

Effective Conversion of Non-Prime Graphs to Prime Graphs**Karnam Gurunadhan Tharunraj, P. Ragukumar****Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore,
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Abstract. A graph G is considered to have a prime labeling when each of its n vertices is assigned a unique label from the set $\{1, 2, 3, 4, \dots, n\}$, ensuring that the labels of any two connected vertices are coprime. In the literature, many graph classes identified as prime graphs and non-prime graphs. In this paper, we focus on converting non-prime graph classes to a prime graph classes by applying corona product with complete graph on one vertex.

1. INTRODUCTION

A graph G can be described as an ordered triple $(V(G), E(G), \psi(G))$ consisting of a non empty set $V(G)$, a set $E(G)$ of edges, and an incidence function $\psi(G)$, that associates with each edge of G an unordered pair of vertices of G [1]. For definitions and terminologies, we refer [1]. Throughout this paper, we consider simple, finite and undirected graphs.

A Graph Labeling is an assignment of integers to vertices or edges or both, subject to certain conditions [3]. There are various types of graph labelings, including Prime labeling, Prime Cordial labeling, Vertex Prime labeling, Magic labeling, Anti-Magic labeling, Harmonious labeling, Lucky labeling.

The concept of prime labeling was introduced by Roger Entringer in the year 1980 and was considered by Tout, Dabboucy and Howalla in the paper titled Prime labeling of graphs [4]. A graph G is considered to have a prime labeling when each of its n vertices is assigned a unique label from the set $\{1, 2, 3, 4, \dots, n\}$, ensuring that the labels of any two connected vertices are coprime. For an extensive survey on graph labeling, one can refer to the dynamic survey by Gallian [3]. We refer [11] to the work on lucky labeling that examines how the lucky number changes under operations such as adding a graph G to K_n or deleting an edge from K_n . Similarly,

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this work studies the impact of graph operations on prime labeling.

TABLE 1. The following table helps us to prove our main results

Graph class	Types	Reference
Cycle (C_n)	Prime Graph	Deretsky et al. (1991) [9]
Wheel (W_n), n is even	Prime Graph	Lee et al. (1988) [10]
Square of a Path (P_n^2), $n \geq 6, n \neq 7$	Non-Prime Graph	Seoud et al. (1999) [5]
Square of a Cycle (C_n^2), $n \geq 4$	Non-Prime Graph	Seoud et al. (1999) [5]
Möbius ladder (M_n), n is even	Non-Prime Graph	Seoud et al. (1999) [5]
$D(P_n)$	Non-Prime Graph	Vaidya et al.(2013) [6]
$D(C_n)$	Non-Prime Graph	Vaidya et al.(2013) [6]
$D(K_{1,n})$	Non-Prime Graph	Vaidya et al.(2013) [6]

From the above table, we observe that there are many graph classes proved to be non-prime graph. This motivated us to raise a question “Are there any graph operations to covert non-prime graph to a prime graph”. In this paper, we will prove that by applying corona product of complete graph on one vertex with various non-prime graph classes are proved to be prime graph.

2. MAIN RESULTS

To prove the following main results , we need the following definition

Definition 2.1. [8] Let G and H be two graphs and let n be the order of G . The corona product, or simply the corona, of graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and n copies of H and then joining by an edge the i^{th} vertex of G to every vertex in the i^{th} copy of H . Given a vertex $g \in G$, the copy of H connected to g is denoted by H_g .

Theorem 2.1. Let $G = P_n^2$ be the square of a path graph with order n . Then the corona product $P_n^2 \circ K_1$ admits prime labeling for all $n \geq 3$.

Proof. Let $G = P_n^2$ be the square of a path graph with vertex set $\{v_1, v_2, v_3, \dots, v_n\}$. In the corona product, for each vertex $v_i \in V(P_n^2)$, a new pendant vertex u_i is attached. Thus the total number of vertices in $P_n^2 \circ K_1$ is $2n$. Let us define the mapping $f : V(P_n^2 \circ K_1) \rightarrow \{1, 2, 3, \dots, 2n\}$. We aim to assign labels from the set $\{1, 2, 3, \dots, 2n\}$ to the vertices of $P_n^2 \circ K_1$ such that adjacent vertices always receive relatively prime labels. We prove this by mathematical induction on n .

Base Case $n = 3$:

Consider the graph $P_3^2 \circ K_1$, where the vertex set is $\{v_1, v_2, v_3\}$ from P_3^2 and $\{u_1, u_2, u_3\}$ are the pendant vertices added in the corona product. The vertices v_1, v_2, v_3 are labeled with the consecutive odd integers 1, 3, 5 respectively. The corresponding pendant vertices u_1, u_2, u_3 are labeled with consecutive even integers 2, 4, 6 respectively. Observe that each pair of adjacent vertices of $P_3^2 \circ K_1$ receives consecutive integers. Therefore, we know that two consecutive non zero integers are relatively prime and two consecutive non zero odd integers are relatively prime. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

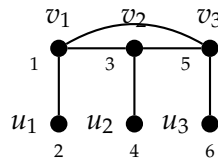


FIGURE 1. prime labeling of $P_3^2 \circ K_1$.

Inductive hypothesis $n = k$:

Assume that for some $n = k \geq 3$, the graph $P_k^2 \circ K_1$ admits prime labeling using labels $\{1, 2, 3, \dots, 2k\}$.

Inductive Step $n = k + 1$:

We now extend this labeling to the graph $P_{k+1}^2 \circ K_1$. The graph $P_{k+1}^2 \circ K_1$ is constructed from $P_k^2 \circ K_1$ by adding one new vertex v_{k+1} to the square of a path P_k^2 , along with a corresponding pendant vertex u_{k+1} . In square of a path, v_{k+1} is adjacent to v_k and v_{k-1} , as per the squaring operation. We assign the next available odd and even integers to these new vertices v_{k+1} receives $2k + 1$ and u_{k+1} receives the label $2k + 2$. To verify the coprime condition is preserved, observe that v_{k+1} is adjacent to v_k , which has the label $2k - 1$ under the inductive hypothesis. Since $2k - 1$ and $2k + 1$ are consecutive odd integers, they are relatively prime. Similarly, v_{k+1} is adjacent to v_{k-1} , which has the label $2k - 3$ under the inductive hypothesis. To verify that these two labels are relatively prime, we apply Euclidean algorithm, $\gcd(2k + 1, 2k - 3) = \gcd(2k - 3, 4)$. Since $2k - 3$ is an odd number, it is not divisible by 2 or 4. Thus, the only divisor between $2k - 3$ and 4 is 1, implying $\gcd(2k + 1, 2k - 3) = 1$. The new vertices v_{k+1} and u_{k+1} receives the labels $2k + 1$ and $2k + 2$ respectively, which are consecutive integers. Therefore, we know that consecutive integers are relatively prime. By the principle of mathematical induction, the corona product of $P_n^2 \circ K_1$ admits prime labeling for all $n \geq 3$. $\square$

Theorem 2.2. Let $G = C_n^2$ be the square of a cycle graph with order n . Then the corona product $C_n^2 \circ K_1$ admits prime labeling for all $n \geq 4$.

Proof. Let $G = C_n^2$ be the square of a cycle graph with vertex set $\{v_1, v_2, v_3, \dots, v_n\}$. In the corona product, for each vertex $v_i \in V(C_n^2)$, a new pendant vertex u_i is attached. Thus, the total number of vertices in $C_n^2 \circ K_1$ is $2n$. Let us define the mapping $f : V(C_n^2 \circ K_1) \rightarrow \{1, 2, 3, \dots, 2n\}$. We aim

to assign labels from the set $\{1, 2, 3, \dots, 2n\}$ to the vertices of $C_n^2 \circ K_1$ such that adjacent vertices always receive relatively prime labels. We prove this by mathematical induction on n .

Base Case $n = 4$:

Consider the graph $C_4^2 \circ K_1$, where the vertex set is $\{v_1, v_2, v_3, v_4\}$ from C_4^2 and $\{u_1, u_2, u_3, u_4\}$ are the pendant vertices added in the corona product. We fix the labels for the first two vertices of C_4^2 and their corresponding pendant vertices as follows, $f(v_1) = 1$, $f(v_2) = 4$, $f(u_1) = 2$ and $f(u_2) = 3$. The vertices v_3, v_4 are labeled with the consecutive odd integers 5, 7 respectively. The corresponding pendant vertices u_3, u_4 are labeled with consecutive even integers 6, 8 respectively. Observe that each pair of adjacent vertices of $C_4^2 \circ K_1$ receives consecutive integers. Therefore, we know that two consecutive non zero integers are relatively prime and two consecutive non zero odd integers are relatively prime. The number 1 is relatively prime to every natural number. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

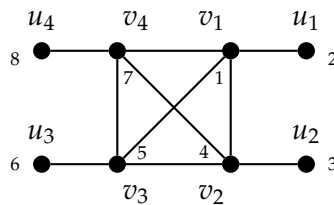


FIGURE 2. Prime labeling of $C_4^2 \circ K_1$.

Inductive hypothesis $n = k$:

Assume that for some $n = k \geq 4$, the graph $C_k^2 \circ K_1$ admits prime labeling using labels $\{1, 2, 3, \dots, 2k\}$.

Inductive Step $n = k + 1$:

We now extend this labeling to $C_{k+1}^2 \circ K_1$. We construct $C_{k+1}^2 \circ K_1$ from $C_k^2 \circ K_1$ by adding a new vertex v_{k+1} between v_k and v_1 , extending the cycle. The resulting cycle becomes $v_1 v_2 v_3 \dots, v_k v_{k+1} v_1$. In the square of this cycle, each vertex is connected to its neighbors and vertices at distance two. Thus, the new vertex v_{k+1} becomes adjacent to v_k, v_1, v_{k-1} and v_2 . Additionally, we attach a new pendant vertex u_{k+1} to v_{k+1} . Since, v_{k+1} is inserted between v_k and v_1 , the distance between v_1 and v_{k-1} become 3, so the edge between v_1 and v_{k-1} that existed in C_k^2 is no longer present in C_{k+1}^2 . Now, we extend the labeling, we assign the next two available integers. Let $f(v_{k+1}) = 2k + 1$ and $f(u_{k+1}) = 2k + 2$. To verify the coprime condition is preserved, observe that v_{k+1} is adjacent to v_k , which has the label $2k - 1$ under the inductive hypothesis. Since $2k - 1$ and $2k + 1$ are consecutive odd integers, they are relatively prime. Similarly, v_{k+1} is adjacent to v_{k-1} , which has the label $2k - 3$ under the inductive hypothesis. To verify that these two labels are relatively prime, we apply Euclidean algorithm, $\gcd(2k + 1, 2k - 3) = \gcd(2k - 3, 4)$. Since $2k - 3$ is an odd number, it is not divisible by 2 or 4. Thus, the only divisor between $2k - 3$ and 4 is 1, implying $\gcd(2k + 1, 2k - 3) = 1$. The new vertices v_{k+1} and u_{k+1} receives the labels $2k + 1$ and $2k + 2$

respectively, which are consecutive integers. Therefore, we know that consecutive integers are relatively prime. By the principle of mathematical induction, the corona product of $C_n^2 \circ K_1$ admits prime labeling for all $n \geq 4$. $\square$

To prove our next result, we need the following definition

Definition 2.2. [7] Consider the graph M_n , where $n \geq 6$, consisting of a polygon of length n and all $n/2$ chords joining opposite pair of vertices. This graph is called the Möbius ladder, and it is defined only for even values of n .

Theorem 2.3. Let $G = M_n$ be the Möbius ladder of order $n \geq 6$, where n is even. Then the corona product $M_n \circ K_1$ admits a prime labeling for all $n \geq 6$.

Proof. Let $G = M_n$ be the Möbius ladder of order $n \geq 6$, where n is even with vertex set $\{v_1, v_2, v_3, \dots, v_n\}$. In the corona product, for each vertex $v_i \in V(M_n)$, a new pendant vertex u_i is attached. Thus the total number of vertices in $M_n \circ K_1$ is $2n$. Let us define the mapping $f : V(M_n \circ K_1) \rightarrow \{1, 2, 3, \dots, 2n\}$. We aim to assign labels from the set $\{1, 2, 3, \dots, 2n\}$ to the vertices of $M_n \circ K_1$ such that adjacent vertices always receive relatively prime labels. We prove this by mathematical induction on n .

Base Case $n = 6$:

Consider the graph $M_6 \circ K_1$, where the vertex set is $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ from M_6 and $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ are the pendant vertices added in the corona product. We fix the labels for the first two vertices of M_6 and their corresponding pendant vertices as follows, $f(v_1) = 1$, $f(v_2) = 4$, $f(u_1) = 2$ and $f(u_2) = 3$. These assignments remain same for all values of $n \geq 6$. We assign the labels to the vertices $\{v_3, v_4, v_5, v_6\}$ as follows, $f(v_i) = 2i - 1, i = 3, 4, 5, 6$ and similarly the labels for their corresponding pendant vertices u_i are assigned as $f(u_i) = 2i, i = 3, 4, 5, 6$. Observe that each pair of adjacent vertices of $M_6 \circ K_1$ receives consecutive integers except 1. Therefore, we know that two consecutive integers are relatively prime and two consecutive odd integers are relatively prime. The number 1 is relatively prime to every natural number. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

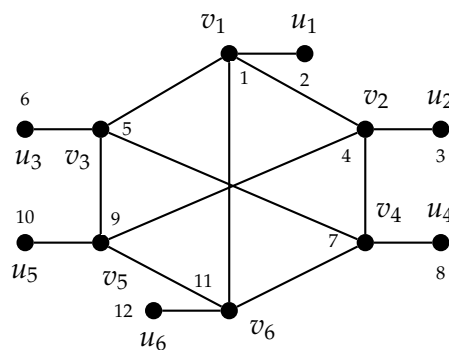


FIGURE 3. Prime labeling of $M_6 \circ K_1$.

Inductive hypothesis $n = k$:

Assume that for some $n = k \geq 6$, the graph $M_k \circ K_1$ admits prime labeling using labels $\{1, 2, 3, \dots, 2k\}$.

Inductive Step $n = k + 1$:

We now extend this labeling to the graph $M_{k+1} \circ K_1$. To construct $M_{k+1} \circ K_1$, we extend $M_k \circ K_1$ by adding 4 new vertices namely v_{k+1} and v_{k+2} , along their corresponding pendant vertices u_{k+1} and u_{k+2} . We now introduce new edges that maintain the Möbius ladder structure. Specifically, we connect v_1 to v_{k+2} and v_2 to v_{k+1} . The newly added vertices v_{k+1} and v_{k+2} are linked to each other, while v_{k+2} connects back to v_k and v_{k+1} connects to v_{k-1} . We remove the following existing edges from $M_k \circ K_1$. The edge between v_1 and v_k and the edge between v_2 and v_{k-1} are removed to maintain the Möbius ladder structure in $M_{k+1} \circ K_1$. Now, we extend the labeling function to these new vertices. Assign labels to the new Möbius ladder vertices using next available two consecutive odd integers as $f(v_{k+1}) = 2k + 1$, $f(v_{k+2}) = 2k + 3$. Assign labels to the new pendant vertices using the next available two consecutive even integers as $f(u_{k+1}) = 2k + 2$ and $f(u_{k+2}) = 2k + 4$. Observe that v_{k+1} and v_{k+2} which are adjacent receives consecutive odd integers, they are relatively prime. The adjacent vertices v_{k+1}, u_{k+1} and v_{k+2}, u_{k+2} receives consecutive integers, they are relatively prime. Now, to verify the gcd of v_2 and v_{k+1} , we apply Euclidean algorithm, $\gcd(f(v_2), f(v_{k+1})) = \gcd(4, 2k + 1)$. Since $2k + 1$ is an odd number, it is not divisible by 4. Thus, the only positive common divisor between $2k + 1$ and 4 are 1, implying $\gcd(4, 2k + 1) = 1$. By the principle of mathematical induction, the corona product of $M_n \circ K_1$ admits prime labeling for all $n \geq 6$. $\square$

To prove our next result, we need the following definition

Definition 2.3. [6] Duplication of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.

Theorem 2.4. The corona product of the graph obtained by duplicating every vertex of the path graph P_n by an edge with K_1 denoted $D(P_n) \circ K_1$, admits a prime labeling for all $n \geq 1$.

Proof. Let P_n be the path graph with vertex set $\{v_1, v_2, v_3, \dots, v_n\}$ and edge set $\{e_1, e_2, e_3, \dots, e_{n-1}\}$. Let the graph obtained by duplicating every vertex of the path graph by edges is denoted as $D(P_n)$. Then $|V(D(P_n))| = 3n$ and $|E(D(P_n))| = 4n - 1$. In the corona product $v_i \in V(D(P_n))$, a new pendant vertex u_i is attached. Thus, the total number of vertices in $D(P_n) \circ K_1$ is $6n$. Let us define the mapping $f : V(D(P_n) \circ K_1) \rightarrow \{1, 2, 3, \dots, 6n\}$. We aim to assign labels from the set $\{1, 2, 3, \dots, 6n\}$ to the vertices of $D(P_n) \circ K_1$ such that adjacent vertices always receive relatively prime labels.

Base Case $n = 1$:

Consider the graph $D(P_1) \circ K_1$, where the vertex set is $\{v_1, v'_1, v''_1\}$ from $D(P_1)$ and $\{u_1, u'_1, u''_1\}$ are the pendant vertices added in the corona product. We assign the labels to the vertices $\{v_1, v'_1, v''_1\}$ as follows, $f(v_1) = 5$, $f(v'_1) = 1$, $f(v''_1) = 3$ and similarly the labels for their corresponding

pendant vertices u_i are assigned as $f(u_1) = 6$, $f(u'_1) = 2$, $f(u''_1) = 4$. Observe that each pair of adjacent vertices of $D(P_1) \circ K_1$ receives consecutive integers except 1. Therefore, we know that two consecutive integers are relatively prime and two consecutive odd integers are relatively prime. The number 1 is relatively prime to every natural number. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

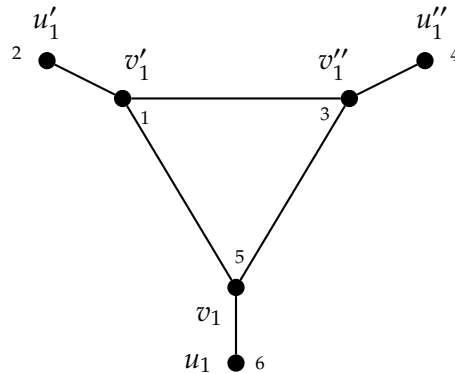


FIGURE 4. Prime labeling of $D(P_1) \circ K_1$.

Inductive hypothesis $n = k$:

Assume that for some $k \geq 1$, the graph $D(P_k) \circ K_1$ admits a prime labeling using the labels $\{1, 2, 3, \dots, 6k\}$.

Inductive Step $n = k + 1$:

To extend this labeling to $D(P_{k+1}) \circ K_1$, we construct the $(k + 1)^{th}$ block and add it to the existing structure. This new block consists of a triangle formed by three vertices, a new path vertex v_{k+1} and two duplication vertices v'_{k+1} and v''_{k+1} . These three form a cycle $v_{k+1}v'_{k+1}v''_{k+1}v_{k+1}$. The vertex v_{k+1} is then connected to the previous path vertex through an edge v_kv_{k+1} , maintaining the underlying path structure. In addition, we attach a pendant vertex to each of the three triangle vertices, $u_{k+1}v_{k+1}$, $u'_{k+1}v'_{k+1}$ and $u''_{k+1}v''_{k+1}$ as per the corona product. Since the labels from $\{1, 2, 3, \dots, 6k\}$ are already used by the existing graph $D(P_k) \circ K_1$, we assign the next six consecutive labels to the newly added vertices as follows $f(v_{k+1}) = 6k + 5$, $f(v'_{k+1}) = 6k + 1$, $f(v''_{k+1}) = 6k + 3$, $f(u_{k+1}) = 6k + 6$, $f(u'_{k+1}) = 6k + 2$ and $f(u''_{k+1}) = 6k + 4$. This assignment ensures that all six new integers are distinct, and together with the earlier labels, cover the full range from 1 to $6(k + 1)$ preserving bijectivity. We now verify that all adjacent vertex pairs involving the newly added block are labeled with relatively prime integers. First, observe that the vertices (u_{k+1}, v_{k+1}) , (u'_{k+1}, v'_{k+1}) and (u''_{k+1}, v''_{k+1}) which are adjacent receives consecutive integers, they are relatively prime. The vertices $(v_{k+1}, v'_{k+1}, v''_{k+1})$ which are adjacent receives consecutive odd integers, they are relatively prime. Now, we check the gcd between v_{k+1} and the previous vertex v_k which has the label $6k - 7$ under the inductive hypothesis. Now, to verify the gcd between v_{k+1} and v_k , we apply the Euclidean algorithm, $\gcd(f(v_k), f(v_{k+1})) = \gcd(6k + 5, 6k - 7) = \gcd((6k - 7), 12) =$

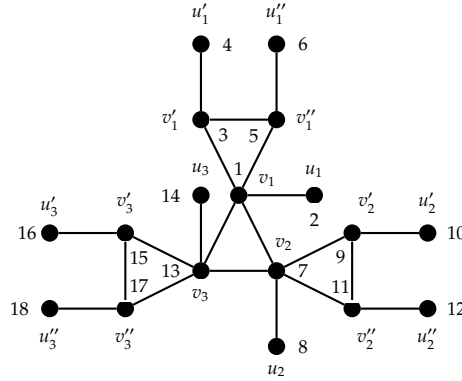
$\gcd(12, (6k-7) \bmod 12)$. Since $6k-7$ and $6k+5$ differ by 12 are not divisible by each other, their gcd is 1. Hence, the prime condition is satisfied for this edge as well. By the principle of mathematical induction, the corona product of $D(P_n) \circ K_1$ admits prime labeling for all $n \geq 1$. $\square$

Theorem 2.5. *The corona product of the graph obtained by duplicating every vertex of the cycle graph C_n by an edge with K_1 denoted $D(C_n) \circ K_1$ admits a prime labeling for all $n \geq 3$.*

Proof. Let C_n be the cycle graph with vertex set $\{v_1, v_2, v_3, \dots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. Construct the graph $D(C_n)$ by duplicating each vertex v_i by two additional vertices v'_i, v''_i such that they form a triangle $v_iv'_iv''_i$ for each $i = 1, 2, 3, \dots, n$. Then, in the corona product $D(C_n) \circ K_1$, we attach a pendant vertex u_i, u'_i, u''_i to each v_i, v'_i, v''_i , respectively. Hence $D(C_n) \circ K_1$ contains $6n$ vertices. Let us define the mapping $f : V(D(C_n) \circ K_1) \rightarrow \{1, 2, 3, \dots, 6n\}$. We aim to assign labels from the set $\{1, 2, 3, \dots, 6n\}$ to the vertices of $D(C_n) \circ K_1$ such that adjacent vertices always receive relatively prime labels.

Base Case $n = 3$:

Consider the graph $D(C_3) \circ K_1$, where $D(C_3)$ is obtained by duplicating each vertex of the cycle graph C_3 using a triangle. That is, for each original vertex v_i in C_3 , two additional vertices v'_i and v''_i are added such that the triangle $v_iv'_iv''_i$ is formed. The three original cycle vertices $\{v_1, v_2, v_3\}$ are connected cyclically, and in the corona product, each of the vertices v_i, v'_i, v''_i is attached to a unique pendant vertex u_i, u'_i, u''_i , respectively. Thus, the graph $D(C_3) \circ K_1$ has 18 vertices in total. We assign the labels to the vertices as follows, for the triangle vertices, define $f(v_i) = 6i - 5, f(v'_i) = 6i - 3, f(v''_i) = 6i - 1$ and for their respective pendant vertices, define $f(u_i) = 6(i - 1) + 2, f(u'_i) = 6(i - 1) + 4, f(u''_i) = 6(i - 1) + 6$ for all $i = 1, 2, 3$. Now, we verify the gcd conditions for adjacent pairs of vertices, each pendant vertex is adjacent to exactly one triangle vertex, and the label of each such pair $(v_i, u_i), (v'_i, u'_i)$, and (v''_i, u''_i) differs by exactly 1, making them consecutive integers. Since any two consecutive integers are relatively prime, their $\gcd = 1$. Within each triangle, the pairs (v_i, v'_i) receive consecutive odd integers. We know that, any two consecutive odd integers are relatively prime, so $\gcd(f(v_i), f(v'_i)) = 1$. The pair (v_i, v''_i) within the triangle satisfies $\gcd(f(v_i), f(v''_i)) = \gcd(6i - 5, 6i - 1) = \gcd(4, 6i - 5)$. Since 4 is even and $6i - 5$ is odd, their $\gcd = 1$. The original cycle adjacencies (v_i, v_{i+1}) , with indices taken modulo 3, are assigned labels that differ by 6. Thus, $\gcd(f(v_i), f(v_{i+1})) = \gcd(6i - 5, 6(i + 1) - 5) = \gcd(6i - 5, 6)$. Since $6i - 5$ is an odd number not divisible by 2 or 3, it shares no common prime factor with 6, so the $\gcd = 1$. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

FIGURE 5. Prime labeling of $D(C_3) \circ K_1$.**Inductive Hypothesis $n = k$:**

Assume that for some $k \geq 3$, the graph $D(C_k) \circ K_1$ admits a prime labeling using the labels from the set $\{1, 2, 3, \dots, 6k\}$.

Inductive Step $n = k + 1$:

To extend the labeling from $D(C_k) \circ K_1$ to $D(C_{k+1}) \circ K_1$, we construct the graph $D(C_{k+1})$ by adding a new vertex v_{k+1} between v_1 and v_k in the original cycle C_k , so that the new cycle becomes $v_1, v_2, \dots, v_k, v_{k+1}, v_1$. This addition maintains the cyclic nature of the graph. To preserve the structure of the duplication operation, we also introduce two new vertices v'_{k+1} and v''_{k+1} such that they form a triangle with v_{k+1} , i.e., $v_{k+1}v'_{k+1}v''_{k+1}v_{k+1}$ is a 3-cycle.

In the corona product $D(C_{k+1}) \circ K_1$, each of the new vertices v_{k+1} , v'_{k+1} , and v''_{k+1} is connected to a pendant vertex u_{k+1} , u'_{k+1} , and u''_{k+1} respectively.

Since the labeling of $D(C_k) \circ K_1$ uses labels from $\{1, 2, 3, \dots, 6k\}$ under the inductive hypothesis, we now assign the next six integers $\{6k + 1, 6k + 2, \dots, 6k + 6\}$ to the newly added vertices as follows:

$$f(v_{k+1}) = 6(k + 1) - 5 = 6k + 1,$$

$$f(v'_{k+1}) = 6(k + 1) - 3 = 6k + 3,$$

$$f(v''_{k+1}) = 6(k + 1) - 1 = 6k + 5,$$

$$f(u_{k+1}) = 6k + 2,$$

$$f(u'_{k+1}) = 6k + 4,$$

$$f(u''_{k+1}) = 6k + 6.$$

We now verify that all new adjacencies satisfy the gcd conditions, each pendant vertex is adjacent to its respective base vertex and receives a consecutive integer. Thus, pairs (v_{k+1}, u_{k+1}) , (v'_{k+1}, u'_{k+1}) , and (v''_{k+1}, u''_{k+1}) receive consecutive labels and are relatively prime. The triangle formed by v_{k+1} , v'_{k+1} , and v''_{k+1} receives consecutive odd labels, which are pairwise relatively prime. For the adjacency between v_{k+1} and v_k (existing from C_k), $f(v_k)$ has label $6k - 5$ under the inductive hypothesis. Then $\gcd(6k - 5, 6k + 1) = \gcd(6k + 1, 6)$, and since $6k + 1$ is odd and not divisible by

2 or 3, their $\gcd = 1$. Similarly, for the edge (v_{k+1}, v_1) , which replaces the previous edge (v_k, v_1) (now removed), we compute $\gcd(6k+1, f(v_1)) = \gcd(6k+1, 1) = 1$. The vertices v_{k+1} and v''_{k+1} are adjacent and receive labels $6k+1$ and $6k+5$, respectively. Using the Euclidean algorithm, $\gcd(6k+1, 6k+5) = \gcd(4, 6k+1) = 1$, since $6k+1$ is odd and not divisible by 4. Therefore, the labels assigned to v_{k+1} and v''_{k+1} are relatively prime. Thus, all new adjacencies preserve the property that adjacent vertices receive relatively prime labels. Therefore, by the principle of mathematical induction, the corona product of $D(C_n) \circ K_1$ admits a prime labeling for all $n \geq 3$. $\square$

Theorem 2.6. *The corona product of the graph obtained by duplicating every vertex of the star graph $K_{1,n}$ by an edge with K_1 denoted $D(K_{1,n}) \circ K_1$ admits a prime labeling for all $n \geq 1$.*

Proof. Let $K_{1,n}$ be the star graph with center vertex v_1 and n leaves $v_2, v_3, \dots, v_{n+1}$. In the graph $D(K_{1,n})$, each vertex is duplicated to form a triangle with two new vertices. That is, for each v_i , we create a triangle with v'_i and v''_i , forming the cycle $v_i v'_i v''_i v_i$. In the corona product $D(K_{1,n}) \circ K_1$, we attach a pendant vertex to each vertex of these triangles. Thus, we have $3(n+1)$ triangle vertices and $3(n+1)$ pendant vertices, totaling $6(n+1)$ vertices. Let us define the mapping $f : V(D(K_{1,n}) \circ K_1) \rightarrow \{1, 2, 3, \dots, 6(n+1)\}$. We aim to assign labels from the set $\{1, 2, 3, \dots, 6(n+1)\}$ to the vertices of $D(K_{1,n}) \circ K_1$ such that adjacent vertices always receive relatively prime labels.

Base Case $n = 1$:

Consider the star graph $K_{1,1}$ consisting of a central vertex v_1 and one leaf vertex v_2 . In the graph $D(K_{1,1})$, both v_1 and v_2 are duplicated to form triangles: (v_1, v'_1, v''_1) and (v_2, v'_2, v''_2) . In the corona product $D(K_{1,1}) \circ K_1$, a pendant vertex is attached to each of the six triangle vertices, resulting in a total of 12 vertices. We assign the labels to the vertices as follows, for the triangle vertices, define $f(v_i) = 6i - 5, f(v'_i) = 6i - 3, f(v''_i) = 6i - 1$ and for their respective pendant vertices, define $f(u_i) = 6(i-1) + 2, f(u'_i) = 6(i-1) + 4, f(u''_i) = 6(i-1) + 6$ for all $i = 1, 2$. This labeling assigns distinct integers from 1 to 12. Each pendant vertex is adjacent to its corresponding triangle vertex and receives a consecutive label, ensuring that the labels are relatively prime. Triangle vertices receive consecutive odd labels, which are also pairwise relatively prime. The pair (v_i, v''_i) within the triangle satisfies $\gcd(f(v_i), f(v''_i)) = \gcd(6i - 5, 6i - 1) = \gcd(4, 6i - 5)$. Since 4 is even and $6i - 5$ is odd, their $\gcd = 1$. The central vertex v_1 is labeled with 1, which is relatively prime to all other labels. Hence, the base case holds as all adjacent vertices receive relatively prime labels.

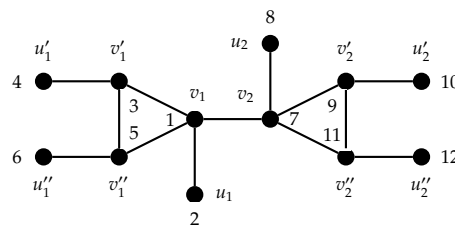


FIGURE 6. Prime labeling of $D(K_{1,1}) \circ K_1$.

Inductive Hypothesis $n = k$:

Assume that for some $k \geq 1$, the graph $D(K_{1,k}) \circ K_1$ admits a prime labeling using the labels from the set $\{1, 2, 3, \dots, 6(k+1)\}$.

Inductive Step $n = k + 1$:

To extend the labeling from $D(K_{1,k}) \circ K_1$ to $D(K_{1,k+1}) \circ K_1$, we add one more leaf vertex to the existing star, denoted by v_{k+1} . We then duplicate this vertex by adding two additional vertices v'_{k+1} and v''_{k+1} to form a triangle $v_{k+1}v'_{k+1}v''_{k+1}$. According to the corona product, we attach pendant vertices $u_{k+1}, u'_{k+1}, u''_{k+1}$ to $v_{k+1}, v'_{k+1}, v''_{k+1}$ respectively. Since the labeling of $D(K_{1,k}) \circ K_1$ uses labels from $\{1, 2, 3, \dots, 6(k+1)\}$ under the inductive hypothesis, we now assign the next six integers $\{6k+1, 6k+2, \dots, 6k+6\}$ to the newly added vertices as follows:

$$f(v_{k+1}) = 6(k+1) - 5 = 6k + 1,$$

$$f(v'_{k+1}) = 6(k+1) - 3 = 6k + 3,$$

$$f(v''_{k+1}) = 6(k+1) - 1 = 6k + 5,$$

$$f(u_{k+1}) = 6k + 2,$$

$$f(u'_{k+1}) = 6k + 4,$$

$$f(u''_{k+1}) = 6k + 6.$$

Observe that each pendant vertex is adjacent to its corresponding triangle vertex and receives a consecutive label, ensuring that the labels are relatively prime. Triangle vertices receive consecutive odd labels, which are also pairwise relatively prime. The vertices v_{k+1} and v''_{k+1} are adjacent and receive labels $6k+1$ and $6k+5$, respectively. Using the Euclidean algorithm, $\gcd(6k+1, 6k+5) = \gcd(4, 6k+1) = 1$, since $6k+1$ is odd and not divisible by 4. Therefore, the labels assigned to v_{k+1} and v''_{k+1} are relatively prime. The central vertex v_1 is labeled with 1, which is relatively prime to all other labels. Thus, all new adjacencies preserve the property that adjacent vertices receive relatively prime labels. Therefore, by the principle of mathematical induction, the corona product of $D(K_{1,n}) \circ K_1$ admits a prime labeling for all $n \geq 1$. $\square$

3. CONCLUSION AND FUTURE SCOPE

In this paper, we identified a graph operation which converts non-prime graphs to prime graphs. Specifically, we consider the non-primes graphs such as Möbius ladder, square of a path, square of a cycle, and graphs formed by duplicating every vertex in the path P_n , cycle C_n , and star $K_{1,n}$. While these graphs are non-prime in their original forms, we demonstrated that their corresponding corona products with K_1 admit prime labelings. These results highlight that the corona product operation can effectively transform non-prime graphs into prime labeled graphs. Similarly, one can think about what are all other graph operations to covert non-prime graphs to prime graphs.

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