z_{p^r} with $r \geq 2$ is an even number,*

$$\omega(\Gamma(M(m,R))) \geq max\{(p^r)^{2m} - p^{2r}, (p^r)^{m^2-1}p^{2r} - p^{2r}\}. \tag{4.5}$$

Proof. Using inequality (4.1) and equation (4.3), then we conclude the result. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Al-Labadi, E.M. Almuhur, Planar of Special Idealization Rings, WSEAS Trans. Math. 19 (2020), 606–609. https://doi.org/10.37394/23206.2020.19.66.

[2] M. Al-Labadi, Some Properties of the Zero-Divisor Graphs of Idealization Ring $R(+)M$, Math. Probl. Eng. 2022 (2022), 6888963. https://doi.org/10.1155/2022/6888963.

[3] M. Al-Labadi, S.M. Alsalem, E. Suleiman, New Structure of D-Algebras Using Permutations, in: International Conference on Mathematical and Statistical Physics, Computational Science, Education and Communication (ICMSCE 2023), SPIE, 2023, pp. 13. https://doi.org/10.1117/12.3011428.

[4] M. Al-Labadi, S.M. Alsalem, E. Suleiman, N. Yerima, On 1-Commutative Permutation BP-Algebras, in: International Conference on Mathematical and Statistical Physics, Computational Science, Education and Communication (ICMSCE 2023), SPIE, 2023, pp. 10. https://doi.org/10.1117/12.3011417.

[5] M. Al-Labadi, S. Khalil, The Idealization Ring, AIP Conf. Proc. 3097 (2024), 080008. https://doi.org/10.1063/5.0209920.

[6] M. Al-Labadi, E.M. Almuhur, W. Audeh, A. Al-Boustanji, Geodetic Number and Domination Number of $\Gamma(R(+)M)$, Bol. Soc. Parana. Mat. 43 (2025), 1–8.

[7] M. Al-Labadi, R. Al-Naimi, W. Audeh, Singular Value Inequalities for Generalized Anticommutators, J. Inequal. Appl. 2025 (2025), 15. https://doi.org/10.1186/s13660-025-03264-z.

[8] W. Audeh, A. Al-Boustanji, M. Al-Labadi, R. Al-Naimi, Singular Value Inequalities of Matrices via Increasing Functions, J. Inequal. Appl. 2024 (2024), 114. https://doi.org/10.1186/s13660-024-03193-3.

[9] W. Audeh, M. Al-Labadi, R. Al-Naimi, Numerical Radius Inequalities via Block Matrices, Acta Sci. Math. (Szeged) (2024). https://doi.org/10.1007/s44146-024-00164-4.

[10] W. Audeh, M. Al-Labadi, Numerical Radius Inequalities for Finite Sums of Operators, Complex Anal. Oper. Theory 17 (2023), 128. https://doi.org/10.1007/s11785-023-01437-6.

[11] W. Audeh, M. Al-Labadi, Some Results About Numerical Radius Inequalities, Int. J. Math. Comput. Sci. 17 (2022), 33–39.

[12] M. Al-Labadi, S. Saranya, Yasmeen, M.I. Idrisi, On Difference Fuzzy Anti λ-Ideal Convergent Double Sequence Spaces, Bull. Math. Anal. Appl. 15 (2023), 34–42.

[13] M. Al-Labadi, S.M. Khalil, E. Almohur, Some Properties of the Circulant Graphs, in: International Conference on Mathematical and Statistical Physics, Computational Science, Education and Communication (ICMSCE 2023), SPIE, 2023, pp. 12. https://doi.org/10.1117/12.3011425.

[14] M. Al-Labadi, S.M. Alsalem, E. Suleiman, N. Yerima, On 1-Commutative Permutation Bp-Algebras, in: International Conference on Mathematical and Statistical Physics, Computational Science, Education and Communication (ICMSCE 2023), SPIE, 2023, pp. 10. https://doi.org/10.1117/12.3011417.

[15] E. Almuhur, H. Miqdad, M.A. Al-labadi, M.I. Idrisi, μ-L-Closed Subsets of Noetherian Generalized Topological Spaces, Int. J. Neutrosophic Sci. 23 (2024), 148–153. https://doi.org/10.54216/ijns.230313.

[16] M. Al-Labadi, S. Khalil, Properties of Idealization Graph Over a Ring, in: Algorithms for Intelligent Systems, Springer Nature Singapore, Singapore, 2024, pp. 365-373. https://doi.org/10.1007/978-981-99-8976-8_31.

[17] M. Al-Labadi, S. Khalil, V.R. Radhika, K. Mohana, Pentapartitioned Neutrosophic Vague Soft Sets and Its Applications, Int. J. Neutrosophic Sci. 25 (2025), 64–83. https://doi.org/10.54216/ijns.250207.

[18] A. Abdollahi, S. Akbari, H. Maimani, Non-Commuting Graph of a Group, J. Algebr. 298 (2006), 468–492. https://doi.org/10.1016/j.jalgebra.2006.02.015.

[19] S. Akbari, A. Mohammadian, H. Radjavi, P. Raja, On the Diameters of Commuting Graphs, Linear Algebr. Appl. 418 (2006), 161–176. https://doi.org/10.1016/j.laa.2006.01.029.

[20] S. Akbari, H. Bidkhori, A. Mohammadian, Commuting Graphs of Matrix Algebras, Commun. Algebr. 36 (2008), 4020–4031. https://doi.org/10.1080/00927870802174538.

[21] I. Beck, Coloring of Commutative Rings, J. Algebr. 116 (1988), 208–226. https://doi.org/10.1016/0021-8693(88)90202-5.

[22] S. Friedland, Simultaneous Similarity of Matrices, Adv. Math. 50 (1983), 189–265. https://doi.org/10.1016/0001-8708(83)90044-0.

[23] M. Gerstenhaber, On Dominance and Varieties of Commuting Matrices, Ann. Math. 73 (1961), 324–348. https://doi.org/10.2307/1970336.

[24] M. Giudici, A. Pope, the Diameters of Commuting Graphs of Linear Groups and Matrix Rings Over the Integers Modulo m, Australasian J. Comb. 48 (2010), 221–230.

[25] S. Khalil, M. Al-Labadi, E. Suleiman, N. Yerima, New Structure of Algebras Using Permutations in Symmetric Groups, J. Discret. Math. Sci. Cryptogr. 27 (2024), 1611–1618. https://doi.org/10.47974/jdmsc-2003.

[26] P. Zhang, G. Chartrand, Introduction to Graph Theory, Tata McGraw-Hill Education, 2006.