Int. J. Anal. Appl. (2026), 24:29

International Journal of Analysis and Applications

Siboid Topological Spaces

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Abstract. In this article we introduce a new notion in topology that we call 'siboid', which yields to siboid topological spaces. We prove several fundamental results regarding siboid topological spaces. We define two new operators $(.)^*$ and Φ and a new map cl^θ that satisfies the Kuratowski closure axioms whenever the siboid satisfies the property $\mathcal{R}$. Moreover, various types of generalized open sets and generalized continuous functions are defined and studied, and 'siboid' is then connected with Boolean algebra.

1. Introduction

Topology is one of the important branches of mathematics. It has a wide range of generalized applications across mathematics, physics, computer science, biology, data science, geography, economics, etc. Throughout the years mathematicians have introduced and extensively studied a number of novel structures, viz., filter [1], ideal [2], grill [3], primal [4], etc. in topology and in its allied areas. These classical structures are some of the most significant topological notions which help in extending classical topology into more abstract and flexible frameworks. The process of inducing topologies is a very interesting field of application of these novel structures. It is seen that most of the topologies induced from filter, ideal, grill and primal are finer than the given topology [5–8].

The concept of a filter in topology was initially introduced by Cartan [1] in 1937. Filters provide a more general framework for defining convergence in topological spaces. The ideal, which serves as the dual notion of a filter, was first proposed by Kuratowski [2]. Topologists have explored this structure through various approaches [7, 9, 10]. The grill is another classical construct in the

Received: Aug. 20, 2025.

2020 *Mathematics Subject Classification.* 54H55.

Key words and phrases. topology; siboid; siboid topological space.

https://doi.org/10.28924/2291-8639-24-2026-29
ISSN: 2291-8639

field of topology. It was introduced by Chóquet in 1947 [3], and subsequently, numerous authors have conducted extensive studies on different aspects related to grills. Certain structures that combine grills and filters have also been investigated [11]. In 2022, Acharjee et al. [4] introduced the concept of a primal, which can be regarded as the dual structure of a grill, and explored many of its fundamental properties. They established a connection between primal topological spaces and standard topological spaces by introducing the $(.)^{\diamond}$ operator. Another widely studied topic in mathematics, computer science, and pattern recognition is proximity space. Al-Omari et al. [12] introduced a new structure called the primal-proximity space by combining these two concepts. Other topological notions, such as generalized primal topological spaces [13], as well as regularity and normality in primal spaces [14], have also been explored using the primal structure. On ideal topological spaces, Hamlett [15] defined the set operator Ψ and investigated its various properties. Additional types of operators related to these structures can be found in [5, 16–19]. Moreover, several new topologies generated through these set operators have been extensively studied by topologists.

To enhance the understanding of topological structures and their interrelationships within the framework of topological systems, it is essential to examine various classes of functions and their properties. One such fundamental concept is that of a continuous function. The behavior and properties of continuous functions in generalized primal topological spaces and grill-filter spaces have been explored in [20, 21]. Furthermore, in recent years, several researchers have investigated numerous generalizations of open sets. Levine [22] first introduced the concepts of semi-open sets and semi-continuity in topological spaces. Later, in 1965, Njåstad [23] proposed various classes of nearly open sets, particularly studying α-open sets and their applications. Mashhour et al. [24] introduced and examined the notions of pre-open sets and pre-continuous functions in 1982, while Abd El-Monsef et al. [25] defined β-open sets and β-continuous mappings in 1983. Similar extensions of open sets and continuous mappings have also been developed in grill topological spaces [26] and primal topological spaces [27].

In this paper, inspired by the existing topological structures, we introduce a new structure named 'siboid'[1] on a finite set. This new structure differs from all the previous traditional structures mentioned above due to the third condition imposed on it. As this structure is new, we study some fundamental properties and further associate the structure with a topological space and name it as 'siboid topological space'. In order to study whether this structure can induce new topologies from the original topology, we define set operators $(.)^*$ and Φ on a siboid topological space. A new map cl^{θ} which satisfies the Kuratowski closure axioms whenever a siboid satisfies the property $\mathcal{R}$ is defined. It can be observed that the topologies resulted from cl^{θ} and Φ are equivalent as well as finer than the original topology. Moreover, we study some generalized open sets in siboid

[1]The name is inspired from Shiva, a major Hindu deity, which is associated with creation, preservation and transformation, and is a part of the Hindu trinity, along with Brahma "the creator" and Vishnu "the preserver".

topological space. Section 4 presents a study of certain generalized continuous maps and their properties, while Section 5 explores the connection between siboid topology and Boolean algebra.

2. Preliminaries

In this section, we mention some preliminary definitions to be used in the next sections.

Definition 2.1. *Let (X, τ) be a topological space and $A \subseteq X$. Then,*

(1) *[28] A is α-open if $A \subseteq int\Big(cl\Big(int(A)\Big)\Big)$,*
(2) *[22] A is semi-open if $A \subseteq cl\Big(int(A)\Big)$,*
(3) *[24] A is pre-open if $A \subseteq int\Big(cl(A)\Big)$,*
(4) *[25]A is β-open if $A \subseteq cl\Big(int\Big(cl(A)\Big)\Big)$.*

Definition 2.2. *Let (X, τ) and (X', τ') be two topological spaces. The function $f\colon (X, \tau) \to (X', \tau')$ is said to be*

(1) *[23] α-continuous if the inverse image of each open set of (X', τ') is α-open in (X, τ),*
(2) *[22] semi-continuous if the inverse image of each open set of (X', τ') is semi-open in (X, τ),*
(3) *[24] pre-continuous if the inverse image of each open set of (X', τ') is pre-open in (X, τ),*
(4) *[25] β-continuous if the inverse image of each open set of (X', τ') is β-open in (X, τ).*

3. Main results

In this section, we introduce a new notion named 'siboid'. Later, we study various fundamental results of siboid topological spaces.

Definition 3.1. *Let X be a non-empty finite set, and $\mathcal{S} \subseteq 2^X$. Then, $\mathcal{S}$ is said to be a siboid on X if the following conditions are satisfied:*

(1) *$X \in \mathcal{S}$,*
(2) *if $A \subset B$ and $A \in \mathcal{S}$, then $B \in \mathcal{S}$,*
(3) *if $A, B \in \mathcal{S}$ and $A \subset B \subset X$, then $A^C \setminus \{x\} \in \mathcal{S}$ for every $x \in (B \setminus A)$.*

The ordered pair $(X, \mathcal{S})$ is said be a siboid space.

Example 3.1. (1) *Let X be a non-empty finite set and $\mathcal{S}_1 = 2^X \setminus \{\emptyset\}$. Then, $(X, \mathcal{S}_1)$ is a siboid space.*

(2) *Let $X = \{a, b, c\}$ and $\mathcal{S}_2 = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $(X, \mathcal{S}_2\ )$ is a siboid space.*

Theorem 3.1. *If $|X| = n$, where $n \geq 3$, then the number of siboids that can be formed on X is $3 + (2^n - 1)$.*

Proof. Let X be a non-empty finite set with $|X| = n$, where $n \geq 3$. Clearly, the collections X, 2^X, and $2^X \setminus \{\emptyset\}$ are siboids. Other possible siboids are those that include X along with subsets of X containing $(n-1)$ elements. Any siboid that contains a subset with fewer than $(n-1)$ elements of X must necessarily be either 2^X or $2^X \setminus \{\emptyset\}$, due to the defining conditions of a siboid.

Now, the number of subsets of X containing $(n-1)$ elements is $\binom{n}{n-1} = n$. Therefore, the number of siboids formed by taking X together with one subset at a time from these n subsets is $\binom{n}{1}$. Similarly, the number of siboids formed by taking X together with two subsets at a time from these n subsets is $\binom{n}{2}$. Continuing in this manner, the number of siboids formed by taking X together with all n subsets is $\binom{n}{n}$. Hence, the total number of siboids constructed in this way is $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$.

Consequently, the total number of siboids that can be formed on a non-empty finite set X with $|X| = n$, where $n \geq 3$, is $3 + (2^n - 1)$. □

Example 3.2. *Let $X = \{a, b, c\}$. Then, the number of siboids that can be formed on X is $3 + (2^3 - 1) = 10$. These are $\mathcal{S}_1 = \{X\}, \mathcal{S}_2 = 2^X, \mathcal{S}_3 = 2^X \setminus \{\emptyset\}, \mathcal{S}_4 = \{\{a, b\}, X\}, \mathcal{S}_5 = \{\{b, c\}, X\}, \mathcal{S}_6 = \{\{a, c\}, X\}, \mathcal{S}_7 = \{\{a, b\}, \{b, c\}, X\}, \mathcal{S}_8 = \{\{a, b\}, \{a, c\}, X\}, \mathcal{S}_9 = \{\{b, c\}, \{a, c\}, X\}$, and $\mathcal{S}_{10} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$.*

Theorem 3.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two siboids on X. Then, $\mathcal{S}_1 \cup \mathcal{S}_2$ is also a siboid on X.*

Proof. $X \in \mathcal{S}_1 \cup \mathcal{S}_2$, since $X \in \mathcal{S}_1$ and $X \in \mathcal{S}_2$. Again, let $A \subset B$ and A, B are subsets of X such that $A \in \mathcal{S}_1 \cup \mathcal{S}_2$. Then, either $A \in \mathcal{S}_1$ or $A \in \mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are siboids on X, thus either $B \in \mathcal{S}_1$ or $B \in \mathcal{S}_2$. Thus, $B \in \mathcal{S}_1 \cup \mathcal{S}_2$.

Now, we consider $A, B \in \mathcal{S}_1 \cup \mathcal{S}_2$, where $A \subset B \subset X$. Then, any one of the following four cases may occur:

(a) $A, B \in \mathcal{S}_1$, or (b) $A, B \in \mathcal{S}_2$, or (c) $A \in \mathcal{S}_1, B \in \mathcal{S}_2$, or (d) $B \in \mathcal{S}_1, A \in \mathcal{S}_2$.

For the above case (a), if $A, B \in \mathcal{S}_1$, then for every $x \in B \setminus A$, we get $A^C \setminus \{x\} \in \mathcal{S}_1 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$. Similarly, we can get for the above case (b).

For the above case (c), if $A \in \mathcal{S}_1, B \in \mathcal{S}_2$, then obviously $B \in \mathcal{S}_1$ due to condition (2) of Definition 3.1. Thus, $B \in \mathcal{S}_1 \cup \mathcal{S}_2$. Also, $A \in \mathcal{S}_1$ implies $A \in \mathcal{S}_1 \cup \mathcal{S}_2$. Then, for every $x \in B \setminus A$, we have $A^C \setminus \{x\} \in \mathcal{S}_1 \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$. Similarly, we can prove for the case (d). Hence, the theorem is proved. □

Theorem 3.3. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two siboids on X. Then, $\mathcal{M} = \{P \cup Q | P \in \mathcal{S}_1, Q \in \mathcal{S}_2\}$ is a siboid on X.*

Proof. $X = X \cup X \in \mathcal{M}$, since $X \in \mathcal{S}_1$ and $X \in \mathcal{S}_2$.

Again, let $A \subset B$ and $A \in \mathcal{M}$. Then, $A = P \cup Q$ for some $P \in \mathcal{S}_1$ and $Q \in \mathcal{S}_2$.

Now, $A \subset B \Rightarrow P \cup Q \subset B \Rightarrow P \subset B$ and $P \in \mathcal{S}_1$. Thus, $B \in \mathcal{S}_1$. Similarly, we get $B \in \mathcal{S}_2$. Consequently, $B \in \mathcal{M}$.

Now, we consider $A, B \in \mathcal{M}$ and $A \subset B \subset X$. Then, $A = P_1 \cup Q_1$ and $B = P_2 \cup Q_2$ for some $P_1, P_2 \in \mathcal{S}_1$ and $Q_1, Q_2 \in \mathcal{S}_2$.

Now, since $P_1 \subset P_1 \cup Q_1$ and $P_1 \in \mathcal{S}_1$, from condition (2) of Definition 3.1, we get, $P_1 \cup Q_1 \in \mathcal{S}_1$ or $A \in \mathcal{S}_1$. Applying the same condition, $A \subset B$ and $A \in \mathcal{S}_1$ give $B \in \mathcal{S}_1$. Thus, $A, B \in \mathcal{S}_1$ and $A \subset B \subset X$ implies $A^c \setminus \{x\} \in \mathcal{S}_1$ for every $x \in (B \setminus A)$. In a similar manner, we can obtain that $A^c \setminus \{x\} \in \mathcal{S}_2$ for every $x \in (B \setminus A)$. Consequently, $A^c \setminus \{x\} \in \mathcal{M}$ for every $x \in (B \setminus A)$. Thus, $\mathcal{M}$ is a siboid on X. □

Theorem 3.4. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two siboids on X. Then, $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a siboid on X.*

Proof. $X \in \mathcal{S}_1 \cap \mathcal{S}_2$, since $X \in \mathcal{S}_1$ and $X \in \mathcal{S}_2$. Again, let $A \subset B$ and $A \in \mathcal{S}_1 \cap \mathcal{S}_2$. Then, $A \in \mathcal{S}_1$ and $A \in \mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are siboids on X, then $B \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$. So, $B \in \mathcal{S}_1 \cap \mathcal{S}_2$.

Now, we consider $A, B \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $A \subset B \subset X$. Thus, $A, B \in \mathcal{S}_1$ and $A, B \in \mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are siboids and $A \subset B \subset X$, we get $A^c \setminus \{x\} \in \mathcal{S}_1$ and $A^c \setminus \{x\} \in \mathcal{S}_2$ for $x \in (B \setminus A)$. So, $A^c \setminus \{x\} \in \mathcal{S}_1 \cap \mathcal{S}_2$ for $x \in B \setminus A$. Hence, the theorem is proved. □

Remark 3.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two siboids on X. Then, $\mathcal{R}$=$\{P \cap Q | P \in \mathcal{S}_1, Q \in \mathcal{S}_2\}$ is not necessary to be a siboid on X.*

We have the following example to verify the aforementioned Remark 3.1:

Example 3.3. *Let $X = \{a, b, c\}$. We consider $\mathcal{S}_1$= $\{\{a, b\}, X\}$ and $\mathcal{S}_2$= $\{\{b, c\}, X\}$ be two siboids on X. Then, $\mathcal{R}$= $\{\{b\}, \{b, c\}, \{a, b\}, X\}$ is not a siboid on X. If we consider $A = \{b\}$ and $B = \{b, c\}$, then $A \subset B$. Now, $A^c \setminus \{c\} = \{a\} \notin \mathcal{R}$.*

Definition 3.2. *Let (X, τ) be a topological space and $\mathcal{S}$ be a siboid on X. Then, the tuple $(X, \tau, \mathcal{S})$ is called a siboid topological space.*

Example 3.4. *Let $X = \{a, b, c\}$, $\tau = 2^X$, and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $(X, \tau, \mathcal{S})$ is a siboid topological space.*

Now, we define a new kind of topological operator on a siboid topological space $(X, \tau, \mathcal{S})$. The new structure is given below.

Definition 3.3. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. We consider an operator $(.)^*$: $2^X \to 2^X$ defined as $A^* = \{x \in X | A^c \cup U^c \notin \mathcal{S}$ for all $U \in \tau$ and $x \in U\}$, where $A \subseteq X$.*

Example 3.5. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If we consider $A = \{a, b\}$, then $A^* = \{b, c\}$.*

Theorem 3.5. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, the following results hold:*

(1) *if $A^c \in \tau$, then $A^* \subseteq A$,*
(2) $\emptyset^* = \emptyset$,
(3) $X^* \subseteq X$,
(4) *if $A \subseteq B$, then $A^* \subseteq B^*$,*
(5) $A^* \cup B^* \subseteq (A \cup B)^*$,
(6) $(A \cap B)^* \subseteq A^* \cap B^*$,
(7) *A^* is a closed set, i.e., $cl(A^*) = A^*$,*
(8) $(A^*)^* \subseteq A^*$,
(9) $A^* \subseteq cl(A)$,
(10) $cl(A^*) \subseteq cl(A)$.

Proof. (1) Let $A^c \in \tau$ and $x \in A^*$. If possible, let us assume that $x \notin A$. Then, $x \in A^c$. Since $x \in A^*$, thus $A^c \cup U^c \notin \mathcal{S}$ for all $U \in \tau$ and $x \in U$. Thus, $X = A^c \cup (A^c)^c \notin \mathcal{S}$, a contradiction. Hence, $x \in A$. So, $A^* \subseteq A$.

(2) $\emptyset^c = X \in \tau$. Thus, $\emptyset^* \subseteq \emptyset$. Again, $\emptyset \subseteq \emptyset^*$. Thus, $\emptyset^* = \emptyset$.

(3) $X^c = \emptyset \in \tau$. Thus, $X^* \subseteq X$.

(4) Let $A \subseteq B$ and $x \in A^*$. Then, $A^c \cup U^c \notin \mathcal{S}$ for all $U \in \tau$ and $x \in U$. Then, obviously $B^c \cup U^c \notin \mathcal{S}$ for all $U \in \tau$ and $x \in U$. Hence, $A^* \subseteq B^*$.

(5) Let $A \subseteq B$ and $x \notin B^*$. Then, $\exists\, U \in \tau, x \in U$ such that $B^c \cup U^c \in \mathcal{S}$. It means, $A^c \cup U^c \in \mathcal{S}$. Thus, $x \notin A^*$. Hence, $A^* \subseteq B^*$.

(6) Using (4), we can easily get that $A^* \subseteq (A \cup B)^*$ and $B^* \subseteq (A \cup B)^*$. Thus, $A^* \cup B^* \subseteq (A \cup B)^*$.

(7) Proof can be obtained by using (4).

(8) Since, $A \subseteq X$, thus we know that $A^* \subseteq cl(A^*)$. Now, let $x \in cl(A^*)$. Then, $A^* \cap U \neq \emptyset$ for all $U \in \tau$ and $x \in U$. Let $y \in A^* \cap U$. Then, $y \in A^*$ and $y \in U$. Then, $A^c \cup V^c \notin \mathcal{S}$ for all $V \in \tau$ and $y \in V$. Thus, $A^c \cup U^c \notin \mathcal{S}$. Hence, $x \in A^*$. Thus, $cl(A^*) \subseteq A^*$. Thus, $cl(A^*) = A^*$.

(9) From the above proof, we get that A^* is a closed set. Thus, $(A^*)^c \in \tau$. Now, by using (1), we get $(A^*)^* \subseteq A^*$.

(10) Let $x \notin cl(A)$. Then, there exists $U \in \tau$ with $x \in U$ such that $A \cap U = \phi$. Thus, $A^c \cup U^c = (A \cap U)^c = \emptyset^c = X \in \mathcal{S}$. This gives $x \notin A^*$. Hence, $A^* \subseteq cl(A)$.

(11) The proof follows from (7) and (9).

$\square$

Remark 3.2. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. Then, $X \subseteq X^*$ is not true in general.*

The above remark can be verified from the following example:

Example 3.6. *Let us consider a siboid topological space $(X, \tau, \mathcal{S})$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$, and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $X^* = \{b, c\}$. Thus, $X \subseteq X^*$ is not true in this case. Thus, $X^* = X$ is not true in general.*

Remark 3.3. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, $A^* \cap B^* \subseteq (A \cap B)^*$ is not true in general.*

The above remark can be verified from the following example:

Example 3.7. *Let us consider a siboid topological space $(X, \tau, \mathcal{S})$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If we consider $A = \{a, b\}$, and $B = \{b, c\}$, then $A^* = \{b, c\}$, $B^* = \{c\}$, and $(A \cap B)^* = \emptyset$. Hence, $A^* \cap B^* = \{c\} \nsubseteq (A \cap B)^* = \emptyset$.*

Remark 3.4. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, $(A \cup B)^* \subseteq A^* \cup B^*$ is not true in general.*

The above remark can be verified from the following example:

Example 3.8. *Let us consider a siboid topological space* $(X, \tau, \mathcal{S})$*, where* $X = \{a, b, c\}$*,* $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$*, and* $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$*. If we consider* $A = \{a\}$ *and* $B = \{b\}$*, then* $A^* = \emptyset$*,* $B^* = \emptyset$ *and* $(A \cup B)^* = \{b, c\}$*. Hence,* $(A \cup B)^* = \{b, c\} \nsubseteq A^* \cup B^* = \emptyset$*.*

Now, we introduce a property $\mathcal{R}$ for a siboid $\mathcal{S}$ on a non-empty finite set.

Property $\mathcal{R}$**:** Let $\mathcal{S}$ be a siboid on a non-empty finite set X. Then, the property $\mathcal{R}$ is defined as: if for all $P, Q \in \mathcal{S}$, then $P \cap Q \in \mathcal{S}$.

Theorem 3.6. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space with property* $\mathcal{R}$ *and* $A, B \subseteq X$*. Then,* $(A \cup B)^* = A^* \cup B^*$*.*

Proof. From (4) of theorem 3.5, we get $A^* \cup B^* \subseteq (A \cup B)^*$.

Now, let $x \notin A^* \cup B^*$. Then, $x \notin A^*$ and $x \notin B^*$. Thus, there exist $U, V \in \tau$ with $x \in U, x \in V$ such that $A^c \cup U^c \in \mathcal{S}$ and $B^c \cup V^c \in \mathcal{S}$. Let $R = U \cap V$. Then, $R \in \tau$. Due to Definition 3.1, we get $A^c \cup R^c \in \mathcal{S}$ and $B^c \cup R^c \in \mathcal{S}$. Now, $(A \cup B)^c \cup R^c = (A^c \cup R^c) \cap (B^c \cup R^c) \in \mathcal{S}$. Thus, $x \notin (A \cup B)^*$. Hence, $(A \cup B)^* \subseteq A^* \cup B^*$. Thus, $(A \cup B)^* = A^* \cup B^*$. □

Theorem 3.7. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space and* $A, B \subseteq X$*. If* A *is open in X, then* $A \cap B^* \subseteq (A \cap B)^*$*.*

Proof. Let $x \in (A \cap B^*)$. Thus, $x \in A$ and $x \in B^*$. Then, $B^c \cup U^c \notin \mathcal{S}$ for all $U \in \tau$ and $x \in U$. Since $A \in \tau$, thus $A \cap U \in \tau$. Hence, $B^c \cup A^c \notin \mathcal{S}$. Now, $(A \cap B)^c \cup U^c = B^c \cup (A \cap U)^c \notin \mathcal{S}$. Thus, $x \in (A \cap B)^*$. Hence, the theorem is proved. □

Remark 3.5. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space and* $A, B \subseteq X$*. If* A *is open in X, then* $A \cup B^* \subseteq (A \cup B)^*$ *is not true in general.*

Remark 3.6. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space and* $A, B \subseteq X$*. If* A *is open in X, then* $(A \cup B)^* \subseteq A \cup B^*$ *is not true in general.*

The above remarks can be verified from the following example:

Example 3.9. *Let us consider a siboid topological space* $(X, \tau, \mathcal{S})$*, where* $X = \{a, b, c\}$*,* $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$*, and* $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$*. If we consider* $A = \{a\}$*, and* $B = \{b, c\}$*, then* $A \cup B^* = \{a, c\}$ *and* $(A \cup B)^* = \{b, c\}$*. Thus,* $A \cup B^* \nsubseteq (A \cup B)^*$ *and* $(A \cup B)^* \nsubseteq A \cup B^*$*.*

Definition 3.4. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space. We define a map* $cl^{\theta}\colon 2^X \to 2^X$ *as* $cl^{\theta}(A) = A \cup A^*$*, where* $A \subseteq X$*.*

Example 3.10. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space, where* $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ *and* $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$*. If we consider* $A = \{a, b\}$*, then* $A^* = \{b, c\}$*. Hence,* $cl^{\theta}(A) = A \cup A^* = \{a, b\} \cup \{b, c\} = \{a, b, c\} = X$*.*

Theorem 3.8. *Let* $(X, \tau, \mathcal{S})$ *be a siboid topological space and* $A, B \subseteq X$*. Then, the following results hold:*

(1) $cl^{\theta}(\emptyset) = \emptyset$,
(2) $cl^{\theta}(X) = X$,
(3) $A \subseteq cl^{\theta}(A)$,
(4) *if* $A \subseteq B$, *then* $cl^{\theta}(A) \subseteq cl^{\theta}(B)$,
(5) $cl^{\theta}(A) \cup cl^{\theta}(B) \subseteq cl^{\theta}(A \cup B)$,
(6) $cl^{\theta}(A \cap B) \subseteq cl^{\theta}(A) \cap cl^{\theta}(B)$,
(7) $cl^{\theta}(A) \subseteq cl^{\theta}(cl^{\theta}(A))$,
(8) $cl^{\theta}(A) \subseteq cl(A)$.

Proof. (1) Using (2) of Theorem 3.5, we get $cl^{\theta}(\emptyset) = \emptyset \cup \emptyset^* = \emptyset$.
(2) Using (3) of Theorem 3.5, we get $cl^{\theta}(X) = X \cup X^* = X$.
(3) Easy to prove, so the proof is omitted.
(4) Using (4) of Theorem 3.5, we get $cl^{\theta}(A) = A \cup A^* \subseteq B \cup B^* = cl^{\theta}(B)$.
(5) Using (5) of Theorem 3.5, we have $cl^{\theta}(A) \cup cl^{\theta}(B) = (A \cup A^*) \cup (B \cup B^*) = (A \cup B) \cup (A^* \cup B^*) \subseteq (A \cup B) \cup (A \cup B)^* = cl^{\theta}(A \cup B)$.
(6) Using (6) of Theorem 3.5, we have $cl^{\theta}(A \cap B) = (A \cap B) \cup (A \cap B)^* \subseteq (A \cap B) \cup (A^* \cap B^*) = \big((A \cap B) \cup A^*\big) \cap \big((A \cap B) \cup B^*\big) = \big((A \cup A^*) \cap (B \cup A^*)\big) \cap \big((A \cup B^*) \cap (B \cup B^*)\big) \subseteq (A \cup A^*) \cap (B \cup B^*) = cl^{\theta}(A) \cap cl^{\theta}(B)$.
(7) Using (3), we get $cl^{\theta}(A) \subseteq cl^{\theta}\big(cl^{\theta}(A)\big)$.
(8) Let $x \in cl^{\theta}(A) \Rightarrow x \in A \cup A^* \Rightarrow x \in A$ or $x \in A^* \Rightarrow x \in A$ or $x \in cl(A) \Rightarrow x \in A \cup cl(A) \Rightarrow x \in cl(A)$. Thus, $cl^{\theta}(A) \subseteq cl(A)$.

□

Remark 3.7. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A \subseteq X$. Then, $cl(A) \subseteq cl^{\theta}(A)$ is not true in general.*

The above remark can be verified from the following example:

Example 3.11. *Consider a siboid topological space $(X, \tau, \mathcal{S})$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If $A = \{a\}$, then $cl^{\theta}(A) = A \cup A^* = \{a\} \cup \emptyset = \{a\}$, whereas $cl(A) = X$. Thus, $cl(A) \nsubseteq cl^{\theta}(A)$.*

Theorem 3.9. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $A, B \subseteq X$. Then, $cl^{\theta}(A \cup B) = cl^{\theta}(A) \cup cl^{\theta}(B)$ and $cl^{\theta}\big(cl^{\theta}(A)\big) = cl^{\theta}(A)$.*

Proof. Using Theorem 3.6, we get $cl^{\theta}(A \cup B) = (A \cup B) \cup (A \cup B)^* = (A \cup B) \cup (A^* \cup B^*) = (A \cup A^*) \cup (B \cup B^*) = cl^{\theta}(A) \cup cl^{\theta}(B)$.

Now, $cl^{\theta}(A) \subseteq cl^{\theta}\big(cl^{\theta}(A)\big)$ holds from (7) of Theorem 3.8. Again, $cl^{\theta}\big(cl^{\theta}(A)\big) = cl^{\theta}(A \cup A^*) = (A \cup A^*) \cup (A \cup A^*)^* = (A \cup A^*) \cup (A^* \cup (A^*)^*) \subseteq (A \cup A^*) \cup (A^* \cup A^*) = A \cup A^* = cl^{\theta}(A)$. Thus, $cl^{\theta}\big(cl^{\theta}(A)\big) = cl^{\theta}(A)$. □

Remark 3.8. *The map cl^θ as defined above satisfies Kuratowski closure axioms if the siboid $\mathcal{S}$ satisfies property $\mathcal{R}$.*

Theorem 3.10. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. If $B \in \tau$, then $cl^\theta(A) \cap B \subseteq cl^\theta(A \cap B)$.*

Proof. $cl^\theta(A) \cap B = (A \cup A^*) \cap B = (A \cap B) \cup (A^* \cap B)$. Since $B \in \tau$, using Theorem 3.7, we get $cl^\theta(A) \cap B \subseteq (A \cap B) \cup (A \cap B)^* = cl^\theta(A \cap B)$. □

Definition 3.5. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$. We define $\tau^\theta_{\mathcal{S}} = \{A \subseteq X |\ cl^\theta(A^c) = A^c\}$. Then, $\tau^\theta_{\mathcal{S}}$ is a new topology on X, and it is called the siboid topology on X.*

Theorem 3.11. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $\tau^\theta_{\mathcal{S}}$ is the siboid topology on X. Then, $\tau \subseteq \tau^\theta_{\mathcal{S}}$.*

Proof. Let $A \in \tau$. Then, by (1) of Theorem 3.5, we have $(A^c)^* \subseteq A^c$. So, by using Definition 3.4 and (3) of Theorem 3.8, we get $cl^\theta(A^c) = A^c$. Thus, $A \in \tau^\theta_{\mathcal{S}}$. □

Theorem 3.12. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $\tau^\theta_{\mathcal{S}}$ is the siboid topology on X. Then, if $\mathcal{S} = 2^X$, then $\tau^\theta_{\mathcal{S}} = 2^X$.*

Proof. We always have $\tau^\theta_{\mathcal{S}} \subseteq 2^X$. Now, let $A \in 2^X$. Since $\mathcal{S} = 2^X$, we have $A^* = \emptyset$ for any subset A of X. Therefore, $cl^\theta(A^c) = A^c \cup (A^c)^* = A^c$. This means that $A \in \tau^\theta_{\mathcal{S}}$. Hence, $2^X \subseteq \tau^\theta_{\mathcal{S}}$. Thus, we have $\tau^\theta_{\mathcal{S}} = 2^X$. □

Theorem 3.13. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $\tau^\theta_{\mathcal{S}}$ is the siboid topology on X. Then, if $\mathcal{S} = \{X\}$, then $\tau = \tau^\theta_{\mathcal{S}}$.*

Proof. From Theorem 3.11, we know that $\tau \subseteq \tau^\theta_{\mathcal{S}}$. Now, we are to prove that $\tau^\theta_{\mathcal{S}} \subseteq \tau$. Let $A \in \tau^\theta_{\mathcal{S}}$ which implies $cl^\theta(A^c) = A^c$. Then $A^c \cup (A^c)^* = A^c$ which means that $(A^c)^* \subseteq A^c$.

Now, let $x \notin (A^c)^*$. Then, there exists $U \in \tau$ with $x \in U$ such that $(A^c)^c \cup U^c \in \mathcal{S}$, i.e., $A \cup U^c \in \mathcal{S}$. Since $\mathcal{S} = \{X\}$, we have $A \cup U^c = X$. So, $A^c \cap U = \emptyset$. Therefore, $x \notin cl(A^c)$. Thus, we have $cl(A^c) \subseteq (A^c)^* \subseteq A^c$. Hence, $cl(A^c) = A^c$. It means that $A \in \tau$. Thus, $\tau^\theta_{\mathcal{S}} \subseteq \tau$. Consequently, we have $\tau = \tau^\theta_{\mathcal{S}}$. □

Remark 3.9. *The converse of Theorem 3.13 is not true in general.*

The above remark can be verified from the following example:

Example 3.12. *Let us consider a siboid topological space $(X, \tau, \mathcal{S})$, where $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$ and $\mathcal{S} = \{\emptyset, X, \{a\}, \{b\}\}$. Then, $\tau^\theta_{\mathcal{S}} = \{\emptyset, \{a\}, \{b\}, X\}$. Thus, $\tau = \tau^\theta_{\mathcal{S}}$, but $\mathcal{S} \neq \{X\}$.*

Theorem 3.14. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $A \subseteq X$. Then, the following results hold:*

(1) *$A \in \tau^{\theta}_{\mathcal{S}}$ if and only if for $x \in A$, there exists a $U \in \tau$ containing x such that $A \cup U^c \in \mathcal{S}$,*

(2) *if $A \in \mathcal{S}$, then $A \in \tau^{\theta}_{\mathcal{S}}$.*

Proof. (1) Let $A \in \tau^{\theta}_{\mathcal{S}}$. It implies $cl^{\theta}(A^c) = A^c$. So, $A^c \cup (A^c)^* = A^c$. It gives $(A^c)^* \subseteq A^c$. Thus, $A \subseteq \left((A^c)^*\right)^c$. So, if $x \in A$, then $x \in \left((A^c)^*\right)^c$. Thus, $x \notin (A^c)^*$. Then, there exists a $U \in \tau$ such that $x \in U$ and $(A^c)^c \cup U^c \in \mathcal{S} \Rightarrow A \cup U^c \in \mathcal{S}$.

Conversely, let for $x \in A$, there exists a $U \in \tau$ containing x such that $A \cup U^c \in \mathcal{S}$. Thus, $x \notin (A^c)^*$. So, $x \in \left((A^c)^*\right)^c$. It implies $A \subseteq \left((A^c)^*\right)^c$. So, $(A^c)^* \subseteq A^c$. This gives $A^c \cup (A^c)^* = A^c \Rightarrow cl^{\theta}(A^c) = A^c$. Thus, $A \in \tau^{\theta}_{\mathcal{S}}$.

(2) Let $A \in S$ and $x \in A$. If we take $U = X$, then U is an open set containing x. Since $A \in \mathcal{S}$ and $A \cup U^c = A$, we have $A \cup U^c \in S$. From (1), we get $A \in \tau^{\theta}_{\mathcal{S}}$.

□

Theorem 3.15. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$. Then, the family $\mathcal{B}_{\mathcal{S}} = \{T \cap S \mid T \in \tau$ and $S \in \mathcal{S}\}$ is the base for the siboid topology $\tau^{\theta}_{\mathcal{S}}$ on X.*

Proof. Let $B \in \mathcal{B}_{\mathcal{S}}$. Then, there exist $T \in \tau$ and $S \in \mathcal{S}$ such that $B = T \cap S$. Since $\tau \subseteq \tau^{\theta}_{\mathcal{S}}$, we get $T \in \tau^{\theta}_{\mathcal{S}}$. On the other hand, from (2) of Theorem 3.14, we have $S \in \tau^{\theta}_{\mathcal{S}}$. Therefore, $B \in \tau^{\theta}_{\mathcal{S}}$. Consequently, $\mathcal{B}_{\mathcal{S}} \subseteq \tau^{\theta}_{\mathcal{S}}$.

Now, let $A \in \tau^{\theta}_{\mathcal{S}}$ and $x \in A$. Then, from (1) of Theorem 3.14, there exists $U \in \tau$ and $x \in U$ such that $A \cup U^c \in \mathcal{S}$. Now, let $B' = U \cap (A \cup U^c)$. Hence, we have $B' \in \mathcal{B}_{\mathcal{S}}$ such that $x \in B' \subseteq A$. Thus, the family $\mathcal{B}_{\mathcal{S}} = \{T \cap S \mid T \in \tau$ and $S \in \mathcal{S}\}$ is the base for the siboid topology $\tau^{\theta}_{\mathcal{S}}$ on X. □

In the following theorem, $(A^c)^*_{\mathcal{S}}$ and $(A^c)^*_{\mathcal{S}'}$ denote the set $(A^c)^*$ with respect to $(X, \tau, \mathcal{S})$ and $(X, \tau, \mathcal{S}')$, respectively.

Theorem 3.16. *Let $(X, \tau, \mathcal{S})$ and $(X, \tau, \mathcal{S}')$ be two siboid topological spaces with property $\mathcal{R}$. If $\mathcal{S} \subseteq \mathcal{S}'$, then $\tau^{\theta}_{\mathcal{S}} \subseteq \tau^{\theta}_{\mathcal{S}'}$.*

Proof. Let $A \in \tau^{\theta}_{\mathcal{S}}$. Then, $cl^{\theta}(A^c) = A^c \Rightarrow (A^c) \cup (A^c)^*_{\mathcal{S}} = A^c$. It means $(A^c)^*_{\mathcal{S}} \subseteq A^c$. Now, let $x \notin A^c$. Then, we get $x \notin (A^c)^*_{\mathcal{S}}$. Thus, there exists an open set $U \in \tau$ and $x \in U$ such that $(A^c)^c \cup U^c \in \mathcal{S}$, i.e., $A \cup U^c \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{S}'$, we have $A \cup U^c \subseteq \mathcal{S}'$. Therefore, $x \notin (A^c)^*_{\mathcal{S}'}$. Thus, $(A^c)^*_{\mathcal{S}'} \subseteq A^c$ and so, $cl^{\theta}(A^c) = A^c \cup (A^c)^*_{\mathcal{S}'} = A^c$. Hence, $A \in \tau^{\theta}_{\mathcal{S}'}$. Consequently, $\tau^{\theta}_{\mathcal{S}} \subseteq \tau^{\theta}_{\mathcal{S}'}$. □

Theorem 3.17. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. If $A^c \in \mathcal{S}$, then $A^* = \emptyset$.*

Proof. Suppose that $x \in A^*$. Then, for any $U \in \tau$ containing x, we have $A^c \cup U^c \notin \mathcal{S}$. Since $A^c \in \mathcal{S}$ and $A^c \subseteq A^c \cup U^c$, then $A^c \cup U^c \in \mathcal{S}$, but it is a contradiction. Hence, $A^* = \emptyset$. □

Theorem 3.18. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $A, B \subseteq X$. Then, $A^* - B^* = (A - B)^* - B^*$.*

Proof. We have, by Theorem 3.6, $A^* = \left((A \cap B) \cup (A - B)\right)^* = (A \cap B)^* \cup (A - B)^* \subseteq B^* \cup (A - B)^*$. Thus, $A^* - B^* \subseteq (A - B)^* - B^*$. Again, $(A - B)^* \subseteq A^*$ and hence, $(A - B)^* - B^* \subseteq A^* - B^*$. Hence, $A^* - B^* = (A - B)^* - B^*$. □

Corollary 3.1. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $A, B \subseteq X$. If $B^c \in \mathcal{S}$, then $(A \cup B)^* = A^* = (A - B)^*$.*

Proof. Since $B^c \in \mathcal{S}$, from Theorem 3.17, we get $B^* = \emptyset$. Then, by Theorem 3.18, $A^* = (A - B)^*$ and by Theorem 3.6, we get $(A \cup B)^* = A^* = (A - B)^*$. □

Definition 3.6. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. An operator Φ: $2^X \to 2^X$ is defined as $\Phi(A) = \{x \in X : \exists U \in \tau, x \in U$ and $(U - A)^c \in \mathcal{S}\}$, where $A \subseteq X$.*

Example 3.13. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If we consider $A = \{a, b\}$, then $\Phi(A) = \{a, b, c\}$.*

The following theorem studies fundamental properties satisfied by the operator Φ:

Theorem 3.19. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. Then, the following properties hold:*

(1) *if $A \subseteq X$, then $\Phi(A) = \big((A^c)^*\big)^c$,*
(2) *$\Phi(X) = X$,*
(3) *if $A \subseteq X$, then $\Phi(A)$ is open,*
(4) *if $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$,*
(5) *if $A, B \subseteq X$, then $\Phi(A \cap B) \subseteq \Phi(A) \cap \Phi(B)$,*
(6) *if $A, B \subseteq X$, then $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cup B)$,*
(7) *if $U \in \tau$, then $U \subseteq \Phi(U)$,*
(8) *if $A \subseteq X$, then $\Phi(A) \subseteq \Phi\big(\Phi(A)\big)$,*
(9) *if $A \subseteq X$, then $\Phi(A) = \Phi\big(\Phi(A)\big)$ if and only if $(A^c)^* = \big((A^c)^*\big)^*$.*

Proof. (1) Let $x \in \Phi(A)$. Then, $\exists U \in \tau$ containing x such that $(U - A)^c \in \mathcal{S}$. But, $(U - A)^c = (U \cap A^c)^c = U^c \cup A$. Thus, $x \notin (A^c)^*$. This implies $x \in \big((A^c)^*\big)^c$. So, $\Phi(A) \subseteq \big((A^c)^*\big)^c$.

Conversely, let $x \in \big((A^c)^*\big)^c$. Then, $x \notin (A^c)^*$. This implies $\exists V \in \tau$ containing x such that $V^c \cup A = (V - A)^c \in \mathcal{S}$. Hence, $x \in \Phi(A)$. So, $\Phi(A) = \big((A^c)^*\big)^c$.

(2) $\Phi(X) = \big((X^c)^*\big)^c = (\emptyset^*)^c = \emptyset^c = X$.
(3) From (7) of Theorem 3.5, we get $(A^c)^*$ is a closed set. Thus, $\big((A^c)^*\big)^c$ is open. Hence, $\Phi(A)$ is open.
(4) Let $x \in \Phi(A)$. Then, $\exists U \in \tau$ containing x such that $(U - A)^c \in \mathcal{S}$, i.e., $U^c \cup A \in \mathcal{S}$. Thus, $x \notin (A^c)^*$. Since $A \subseteq B$, using (4) of Theorem 3.5, we get $x \notin (B^c)^*$. This implies $\exists V \in \tau$ containing x such that $V^c \cup B = (V - B)^c \in \mathcal{S}$. Thus, $x \in \Phi(B)$. Hence, $\Phi(A) \subseteq \Phi(B)$.
(5) It follows from (4) that $\Phi(A \cap B) \subseteq \Phi(A)$ and $\Phi(A \cap B) \subseteq \Phi(B)$. Hence, $\Phi(A \cap B) \subseteq \Phi(A) \cap \Phi(B)$.
(6) It follows from (4) that $\Phi(A) \subseteq \Phi(A \cup B)$ and $\Phi(B) \subseteq \Phi(A \cup B)$. Hence, $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cup B)$.
(7) If $U \in \tau$, then from (1) of Theorem 3.5, we get $(U^c)^* \subseteq U^c$. Hence, using (1), $U \subseteq ((U^c)^*)^c = \Phi(U)$.

(8) Proof can be obtained by using (3) and (7).

(9) Using (1), $\Phi\Big(\Phi(A)\Big) = \left(\left(\left(\Phi(A)\right)^c\right)^*\right)^c = \left(\left(\left(\left((A^c)^*\right)^c\right)^c\right)^*\right)^c = \left(\left((A^c)^*\right)^*\right)^c$. Thus, $\Phi(A) = \Phi\Big(\Phi(A)\Big)$ if and only if $\left((A^c)^*\right)^c = \left(\left((A^c)^*\right)^*\right)^c$, i.e., $(A^c)^* = \left((A^c)^*\right)^*$. □

Remark 3.10. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. If $U \notin \tau$, then $U \subseteq \Phi(U)$ is not true in general for $U \subseteq X$.*

The above remark can be verified from the following example:

Example 3.14. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{S} = \{X\}$. Then, for $U = \{a, c\} \notin \tau$, we have $\Phi(U) = \{a\}$. Thus, in this case $U \nsubseteq \Phi(U)$.*

Remark 3.11. $\Phi(\emptyset) \neq \emptyset$ *in general.*

The above remark can be verified from the following example:

Example 3.15. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $\Phi(\emptyset) = \{a\} \neq \emptyset$.*

Remark 3.12. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, $\Phi(A) \cap \Phi(B) \subseteq \Phi(A \cap B)$ is not true in general.*

The above remark can be verified from the following example:

Example 3.16. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If we consider $A = \{a, b\}$ and $B = \{b, c\}$, then $\Phi(A) = \{a, b, c\}, \Phi(B) = \{a, b, c\}$ and $\Phi(A \cap B) = \{a, b\}$. Hence, $\Phi(A) \cap \Phi(B) = \{a, b, c\} \nsubseteq \Phi(A \cap B) = \{a, b\}$.*

Remark 3.13. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, $\Phi(A \cup B) \subseteq \Phi(A) \cup \Phi(B)$ is not true in general.*

The above remark can be verified from the following example:

Example 3.17. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. If we consider $A = \{a\}$ and $B = \{b\}$, then $\Phi(A) = \{a, b\}, \Phi(B) = \{a, b\}$ and $\Phi(A \cup B) = \{a, b, c\}$. Hence, $\Phi(A \cup B) = \{a, b, c\} \nsubseteq \Phi(A) \cup \Phi(B) = \{a, b\}$.*

Theorem 3.20. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and $A, B \subseteq X$. Then, $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.*

Proof. Using (1) of Theorem 3.19 and Theorem 3.6, we get $\Phi(A \cap B) = \left(\left((A \cap B)^c\right)^*\right)^c = \left((A^c \cup B^c)^*\right)^c = \left((A^c)^* \cup (B^c)^*\right)^c = \left((A^c)^*\right)^c \cap \left((B^c)^*\right)^c = \Phi(A) \cap \Phi(B)$. □

Theorem 3.21. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A, B \in \tau$, then $\Phi(A \cup B) \subseteq \Phi\Big(A \cup \Phi(B)\Big) \subseteq \Phi\Big(\Phi(A) \cup \Phi(B)\Big) \subseteq \Phi\Big(\Phi(A \cup B)\Big)$.*

Proof. From (7) of Theorem 3.19, we get $A \subseteq \Phi(A)$ and $B \subseteq \Phi(B)$. Thus, $A \cup B \subseteq A \cup \Phi(B) \subseteq \Phi(A) \cup \Phi(B)$. Using (4) of Theorem 3.19, we get $\Phi(A \cup B) \subseteq \Phi\Big(A \cup \Phi(B)\Big) \subseteq \Phi\Big(\Phi(A) \cup \Phi(B)\Big)$. Again, $A \subseteq A \cup B$ and $B \subseteq A \cup B$ give $\Phi(A) \subseteq \Phi(A \cup B)$ and $\Phi(B) \subseteq \Phi(A \cup B)$. Thus, $\Phi(A) \cup \Phi(B) \subseteq \Phi(A \cup B) \Rightarrow \Phi\Big(\Phi(A) \cup \Phi(B)\Big) \subseteq \Phi\Big(\Phi(A \cup B)\Big)$. □

Theorem 3.22. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. If $\mathcal{S} = 2^X$, then $\Phi\Big(\Phi(A)\Big) = \Phi(A) = X$ for all $A \subseteq X$.*

Proof. If $\mathcal{S} = 2^X$, then $(A^c)^* = \emptyset$. From (1) of Theorem 3.19, we get $\Phi(A) = \Big((A^c)^*\Big)^c = \emptyset^c = X$ and $\Phi\Big(\Phi(A)\Big) = \Phi(X) = \Big((X^c)^*\Big)^c = (\emptyset^*)^c = \emptyset^c = X$, for all $A \subseteq X$. □

Corollary 3.2. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. If $\mathcal{S} = 2^X$, then $A \subseteq \Phi(A)$, for all $A \subseteq X$.*

Proof. If $\mathcal{S} = 2^X$, then $\Phi(A) = X$. Hence $A \subseteq X = \Phi(A)$, for all $A \subseteq X$. □

Corollary 3.3. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. If $\mathcal{S} = 2^X$, then $(A^c)^* = \Big((A^c)^*\Big)^* = \emptyset$, for all $A \subseteq X$.*

Proof. If $\mathcal{S} = 2^X$, then $\Phi(A) = \Phi\Big(\Phi(A)\Big) = X$. Using (9) of Theorem 3.19, we get $(A^c)^* = \Big((A^c)^*\Big)^* = \emptyset$, for all $A \subseteq X$. □

Theorem 3.23. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$. Then, $\psi = \{A \subseteq X : A \subseteq \Phi(A)\}$ is a topology on X induced by the operator Φ.*

Proof. Let $\psi = \{A \subseteq X : A \subseteq \Phi(A)\}$. Since $\emptyset \subseteq \Phi(\emptyset)$, we have $\emptyset \in \psi$. From (2) of Theorem 3.19, we have, $X \subseteq \Phi(X)$. Thus, $X \in \psi$.

Let $A, B \in \psi$. Then, $A \subseteq \Phi(A)$ and $B \subseteq \Phi(B)$. Thus, from Theorem 3.20, $A \cap B \subseteq \Phi(A) \cap \Phi(B) = \Phi(A \cap B)$. Hence, $A \cap B \in \psi$.

Let $\{A_i : i \in \Delta$ and Δ is an index set$\} \subseteq \psi$. Then, for each $i \in \Delta$, we have $A_i \subseteq \Phi(A_i)$. Now, $\bigcup\limits_{i\in\Delta} A_i \subseteq \bigcup\limits_{i\in\Delta} \Phi(A_i) \subseteq \Phi(\bigcup\limits_{i\in\Delta} A_i)$. Thus, $\bigcup\limits_{i\in\Delta} A_i \in \psi$. Thus, ψ is a topology on X induced by the operator Φ. □

Theorem 3.24. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$. Then, $\tau^\theta_{\mathcal{S}} = \psi$.*

Proof. Let $A \in \tau^\theta_{\mathcal{S}}$. Then, $cl^\theta(A^c) = A^c \Rightarrow A^c \cup (A^c)^* = A^c \Rightarrow (A^c)^* \subseteq A^c \Rightarrow A \subseteq \Big((A^c)^*\Big)^c = \Phi(A) \Rightarrow A \in \psi$. Thus, $\tau^\theta_{\mathcal{S}} \subseteq \psi$.

Conversely, let $A \in \psi$. Then, $A \subseteq \Phi(A)$. Using (1) of Theorem 3.19, we get $A \subseteq \Big((A^c)^*\Big)^c \Rightarrow (A^c)^* \subseteq A^c \Rightarrow A^c \cup (A^c)^* = A^c \Rightarrow cl^\theta(A^c) = A^c \Rightarrow A \in \tau^\theta_{\mathcal{S}} \Rightarrow \psi \subseteq \tau^\theta_{\mathcal{S}}$. Hence, we get $\tau^\theta_{\mathcal{S}} = \psi$. □

Now, we define five types of generalized open sets on siboid topological space in light of the operator cl^θ.

Definition 3.7. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A \subseteq X$. Then, the subset A is said to be a:*

(1) *$\mathcal{S}$-α-open set if $A \subseteq int\Big(cl^{\theta}\Big(int(A)\Big)\Big)$,*
(2) *$\mathcal{S}$-semi-open set if $A \subseteq cl^{\theta}\Big(int(A)\Big)$,*
(3) *$\mathcal{S}$-pre-open set if $A \subseteq int\Big(cl^{\theta}(A)\Big)$,*
(4) *$\mathcal{S}$-β-open set if $A \subseteq cl^{\theta}\Big(int\Big(cl^{\theta}(A)\Big)\Big)$,*
(5) *$\mathcal{S}$-regular-open set if $A = int\Big(cl^{\theta}(A)\Big)$.*

Theorem 3.25. *In a siboid topological space $(X, \tau, \mathcal{S})$, the following results are true:*

(1) *each $\mathcal{S}$-α-open set is an α-open set,*
(2) *each $\mathcal{S}$-semi-open set is a semi-open set,*
(3) *each $\mathcal{S}$-pre-open is a pre-open set,*
(4) *each $\mathcal{S}$-β-open set is a β-open set.*

Proof. (1) Let $A \subseteq X$ be a $\mathcal{S}$-α-open set. Then, $A \subseteq int\Big(cl^{\theta}\Big(int(A)\Big)\Big) = int\Big(int(A) \cup \Big(int(A)\Big)^*\Big) \subseteq int\Big(int(A) \cup cl\Big(int(A)\Big)\Big) \subseteq int\Big(cl\Big(int(A)\Big)\Big)$. Thus, A is an α-open set.

(2) Let $A \subseteq X$ be a $\mathcal{S}$-semi-open set. Then, $A \subseteq cl^{\theta}\Big(int(A)\Big) = int(A) \cup \Big(int(A)\Big)^* \subseteq int(A) \cup cl\Big(int(A)\Big) = cl\Big(int(A)\Big)$. Thus, A is a semi-open set.

(3) Let $A \subseteq X$ is a $\mathcal{S}$-pre-open set. Then, $A \subseteq int\Big(cl^{\theta}(A)\Big) = int(A \cup A^*) \subseteq int\Big(A \cup cl(A)\Big) = int\Big(cl(A)\Big)$. Thus, A is a pre-open set.

(4) Let $A \subseteq X$ be a $\mathcal{S}$-β-open set. Then, $A \subseteq cl^{\theta}\Big(int\Big(cl^{\theta}(A)\Big)\Big) = cl^{\theta}\Big(int(A \cup A^*)\Big) \subseteq cl^{\theta}\Big(int\Big(A \cup cl(A)\Big)\Big) = cl^{\theta}(int\Big(cl(A)\Big)) \subseteq cl\Big(int\Big(cl(A)\Big)\Big)$. Thus, A is a β-open set.

□

Remark 3.14. *The converse of Theorem 3.25 is not true in general. This is demonstrated in the following two examples:*

Example 3.18. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then,*

(1) *$A = \{a, b\}$ is an α-open set which is not $\mathcal{S}$-α-open, since, $A \subseteq int\Big(cl\Big(int(A)\Big)\Big) = X$ but $A \nsubseteq int\Big(cl^{\theta}\Big(int(A)\Big)\Big) = \{a\}$.*
(2) *$A = \{a, b\}$ is a semi-open set which is not $\mathcal{S}$-semi-open, since, $A \subseteq cl(int(A)) = X$ but $A \nsubseteq cl^{\theta}(int(A)) = \{a\}$.*

Example 3.19. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = 2^X$. Then,*

(1) *$A = \{a, b\}$ is a pre-open set which is not $\mathcal{S}$-pre-open, since, $A \subseteq int\Big(cl(A)\Big) = X$ but $A \nsubseteq int\Big(cl^{\theta}(A)\Big) = \{a\}$.*

(2) *$A = \{a, b\}$ is a β-open set which is not $\mathcal{S}$-β-open, since, $A \subseteq cl\Big(int\big(cl(A)\big)\Big) = X$ but $A \nsubseteq cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big) = \{a\}$.*

Theorem 3.26. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and $A \subseteq X$. Then, the following results are true:*

(1) *A is $\mathcal{S}$-semi-open and $\mathcal{S}$-pre-open if A is $\mathcal{S}$-α-open,*
(2) *A is $\mathcal{S}$-β-open if A is $\mathcal{S}$-semi-open,*
(3) *A is $\mathcal{S}$-β-open if A is $\mathcal{S}$-pre-open.*

Proof. (1) Let A is $\mathcal{S}$-α-open. Hence, $A \subseteq int\Big(cl^{\theta}\big(int(A)\big)\Big)$. Then, $A \subseteq cl^{\theta}\big(int(A)\big)$. Hence, A is $\mathcal{S}$-semi-open. Again, $int(A) \subseteq A \Rightarrow cl^{\theta}\big(int(A)\big) \subseteq cl^{\theta}(A)$. Thus, $A \subseteq int\Big(cl^{\theta}\big(int(A)\big)\Big) \subseteq int\big(cl^{\theta}(A)\big)$. So, A is $\mathcal{S}$-pre-open.

(2) Let A is $\mathcal{S}$-semi-open. Thus, we get $A \subseteq cl^{\theta}\big(int(A)\big)$. Since $A \subseteq cl^{\theta}(A)$, it gives $A \subseteq cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)$. So, A is $\mathcal{S}$-β-open.

(3) Since A is $\mathcal{S}$-pre-open, we have $A \subseteq int(cl^{\theta}(A))$. By using (3) of theorem 3.8, we have $A \subseteq int\big(cl^{\theta}(A)\big) \subseteq cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)$. Hence, A is $\mathcal{S}$-β-open.

□

Remark 3.15. *The converse of (2) of Theorem 3.26 is not true in general.*

The above remark is verified in the following example:

Example 3.20. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then, $A = \{a, b\}$ is $\mathcal{S}$-β-open but not $\mathcal{S}$-semi-open since $A \subseteq cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big) = X$ but $A \nsubseteq cl^{\theta}\big(int(A)\big) = \{a\}$.*

Remark 3.16. *A $\mathcal{S}$-pre-open set is not $\mathcal{S}$-semi-open in general.*

The above remark can be verified from the following example:

Example 3.21. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $A = \{a, b\}$ is $\mathcal{S}$-pre-open since $A \subseteq int\big(cl^{\theta}(A)\big) = X$, but A is not $\mathcal{S}$-semi-open since $A \nsubseteq cl^{\theta}\big(int(A)\big) = \{a\}$.*

Theorem 3.27. *In a siboid topological space $(X, \tau, \mathcal{S})$, the following results hold:*

(1) *the arbitrary union of $\mathcal{S}$-α-open sets is $\mathcal{S}$-α-open,*
(2) *the arbitrary union of $\mathcal{S}$-semi-open sets is $\mathcal{S}$-semi-open,*
(3) *the arbitrary union of $\mathcal{S}$-pre-open sets is $\mathcal{S}$-pre-open,*
(4) *the arbitrary union of $\mathcal{S}$-β-open sets is $\mathcal{S}$-β-open.*

Proof. Let $(X, \tau, \mathcal{S})$ be a siboid topological space.

(1) Let $\{A_i : i \in \Delta$ and Δ is an index set$\}$ be an arbitrary collection of $\mathcal{S}$-α-open sets. Then, $A_i \subseteq int\Big(cl^\theta\Big(int(A_i)\Big)\Big), \forall i \in \Delta$.
Hence,

$$\begin{aligned}\bigcup_{i\in\Delta} A_i &\subseteq \bigcup_{i\in\Delta} int\Big(cl^\theta\Big(int(A_i)\Big)\Big) \\ &\subseteq int\Big(\bigcup_{i\in\Delta} cl^\theta\Big(int(A_i)\Big)\Big) \\ &\subseteq int\Big(cl^\theta\Big(\bigcup_{i\in\Delta} int(A_i)\Big)\Big) \\ &\subseteq int\Big(cl^\theta\Big(int(\bigcup_{i\in\Delta} A_i)\Big)\Big)\end{aligned}$$

Thus, $\bigcup\limits_{i\in\Delta} A_i$ is $\mathcal{S}$-α-open.

We can prove (2), (3), and (4) in a similar manner. □

Theorem 3.28. *Intersection of two $\mathcal{S}$-regular-open sets in a siboid topological space $(X, \tau, \mathcal{S})$ is $\mathcal{S}$-regular open.*

Proof. Consider A and B be two $\mathcal{S}$-regular-open sets in a siboid topological space $(X, \tau, \mathcal{S})$. Then, $A = int\Big(cl^\theta(A)\Big)$ and $B = int\Big(cl^\theta(B)\Big)$. Since A and B are open sets in (X, τ), $A \cap B$ is also an open set.

Now, $A \cap B \subseteq cl^\theta(A \cap B) \Rightarrow int(A \cap B) \subseteq int\Big(cl^\theta(A \cap B)\Big) \Rightarrow A \cap B \subseteq int\Big(cl^\theta(A \cap B)\Big)$. Again, $int\Big(cl^\theta(A \cap B)\Big) \subseteq int\Big(cl^\theta(A) \cap cl^\theta(B)\Big) = int\Big(cl^\theta(A)\Big) \cap int\Big(cl^\theta(B)\Big) = A \cap B$. Consequently, we get $A \cap B = int\Big(cl^\theta(A \cap B)\Big)$. Hence, $A \cap B$ is a $\mathcal{S}$-regular-open set. □

Corollary 3.4. *If A and B are $\mathcal{S}$-regular-open sets in a siboid topological space $(X, \tau, \mathcal{S})$, then, $int\Big(cl^\theta(A \cap B)\Big) = int\Big(cl^\theta(A)\Big) \cap int\Big(cl^\theta(B)\Big)$.*

Proof. Since $A = int\Big(cl^\theta(A)\Big)$ and $B = int\Big(cl^\theta(B)\Big)$, the result follows from Theorem 3.28. □

Remark 3.17. *The union of two $\mathcal{S}$-regular-open sets in a siboid topological space $(X, \tau, \mathcal{S})$ is not always a $\mathcal{S}$-regular-open set.*

The above remark can be verified from the following example:

Example 3.22. *Consider $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then, $A = \{a\}$ and $B = \{b\}$ are $\mathcal{S}$-regular-open sets since $int\Big(cl^\theta(A)\Big) = \{a\} = A$ and $int\Big(cl^\theta(B)\Big) = \{b\} = B$, respectively. But, $A \cup B = \{a, b\}$ is not a $\mathcal{S}$-regular-open set since $int\Big(cl^\theta(A \cup B)\Big) = X \neq A \cup B$.*

Theorem 3.29. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space with property $\mathcal{R}$ and A and B be two $\mathcal{S}$-regular-open sets. Then, $int\Big(cl^\theta(A \cup B)\Big)$ is a $\mathcal{S}$-regular-open set.*

Proof. A and B are $\mathcal{S}$-regular-open sets. Thus, $A = int\big(cl^{\theta}(A)\big)$ and $B = int\big(cl^{\theta}(B)\big)$. Now, by using Theorem 3.9, we get $int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A\cup B)\big)\Big)\Big) \subseteq int\Big(cl^{\theta}\big(cl^{\theta}(A\cup B)\big)\Big) = int\big(cl^{\theta}(A\cup B)\big)$.

Also, $int\big(cl^{\theta}(A\cup B)\big) = int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\cup int\big(cl^{\theta}(B)\big)\Big)\Big) \subseteq int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\cup cl^{\theta}(B)\big)\Big)\Big) = int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A\cup B)\big)\Big)\Big)$. Thus, we get $int\big(cl^{\theta}(A\cup B)\big) = int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A\cup B)\big)\Big)\Big)$. This gives $int\big(cl^{\theta}(A\cup B)\big)$ is a $\mathcal{S}$-regular-open set. □

Theorem 3.30. *Let A be a $\mathcal{S}$-regular-open set in a siboid topological space $(X,\tau,\mathcal{S})$. Then, $int(A^c)$ is a $\mathcal{S}$-regular-open set.*

Proof. A is a $\mathcal{S}$-regular-open set. Thus, A is an open set. So, $(A^c)^c \in \tau$. By (1) of Theorem 3.5, $(A^c)^* \subseteq A^c$. Consequently $cl^{\theta}(A^c) = A^c \cup (A^c)^* = A^c$.

Now, $int(A^c) \subseteq cl^{\theta}\big(int(A^c)\big) \Rightarrow int\big(int(A^c)\big) \subseteq int\Big(cl^{\theta}\big(int(A^c)\big)\Big) \Rightarrow int(A^c) \subseteq int\Big(cl^{\theta}\big(int(A^c)\big)\Big)$. Again, $int\Big(cl^{\theta}\big(int(A^c)\big)\Big) \subseteq int\big(cl^{\theta}(A^c)\big) = int(A^c)$. Thus, $int\Big(cl^{\theta}\big(int(A^c)\big)\Big) = int(A^c)$. So, $int(A^c)$ is a $\mathcal{S}$-regular-open set. □

Theorem 3.31. *Let $(X,\tau,\mathcal{S})$ be a siboid topological space and $A, B \subseteq X$. Then, the following results hold:*

(1) *$cl^{\theta}(A) = cl^{\theta}\big(int(A)\big) \Rightarrow A$ is $\mathcal{S}$-semi-open,*
(2) *A is $\mathcal{S}$-semi-open $\Leftrightarrow$ there exists $B \in \tau$ such that $B \subseteq A \subseteq cl^{\theta}(B)$,*
(3) *if A is $\mathcal{S}$-semi-open and $B \in \tau$, then $A \cap B$ is $\mathcal{S}$-semi-open.*

Proof. (1) Let $cl^{\theta}(A) = cl^{\theta}\big(int(A)\big)$. Since $A \subseteq cl^{\theta}(A)$, we get $A \subseteq cl^{\theta}\big(int(A)\big)$. Hence, A is $\mathcal{S}$-semi-open.

(2) Let A is $\mathcal{S}$-semi-open. Thus, $A \subseteq cl^{\theta}\big(int(A)\big)$. If we consider $int(A) = B$, then $B \in \tau$, since $int(A)$ is an open set of X and $B = int(A) \subseteq A$, which is contained in $cl^{\theta}(B)$. So, $B \subseteq A \subseteq cl^{\theta}(B)$.

Conversely, suppose that there exists $B \in \tau$ such that $B \subseteq A \subseteq cl^{\theta}(B)$. Thus, $B \subseteq int(A)$. By (4) of Theorem 3.8, $cl^{\theta}(B) \subseteq cl^{\theta}\big(int(A)\big)$. This implies $A \subseteq cl^{\theta}\big(int(A)\big)$. So, A is $\mathcal{S}$-semi-open.

(3) Since A is $\mathcal{S}$-semi-open, we get $A \subseteq cl^{\theta}\big(int(A)\big)$. Hence, from Theorem 3.10, we get $A \cap B \subseteq cl^{\theta}\big(int(A)\big) \cap B \subseteq cl^{\theta}\big(int(A) \cap B\big)$. As $B \in \tau$, we can write $B = int(B)$. Thus, $A \cap B \subseteq cl^{\theta}\big(int(A) \cap int(B)\big) = cl^{\theta}\big(int(A \cap B)\big)$. So, $A \cap B$ is $\mathcal{S}$-semi-open.

□

Definition 3.8. *Let $(X,\tau,\mathcal{S})$ be a siboid topological space. Then, a set $A \subseteq X$ is called $\mathcal{S}$-α-closed (respectively, $\mathcal{S}$-semi-closed, $\mathcal{S}$-pre-closed, $\mathcal{S}$-β-closed, $\mathcal{S}$-regular-closed) if $X - A$ is $\mathcal{S}$-α-open (respectively, $\mathcal{S}$-semi-open, $\mathcal{S}$-pre-open, $\mathcal{S}$-β-open, $\mathcal{S}$-regular-open).*

Theorem 3.32. *Let $(X,\tau,\mathcal{S})$ be a siboid topological space and $A \subseteq X$. Then, the following results hold:*

(1) *if A is a $\mathcal{S}$-α-closed set, then* $cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big) \subseteq A$,
(2) *if A is a $\mathcal{S}$-semi-closed set, then* $int\big(cl^{\theta}(A)\big) \subseteq A$,
(3) *if A is a $\mathcal{S}$-pre-closed set, then* $cl^{\theta}\big(int(A)\big) \subseteq A$,
(4) *if A is a $\mathcal{S}$-β-closed set, then* $int\Big(cl^{\theta}\big(int(A)\big)\Big) \subseteq A$,
(5) *if A is a $\mathcal{S}$-regular-closed set and a clopen set, then* $A = cl^{\theta}\big(int(A)\big)$.

Proof. (1) $(X - A)$ is $\mathcal{S}$-α-open set, whenever A is $\mathcal{S}$-α-closed. By Theorem 3.25, $(X - A)$ is an α-open set. Thus, $(X - A) \subseteq int\Big(cl\big(int(X - A)\big)\Big) = int\Big(cl\big(X - cl(A)\big)\Big) = int\Big(X - int\big(cl(A)\big)\Big) = X - \Big(cl\Big(int\big(cl(A)\big)\Big)\Big) \subseteq X - \Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)\Big) \Rightarrow cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big) \subseteq A$.

Similarly, we can obtain the proofs of (2), (3), and (4).

(5) Let A is a $\mathcal{S}$-regular-closed and clopen set in (X, τ). Then, $(X - A)$ is a $\mathcal{S}$-regular-open set. Then, $(X - A) = int(cl^{\theta}(X - A))$. Since $A \in \tau$, using (1) of Theorem 3.5, we get $(X - A)^* \subseteq (X - A) \Rightarrow cl^{\theta}(X - A) = (X - A) = cl(X - A)$. Thus, $X - A = int(cl(X - A)) = int(X - int(A)) = X - cl(int(A))$. This gives $A = cl(int(A))$.

Again, since $(X - A) \in \tau$, using (1) of Theorem 3.5, we get $A^* \subseteq A \Rightarrow cl^{\theta}(A) = A = cl(A) \Rightarrow cl^{\theta}(int(A)) = cl(int(A))$. Hence, $A = cl^{\theta}(int(A))$. □

Definition 3.9. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. Then, a subset $A \subseteq X$ is said to be a $\mathcal{S}$-dense set if $cl^{\theta}(A) = X$.*

Theorem 3.33. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space. Then, every $\mathcal{S}$-pre-open set can be written as the intersection of a $\mathcal{S}$-regular-open set and a $\mathcal{S}$-dense set.*

Proof. Consider $A \subseteq X$ be a $\mathcal{S}$-pre-open set. So, $A \subseteq int\big(cl^{\theta}(A)\big)$.

Then,

$$\begin{aligned} A &= int\big(cl^{\theta}(A)\big) \cap A \\ &= \Big(int\big(cl^{\theta}(A)\big) \cap A\Big) \cup \Big(int\big(cl^{\theta}(A)\big) \cap \Big(X - int\big(cl^{\theta}(A)\big)\Big)\Big) \\ &= int\big(cl^{\theta}(A)\big) \cap \Big(A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big)\Big) \end{aligned}$$

Now, $A \subseteq int\big(cl^{\theta}(A)\big)$ gives $int\big(cl^{\theta}(A)\big) \subseteq int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)\Big)$ and $int\big(cl^{\theta}(A)\big) \subseteq cl^{\theta}(A)$ gives $int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)\Big) \subseteq int\big(cl^{\theta}(A)\big)$. Thus, $int\big(cl^{\theta}(A)\big) = int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\big)\Big)\Big)$. So, $int\big(cl^{\theta}(A)\big)$ is a $\mathcal{S}$-regular-open set.

Again, $A \subseteq A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big) \Rightarrow cl^{\theta}(A) \subseteq cl^{\theta}\Big(A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big)\Big)$. It gives $int\big(cl^{\theta}(A)\big) \subseteq cl^{\theta}\Big(A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big)\Big)$. Also, $X - int\big(cl^{\theta}(A)\big) \subseteq A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big) \subseteq cl^{\theta}\Big(A \cup \Big(X - int\big(cl^{\theta}(A)\big)\Big)\Big)$.

Consequently, $cl^{\theta}\Big(A \cup \Big(X - int\Big(cl^{\theta}(A)\Big)\Big)\Big) = X$. Thus, $A \cup \Big(X - int\Big(cl^{\theta}(A)\Big)\Big)$ is a $\mathcal{S}$-dense set. Hence, A can be written as the intersection of a $\mathcal{S}$-regular-open set and a $\mathcal{S}$-dense set. □

4. Continuity on siboid topological spaces

In this section, we introduce some new types of functions in a topological space from the perspective of a siboid.

Definition 4.1. *A function $f\colon (X, \tau, \mathcal{S}) \to (X', \tau')$ is said to be $\mathcal{S}$-α-continuous (respectively, $\mathcal{S}$-semi-continuous, $\mathcal{S}$-pre-continuous, $\mathcal{S}$-β-continuous) if the inverse image of every open set in the topology τ' is $\mathcal{S}$-α-open (respectively, $\mathcal{S}$-semi-open, $\mathcal{S}$-pre-open, $\mathcal{S}$-β-open) in τ with respect to a siboid $\mathcal{S}$.*

Theorem 4.1. *Let $f\colon (X, \tau, \mathcal{S}) \to (X', \tau')$ be a function. Then, f is $\mathcal{S}$-semi-continuous and $\mathcal{S}$-pre-continuous if f is $\mathcal{S}$-α-continuous .*

Proof. Consider f is $\mathcal{S}$-α-continuous and $A \in \tau'$. Then, $f^{-1}(A)$ is $\mathcal{S}$-α-open in X. Also, by (1) of Theorem 3.26, $f^{-1}(A)$ is $\mathcal{S}$-semi-open and $\mathcal{S}$-pre-open in X. Thus, f is $\mathcal{S}$-semi-continuous and $\mathcal{S}$-pre-continuous. □

Theorem 4.2. *Let $f\colon (X, \tau, \mathcal{S}) \to (X', \tau')$ be a function. If f is a continuous function, then it is a $\mathcal{S}$-semi-continuous function.*

Proof. Let $A \in \tau'$. Since f is continuous, $f^{-1}(A) \in \tau$. Thus, $f^{-1}(A) = int\Big(f^{-1}(A)\Big) \subseteq cl^{\theta}\Big(int\Big(f^{-1}(A)\Big)\Big)$. Thus, $f^{-1}(A)$ is $\mathcal{S}$-semi-open. Hence, f is $\mathcal{S}$-semi-continuous. □

Remark 4.1. *The converse of Theorem 4.2 is not true in general.*

The above remark is verified by the following example:

Example 4.1. *Consider $(X, \tau, \mathcal{S})$ be a siboid topological space, where $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{S} = \{\{a, c\}, X\}$. Let (X', τ') be a topological space where $X' = \{x, y\}$ and $\tau' = \{\emptyset, \{x\}, X'\}$.*

Now, we consider $f{:}(X, \tau, \mathcal{S}) \to (X', \tau')$ be a function defined by $f(a) = f(b) = x$ and $f(c) = y$. Then, f is $\mathcal{S}$-semi-continuous since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X') = X$ and $f^{-1}(\{x\}) = \{a, b\}$. Here, $\emptyset, X$ and $\{a, b\}$ are $\mathcal{S}$-semi-open sets in X. However, f is not continuous because $\{x\} \in \tau'$ but $f^{-1}(\{x\}) = \{a, b\} \notin \tau$.

Definition 4.2. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and (X', τ') be a topological space. A function $f{:}(X', \tau') \to (X, \tau, \mathcal{S})$ is called a $\mathcal{S}$-semi-open function (respectively, $\mathcal{S}$-semi-closed function) if for any $E \in \tau'$ (respectively, $(X' - F) \in \tau'$), $f(E)$ is a $\mathcal{S}$-semi-open set (respectively, $f(X' - F)$ is a $\mathcal{S}$-semi-closed set) in $(X, \tau, \mathcal{S})$.*

Theorem 4.3. *Let $(X, \tau, \mathcal{S})$ be a siboid topological space and (X', τ') be a topological space. Then, a function $f{:}(X', \tau') \to (X, \tau, \mathcal{S})$ is a $\mathcal{S}$-semi-open function if and only if for all $x \in X'$ and for any neighborhood E' of x, there exists a $\mathcal{S}$-semi-open set $E \subseteq X$ satisfying $f(x) \in E \subseteq f(E')$.*

Proof. Given, $f:(X',\tau') \to (X,\tau,\mathcal{S})$ be a $\mathcal{S}$-semi-open function. Suppose that $x \in X'$ and E' be a neighborhood of x. Hence, there exists $F' \in \tau'$ satisfying $x \in F' \subseteq E'$. Since f is $\mathcal{S}$-semi-open function, then $f(F')$ is a $\mathcal{S}$-semi-open set in X. If we take $f(F') = E$, then we get $f(x) \in E \subseteq f(E')$.

For the converse part, we consider $E' \in \tau'$. Thus, for any $x \in E'$, there exists $\mathcal{S}$-semi-open set $E_x \subseteq X$ satisfying $f(x) \in E_x \subseteq f(E')$. Hence, $f(E') = \bigcup\{E_x : x \in E'\}$. By (2) of theorem 3.27, $f(E')$ is a $\mathcal{S}$-semi-open set in X. So, f is a $\mathcal{S}$-semi-open function. □

Theorem 4.4. *Let f: $(X',\tau') \to (X,\tau,\mathcal{S})$ be a bijective $\mathcal{S}$-semi-open function. If $E \subseteq X$ and $F' \subseteq X'$ is a closed set satisfying $f^{-1}(E) \subseteq F'$, then there exists a $\mathcal{S}$-semi-closed set G in X such that $E \subseteq G$.*

Proof. Let f: $(X',\tau') \to (X,\tau,\mathcal{S})$ be a $\mathcal{S}$-semi-open function, $E \subseteq X$ and $F' \subseteq X'$ be a closed set such that $f^{-1}(E) \subseteq F'$. Since F' is closed set, $(X' - F')$ is open in (X',τ'). Thus, $f(X' - F')$ is a $\mathcal{S}$-semi-open set. Hence, $X - f(X' - F')$ is a $\mathcal{S}$-semi-closed set in X. Given f is a bijection, hence $f(F') \subseteq X - f(X' - F')$. If we take $G = X - f(X' - F')$, then we get $E \subseteq f(F') \subseteq G$. □

Theorem 4.5. *Let f: $(X',\tau') \to (X,\tau,\mathcal{S})$ be a bijection. Then, f^{-1} is $\mathcal{S}$-semi-continuous if and only if f is $\mathcal{S}$-semi-open.*

Proof. Let f^{-1} is a $\mathcal{S}$-semi-continuous function. Let $A' \in \tau'$. Thus, we get $(f^{-1})^{-1}(A') = f(A')$ is a $\mathcal{S}$-semi-open set in $(X,\tau,\mathcal{S})$. Hence, f is a $\mathcal{S}$-semi-open function.

For the converse part, consider f to be a $\mathcal{S}$-semi-open set and $A' \in \tau'$. Then, $(f^{-1})^{-1}(A') = f(A')$ is a $\mathcal{S}$-semi-open set. Hence, f^{-1} is $\mathcal{S}$-semi-continuous. □

Definition 4.3. *A function $f : (X,\tau,\mathcal{S}) \to (X',\tau')$ is said to be $\mathcal{S}$-regular-continuous if the inverse image of every open set in (X',τ') is $\mathcal{S}$-regular-open in $(X,\tau,\mathcal{S})$.*

Theorem 4.6. *Let f: $(X,\tau,\mathcal{S}) \to (X',\tau')$ be a function. Then, the following results are equivalent:*

(1) *f is $\mathcal{S}$-regular-continuous,*
(2) *f is $\mathcal{S}$-pre-continuous as well as the inverse image of every open set in (X',τ') is $\mathcal{S}$-semi-closed in $(X,\tau,\mathcal{S})$.*

Proof. (1) ⇒ (2) Let f is $\mathcal{S}$-regular-continuous. Then, for $A \in \tau'$, we get $f^{-1}(A)$ is $\mathcal{S}$-regular-open in $(X,\tau,\mathcal{S})$. That is, $f^{-1}(A) = int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big)$.

From this, $f^{-1}(A) \subseteq int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big)$ gives f is $\mathcal{S}$-pre-continuous. On the other hand, $int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big) \subseteq f^{-1}(A)$ gives that the inverse image of an open set $A \in \tau'$ is $\mathcal{S}$-semi-closed.

(2) ⇒ (1) Let f is $\mathcal{S}$-pre-continuous and the inverse image of every open set in (X',τ') is $\mathcal{S}$-semi closed in $(X,\tau,\mathcal{S})$. Thus, for $A \in \tau'$ we get, $f^{-1}(A) \subseteq int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big)$ and $int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big) \subseteq f^{-1}(A)$. This gives $f^{-1}(A) = int\Big(cl^{\theta}\Big(f^{-1}(A)\Big)\Big)$. Hence, f is $\mathcal{S}$-regular-continuous. □

5. Connection of siboid topological space with Boolean algebra

In this section, we try to establish the relation between siboid topological space and Boolean algebra.

Theorem 5.1. *The set, say $\mathcal{M}$, of all $\mathcal{S}$-regular-open sets which are also clopen in a siboid toplogical space $(X, \tau, \mathcal{S})$ with property $\mathcal{R}$ forms a Boolean algebra $\mathcal{B} = (\mathcal{M}, \wedge, \vee, 0, 1,')$ with respect to the Boolean elements and operations defined by:*

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= X, \\ A \wedge B &= A \cap B, \\ A \vee B &= int\Big(cl^{\theta}(A \cup B)\Big), \\ A' &= int(X - A), \end{aligned}$$

where $A, B \in \mathcal{M}$.

Proof. $\emptyset$ and X are $\mathcal{S}$-regular-open sets since $int\Big(cl^{\theta}(\emptyset)\Big) = \emptyset$ and $int\Big(cl^{\theta}(X)\Big) = X$. If A and B are $\mathcal{S}$-regular-open sets, then using Theorems 3.28 and 3.29, $A \cap B$ and $int\Big(cl^{\theta}(A \cup B)\Big)$ are $\mathcal{S}$-regular-open sets. Again, from Theorem 3.30, we get $int(X - A)$ is a $\mathcal{S}$-regular-open set. Now, we shall show that these operations satisfy the axioms of Boolean algebra [29].

(1) $\emptyset' = int(X - \emptyset) = int(X) = X$ and $X' = int(X - X) = int(\emptyset) = \emptyset$.

(2) (Identity laws)
$A \wedge \emptyset = A \cap \emptyset = \emptyset$ and $A \vee X = int\Big(cl^{\theta}(A \cup X)\Big) = int\Big(cl^{\theta}(X)\Big) = int(X) = X$.

(3) (Boundedness laws)
$A \wedge X = A \cap X = A$ and $A \vee \emptyset = int\Big(cl^{\theta}(A \cup \emptyset)\Big) = int\Big(cl^{\theta}(A)\Big) = A$, since A is a $\mathcal{S}$-regular-open set.

(4) (Complement laws)
$A \wedge A' = A \cap A' = int(A) \cap int(X - A) = int\Big(A \cap (X - A)\Big) = int(\emptyset) = \emptyset$.
$A \vee A' = int\Big(cl^{\theta}(A \cup A')\Big) = int\Big(cl^{\theta}\Big(A \cup int(X - A)\Big)\Big) = int\Big(cl^{\theta}(A) \cup cl^{\theta}\Big(int(X - A)\Big)\Big) = int\Big((A \cup A^{*}) \cup (X - A)\Big) = int\Big(\Big(A \cup (X - A)\Big) \cup A^{*}\Big) = int(X \cup A^{*}) = int(X) = X$.

(5) (Involution law)
$(A')' = int\Big(X - int(X - A)\Big) = X - cl\Big(int(X - A)\Big) \subseteq X - cl^{\theta}\Big(int(X - A)\Big) = X - (X - A) = A$.
Again, $A = int\Big(cl^{\theta}(A)\Big) \subseteq cl^{\theta}(A) \subseteq cl(A) = X - int(X - A) \Rightarrow int(A) \subseteq int\Big(X - int(X - A)\Big) \Rightarrow A \subseteq int\Big(X - int(X - A)\Big) \Rightarrow A \subseteq (A')'$. Thus, $(A')' = A$.

(6) (Idempotent laws)
$A \wedge A = A \cap A = A$ and $A \vee A = int\Big(cl^{\theta}(A \cup A)\Big) = int\Big(cl^{\theta}(A)\Big) = A$.

(7) (De Morgan's laws)
$(A \wedge B)' = int\Big(X - (A \cap B)\Big) = int\Big((X - A) \cup (X - B)\Big)$. Now, A is $\mathcal{S}$-regular-open set implies $(X - A)$ is $\mathcal{S}$-regular-closed set. This gives $cl^{\theta}\Big(int(X - A)\Big) = (X - A)$. Similarly, we get $cl^{\theta}\Big(int(X - B)\Big) = (X - B)$. Hence, we have

$$\begin{aligned} A' \vee B' &= \mathrm{int}(X - A) \vee \mathrm{int}(X - B) \\ &= \mathrm{int}\Big(cl^{\theta}(\mathrm{int}(X - A) \cup \mathrm{int}(X - B))\Big) \end{aligned}$$

$$\begin{aligned} &= \operatorname{int}\Big(cl^{\theta}(\operatorname{int}(X-A)) \cup cl^{\theta}(\operatorname{int}(X-B))\Big) \\ &= \operatorname{int}((X-A) \cup (X-B)). \end{aligned}$$

Thus, $(A \wedge B)' = A' \vee B'$.

Again, A and B are clopen sets. So, using (3) and (8) of Theorem 3.8, we get $cl^{\theta}(A) = cl(A)$ and $cl^{\theta}(B) = cl(B)$.

Now,

$$\begin{aligned} A' \wedge B' &= int(X-A) \cap int(X-B) \\ &= int\Big((X-A) \cap (X-B)\Big) \\ &= int\Big(X - (A \cup B)\Big) \\ &= int\Big(X - int(A \cup B)\Big) \\ &= int\Big(X - int\Big(cl(A) \cup cl(B)\Big)\Big) \\ &= int\Big(X - int\Big(cl^{\theta}(A) \cup cl^{\theta}(B)\Big)\Big) \\ &= int\Big(X - int\Big(cl^{\theta}(A \cup B)\Big)\Big) \\ &= (A \vee B)'. \end{aligned}$$

Thus, $A' \wedge B' = (A \vee B)'$.

(8) (Commutative laws)

$A \wedge B = A \cap B = B \cap A = B \wedge A$ and $A \vee B = int\Big(cl^{\theta}(A \cup B)\Big) = int\Big(cl^{\theta}(B \cup A)\Big) = B \vee A$.

(9) (Associative laws)

$$\begin{aligned} & A \vee (B \vee C) \\ = \;& int\Big(cl^{\theta}\Big(A \cup int\Big(cl^{\theta}(B \cup C)\Big)\Big)\Big) \\ = \;& int\Big(cl^{\theta}\Big(int\Big(cl^{\theta}(A)\Big) \cup int\Big(cl^{\theta}(B \cup C)\Big)\Big)\Big) \\ \subseteq \;& int\Big(cl^{\theta}\Big(int\Big(cl^{\theta}(A) \cup cl^{\theta}(B \cup C)\Big)\Big)\Big) \\ = \;& int\Big(cl^{\theta}\Big(int\Big(cl^{\theta}\Big(A \cup (B \cup C)\Big)\Big)\Big)\Big) \\ = \;& int\Big(cl^{\theta}\Big(int\Big(cl^{\theta}\Big((A \cup B) \cup C\Big)\Big)\Big)\Big) \\ \subseteq \;& int\Big(cl^{\theta}\Big(cl^{\theta}\Big((A \cup B) \cup C\Big)\Big)\Big) \\ = \;& int\Big(cl^{\theta}\Big((A \cup B) \cup C\Big)\Big) \end{aligned}$$

$$\begin{aligned}
&= int\Big(cl^{\theta}\Big(\big(int\big(cl^{\theta}(A)\big)\cup int\big(cl^{\theta}(B)\big)\big)\cup C\Big)\Big)\\
&\subseteq int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A)\cup cl^{\theta}(B)\big)\cup C\Big)\Big)\\
&= int\Big(cl^{\theta}\Big(int\big(cl^{\theta}(A\cup B)\big)\cup C\Big)\Big)\\
&= (A\vee B)\vee C
\end{aligned}$$

In a similar manner, we can show $(A \vee B) \vee C \subseteq A \vee (B \vee C)$. Consequently, $A \vee (B \vee C) = (A \vee B) \vee C$. Also, $A \wedge (B \wedge C) = A \wedge (B \cap C) = A \cap (B \cap C) = (A \cap B) \cap C = (A \wedge B) \cap C = (A \wedge B) \wedge C$.

(10) (Distributive laws)

$$\begin{aligned}
A\wedge(B\vee C) &= A\wedge int\big(cl^{\theta}(B\cup C)\big)\\
&= A\cap int\big(cl^{\theta}(B\cup C)\big)\\
&= int\big(cl^{\theta}(A)\big)\cap int\big(cl^{\theta}(B\cup C)\big)\\
&= int\Big(cl^{\theta}\big(A\cap(B\cup C)\big)\Big)\\
&= int\Big(cl^{\theta}\big((A\cap B)\cup(A\cap C)\big)\Big)\\
&= int\big(cl^{\theta}(A\cap B)\big)\cup int\big(cl^{\theta}(A\cap C)\big)\\
&= (A\cap B)\vee(A\cap C)\\
&= (A\wedge B)\vee(A\wedge C)
\end{aligned}$$

Since B and C are clopen, thus $B \cap C$ is also clopen. Using (3) and (8) of Theorem 3.8, we get $B \cap C = int(B \cap C) = cl(B \cap C) = cl^{\theta}(B \cap C)$. Also, $B = int(B) = cl(B) = cl^{\theta}(B)$ and $C = int(C) = cl(C) = cl^{\theta}(C)$.

Thus,

$$\begin{aligned}
A\vee(B\wedge C) &= int\Big(cl^{\theta}\big(A\cup(B\cap C)\big)\Big)\\
&= int\big(cl^{\theta}(A)\cup cl^{\theta}(B\cap C)\big)\\
&= int\big(cl^{\theta}(A)\cup int(B\cap C)\big)\\
&= int\Big(cl^{\theta}(A)\cup\big(int(B)\cap int(C)\big)\Big)\\
&= int\Big(\big(cl^{\theta}(A)\cup int(B)\big)\cap\big(cl^{\theta}(A)\cup int(C)\big)\Big)\\
&= int\Big(\big(cl^{\theta}(A)\cup cl^{\theta}(B)\big)\cap\big(cl^{\theta}(A)\cup cl^{\theta}(C)\big)\Big)\\
&= int\Big(\big(cl^{\theta}(A\cup B)\big)\cap\big(cl^{\theta}(A\cup C)\big)\Big)
\end{aligned}$$

$$\begin{aligned} &= int\big(cl^{\theta}(A \cup B)\big) \cap int\big(cl^{\theta}(A \cup C)\big) \\ &= (A \vee B) \wedge (A \vee C) \end{aligned}$$

□

Theorem 5.2. *The Boolean algebra $\mathcal{B} = (\mathcal{M}, \wedge, \vee, 0, 1,')$ defined in Theorem 5.1 is a bounded, distributive and complemented lattice.*

Proof. Let $\mathcal{B} = (\mathcal{M}, \wedge, \vee, 0, 1,')$ be the Boolean algebra as defined in Theorem 5.1, formed by the set of all $\mathcal{S}$-regular-open sets which are also clopen sets of a siboid topological space $(X, \tau, \mathcal{S})$ with property $\mathcal{R}$. We show that the relation $A \leq B$ defined by $A \wedge B = A$ or $A \vee B = B$ is a partial order on the class of $\mathcal{S}$-regular open sets which are also clopen sets of $(X, \tau, \mathcal{S})$.

(1) *Reflexivity:* $A \leq A$ as $A \wedge A = A$ or $A \vee A = A$ for all $A \in \mathcal{M}$.
(2) *Antisymmetry:* Let $A \leq B$ and $B \leq A$ where $A, B \in \mathcal{M}$. Then, $A \wedge B = A$ and $B \wedge A = B$. So, $A \cap B = A$ and $B \cap A = B$. Since $A \cap B = B \cap A$, this gives $A = B$.
(3) *Transitivity:* Let $A \leq B$ and $B \leq C$, where $A, B, C \in \mathcal{M}$. Then, $A \wedge B = A$ and $B \wedge C = B$. Hence, $A \cap B = A$ and $B \cap C = B$. This implies $A \subseteq B$ and $B \subseteq C$. Thus, we get $A \subseteq C$. Hence, $A \cap C = A$. Thus, $A \wedge C = A$. Hence, $A \leq C$.

Now, $\mathcal{B}$ satisfies the idempotent, associative and commutative laws as shown in Theorem 5.1. Here, $\mathcal{B}$ satisfies the absorption laws as well, since $A \wedge (A \vee B) = (A \vee \emptyset) \wedge (A \vee B) = A \vee (\emptyset \wedge B) = A \vee \emptyset = A$ and $A \vee (A \wedge B) = (A \wedge X) \vee (A \wedge B) = A \wedge (X \vee B) = A \wedge X = A$. Thus, $\mathcal{B}$ forms a lattice where $\vee$ and $\wedge$ are join and meet operations, respectively.

In this lattice, $A \wedge \emptyset = \emptyset$ implies $\emptyset \leq A$ and $A \vee X = X$ implies $A \leq X$, for any $A \in \mathcal{M}$. Thus, $\mathcal{B}$ is a bounded lattice. Furthermore, Theorem 5.1 shows that complement laws and distributive laws are satisfied by the elements of $\mathcal{B}$. Accordingly, we have $\mathcal{B}$ is a bounded, distributive and complemented lattice. □

6. Open questions

Naturally, in any emerging area of research, the number of open questions frequently surpasses the progress achieved. Our aim is to build upon this work by investigating additional applications of the siboid and assessing its potential significance within topology. In this context, we emphasize three particular questions of interest:

Question 1. Is the converse of statement (3) in Theorem 3.26 valid? If so, under what conditions does it hold?

Question 2. In a siboid topological space $(X, \tau, \mathcal{S})$, does every $\mathcal{S}$-semi-open set also qualify as a $\mathcal{S}$-pre-open set?

Question 3. In a siboid topological space $(X, \tau, \mathcal{S})$, is every $\mathcal{S}$-pre-open set necessarily $\mathcal{S}$-semi-open?

7. Conclusion

In this paper, we defined a new mathematical structure on a finite set called 'siboid' and we connected it to a topological space to get 'siboid topological space'. We further investigated some basic characteristics and introduced two operators $(.)^*$ and Φ on a siboid topological space. Additionally, a new topology, the siboid topology, was established using a new map cl^θ that fulfils the Kuratowski closure axioms whenever a siboid satisfies the property $\mathcal{R}$. The siboid topology is finer than the original topology of the siboid topological space. We also demonstrated that the topology induced by the operator Φ is equivalent to the siboid topology. Additionally, we defined five generalized open sets on the siboid topological space and examined a number of their characteristics. In section 4, we examined several kinds of continuity maps on siboid topological space. In section 5, we attempted to use a lattice structure to link this siboid topological space with Boolean algebra.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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