International Journal of Analysis and Applications

# T-Closed-Coessential and T-Closed-Coclosed Submodules

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ABSTRACT. Many generalizations have been provided in the field of abstract algebra specially in modules theory in order to presents some new concepts and study some of their properties. The goal of this article is to present new concepts of submodules as a generalization of previous studies. In particular, we provided the concepts of T-closed-coessential submodule and T-closed-coclosed submodule in the current study. More precisely, let T be a submodule of the R-module $\mathcal{F}$ and $\mathcal{N}, \mathcal{K}$ are its submodules in which $\mathcal{K} \subseteq \mathcal{N}$, $\mathcal{K}$ is named as T-closed-co-essential ($\mathcal{K} \leq_{Tc.ce} \mathcal{N}$) of $\mathcal{N}$ in $\mathcal{F}$ if, $\frac{\mathcal{N}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$. Furthermore, $\mathcal{N}$ is called T-closed-coclosed ($\mathcal{N} \leq_{Tc.cc} \mathcal{F}$) if $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ in $\mathcal{F}$ implies $\mathcal{K} = \mathcal{N}$. Several basic properties of these concepts have studies and proved. Moreover, many related examples have been presented in order to illustrate the mentioned concepts.

## 1. Introduction

During this article the module $\mathcal{F}$ is unitary left R-module with the ring that have identity. Consider the submodule $\mathcal{N}$ of $\mathcal{F}$, then its termed as small submodule ($\mathcal{N} \ll \mathcal{F}$) if, in case each submodule $\mathcal{K}$ of $\mathcal{F}$ for which $\mathcal{N} + \mathcal{K} = \mathcal{F}$ gives that $\mathcal{K} = \mathcal{F}$[1]. For many discussion about this type of submodules, we recommended to the reader the sources [2-4]. The definition of small submodule has been generalized and studied by many authors such as [5] and [6]. Beyranvand and Moradi [7] had also generalized the mentioned definition in the following way: Consider T be any submodule from the R-module $\mathcal{F}$, then $\mathcal{N} \leq \mathcal{F}$ is said to be T-small in condition that any submodue $\mathcal{K}$ of $\mathcal{F}$, $T \subseteq \mathcal{N} + \mathcal{K}$ implies $T \subseteq \mathcal{K}$. Furthermore, the concept of Semi-T-small

Received Aug. 6, 2025

*2020 Mathematics Subject Classification.* 16D10.

*Key words and phrases.* closed submodule; T-small submodule; coessential submodule; small submodule; closed-coclosed submodule.

https://doi.org/10.28924/2291-8639-23-2025-299

ISSN: 2291-8639

submodules has been introduced by Elewi [8] as a generalization to the concept of T-small by considering the Jacobson radical of the module. In same year, Elwei [9] had also discussed the idea of semi-T-smallness submodules with the hollow and lifting modules respectively. In addition, some other generalizations of small submodules can be existed in [10]. On the other hand, the submodule $\mathcal{N}$ from $\mathcal{F}$ is said to be essential ($\mathcal{N} \leq_e \mathcal{F}$) in $\mathcal{F}$, in case that each submodule $\mathcal{K} \leq \mathcal{F}$, $\mathcal{N} \cap \mathcal{K} \neq 0$ [11]. This definition has been generalized by Safaeeyan and Shirazi [12] by providing the concept of T-essential submodule as follows. Suppose T be any submodule from the R-module $\mathcal{F}$, then $\mathcal{N} \leq \mathcal{F}$ is said to be T-essential in condition that $\mathcal{N} \nsubseteq \mathrm{T}$ and any submodue $\mathcal{K}$ of $\mathcal{F}$, $\mathcal{N} \cap \mathcal{K} \subseteq \mathrm{T}$ implies $\mathcal{K} \subseteq \mathrm{T}$. Moreover, many researchers have been developed this definition in different directions. For instance, Mohammad and Yaseen [13] presented the concept of ST-essential submodules and studied various properties about it. While, the notation of e*-essential submodules has been investigated with some properties of it by [14]. A submodule $\mathcal{N}$ of $\mathcal{F}$ is named as closed ($\mathcal{N} \leq_c \mathcal{F}$), if it has no proper essential extension in $\mathcal{F}$ [15]. Let $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$, then $\mathcal{K}$ is referred as coessential ($\mathcal{K} \leq_{ce} \mathcal{N}$) of $\mathcal{N}$ in $\mathcal{F}$ if, $\frac{\mathcal{N}}{\mathcal{K}} \ll \frac{\mathcal{F}}{\mathcal{K}}$ [11]. While, the same submodule is named as coclosed ($\mathcal{K} \leq_{cc} \mathcal{N}$), if it has no proper coessential submodule in $\mathcal{F}$ [15]. Very recently, some studies have been considered these types of submodules such like [16] which discussed the concept of ET-coessential submodules and ET-coclosed submodules. Besides that, the concept of closed coessential submodules and closed coclosed submodules are given by Yahyaa and Yaseen [17]. Motivated by the prior studies, we generalized the work of [17] by providing the concepts of T-closed-coessential and T-closed-coclosed submodules with some properties about them.

## 2. Basic Concepts

This section included some basic definitions that are needed in this work which are as follows.

**Definition 2.1 [17]** Suppose $\mathcal{F}$ is the R-module with $\mathcal{N}, \mathcal{K}$ its submodules in which $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$. The submodule $\mathcal{K}$ is termed as closed-co-essential from $\mathcal{N}$ in $\mathcal{F}$ ($\mathcal{K} \leq_{c.ce} \mathcal{N}$), if $\frac{\mathcal{N}}{\mathcal{K}} \ll_c \frac{\mathcal{F}}{\mathcal{K}}$.

**Definition 2.2 [17]** Suppose $\mathcal{F}$ is the R-module with $\mathcal{N}, \mathcal{K}$ its submodules in which $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$. The submodule $\mathcal{K}$ is referred as closed-co-closed from $\mathcal{N}$ in $\mathcal{F}$ ($\mathcal{K} \leq_{c.cc} \mathcal{N}$), if $\mathcal{K} \leq_{c.ce} \mathcal{N}$ with $\mathcal{K} = \mathcal{N}$.

## 3. Main Results

This section deals with the main results of this study which are given as follows.

**Definition 3.1** Suppose $\mathcal{F}$ is the R-module and $\mathrm{T} \leq \mathcal{F}$. The submodule $\mathcal{K}$ of $\mathcal{F}$ is termed T-closed-coessential from $\mathcal{N}$ in $\mathcal{F}$ ($\mathcal{K} \leq_{Tc.ce} \mathcal{N}$), if $\frac{\mathcal{N}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$ where $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$.

**Remarks and Examples**

(1) Consider $Z_6$ as Z-module and let T = {0,3} and $\mathcal{K}$ = {0} with $\mathcal{N}$ = {0,3}. Then, $\frac{\{0,3\}}{\{0\}} \cong \{0,3\} \ll_{Tc} \frac{Z_6}{\{0\}} \cong Z_6$.

(2) Consider $Z_4$ as Z-module and let T = {0,2}, $\mathcal{K}$ = {0} and $\mathcal{N}$ = {0,2}. Then, $\frac{\{0,2\}}{\{0\}} \cong \{0,2\} \ll_{Tc} \frac{Z_4}{\{0\}} \cong Z_4$.

(3) Consider $Z_8$ as Z-module and let T = {0,4}, $\mathcal{K}$ = {0} and $\mathcal{N}$ = {0,2,4,6}. Then, $\frac{\{0,2,4,6\}}{\{0\}} \cong \{0,2,4,6\} \ll_{Tc} \frac{Z_8}{\{0\}} \cong Z_8$.

(4) Consider $Z_{12}$ as Z-module and let T = {0,6}, $\mathcal{K}$ = {0} and $\mathcal{N}$ = {0,2,4,6,8,10}. Then, $\frac{\{0,2,4,6,8,10\}}{\{0\}} \cong \{0,2,4,6,8,10\} \ll_{Tc} \frac{Z_{12}}{\{0\}} \cong Z_{12}$.

(5) Consider $Z_{24}$ as Z-module and let T $=< 6 >$, $\mathcal{K}$ = {0} and $\mathcal{N} =< 2 >$. Then, $\frac{<2>}{\{0\}} \cong < 2 > \ll_{Tc} \frac{Z_{24}}{\{0\}} \cong Z_{24}$.

(6) If T = {0}, then any submodule of $\mathcal{F}$ is T-closed-coessential in $\mathcal{F}$.

**Proposition 3.1** Consider the R-module $\mathcal{F}$ and T ≤ $\mathcal{F}$ with $\mathcal{K}, \mathcal{N}$ its submodules from $\mathcal{F}$ in which $\mathcal{K} \leq \mathcal{N}$. Then, $\mathcal{K} \ll_{Tc} \mathcal{F}$ iff $\{0\} \leq_{Tc.ce} \mathcal{K}$.

Proof: Let $\mathcal{K} \ll_{Tc} \mathcal{F}$, then $\frac{\mathcal{K}}{\{0\}} \ll_{Tc} \frac{\mathcal{F}}{\{0\}}$ and by Definition 3.1, $\{0\} \leq_{Tc.ce} \mathcal{K}$ in $\mathcal{F}$. Conversely, let $\mathrm{T} \subseteq \mathcal{K} + \dot{\mathrm{X}}$ where $\dot{\mathrm{X}} \leq_c \mathcal{F}$ to show that $\mathrm{T} \subseteq \dot{\mathrm{X}}$. Thus, $\frac{\mathrm{T}}{\{0\}} \subseteq \frac{\mathcal{K}+\dot{\mathrm{X}}}{\{0\}}$ implies $\frac{\mathrm{T}}{\{0\}} \subseteq \frac{\mathcal{K}}{\{0\}} + \frac{\dot{\mathrm{X}}}{\{0\}}$. Since $\{0\} \leq_{Tc.ce} \mathcal{K}$ in $\mathcal{F}$, then by Definition 3.1, $\frac{\mathcal{K}}{\{0\}} \ll_{Tc} \frac{\mathcal{F}}{\{0\}}$ which gives $\frac{\mathrm{T}}{\{0\}} \subseteq \frac{\dot{\mathrm{X}}}{\{0\}}$. Hence, $\mathrm{T} \subseteq \dot{\mathrm{X}}$. Therefore, $\mathcal{K} \ll_{Tc} \mathcal{F}$. ■

**Proposition 3.2** Consider the R-module $\mathcal{F}$ and T ≤ $\mathcal{F}$ with $\mathcal{N}, \mathcal{K}, \mathcal{L}$ are submodules of $\mathcal{F}$ in which $\mathcal{N} \leq \mathcal{K} \leq \mathcal{L} \leq \mathcal{F}$. Then, $\mathcal{K} \leq_{Tc.ce} \mathcal{L}$ iff $\frac{\mathcal{K}}{\mathcal{N}} \leq_{Tc.ce} \frac{\mathcal{L}}{\mathcal{N}}$ in $\frac{\mathcal{F}}{\mathcal{N}}$.

Proof: Let $\mathcal{K} \leq_{Tc.ce} \mathcal{L}$, then by Definition 3.1, $\frac{\mathcal{L}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$. Since $\frac{\mathcal{L}}{\mathcal{K}} \cong \frac{\mathcal{L}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$ and $\frac{\mathcal{F}}{\mathcal{K}} \cong \frac{\mathcal{F}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$ , then by Third Isomorphism Theorem $\frac{\mathcal{L}/\mathcal{N}}{\mathcal{K}/\mathcal{N}} \ll_{Tc} \frac{\mathcal{F}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$. That is $\frac{\mathcal{K}}{\mathcal{N}} \leq_{Tc.ce} \frac{\mathcal{L}}{\mathcal{N}}$ in $\frac{\mathcal{F}}{\mathcal{N}}$. Conversely, let $\frac{\mathcal{K}}{\mathcal{N}} \leq_{Tc.ce} \frac{\mathcal{L}}{\mathcal{N}}$ in $\frac{\mathcal{F}}{\mathcal{N}}$ , then by Definition 3.1, $\frac{\mathcal{L}/\mathcal{N}}{\mathcal{K}/\mathcal{N}} \ll_{Tc} \frac{\mathcal{F}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$. By Third Isomorphism Theorem $\frac{\mathcal{L}}{\mathcal{K}} \cong \frac{\mathcal{L}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$ and $\frac{\mathcal{F}}{\mathcal{K}} \cong \frac{\mathcal{F}/\mathcal{N}}{\mathcal{K}/\mathcal{N}}$. That is $\frac{\mathcal{L}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$ which gives that $\mathcal{K} \leq_{Tc.ce} \mathcal{L}$. ■

**Proposition 3.3** Consider the R-module $\mathcal{F}$ and T ≤ $\mathcal{F}$ with $\mathcal{N}, \mathcal{K}, \mathcal{L}$ are submodules of $\mathcal{F}$ for which $\mathcal{K} \leq \mathcal{N} \leq \mathcal{L}$. If $\mathcal{K} \leq_{Tc.ce} \mathcal{L}$, then $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$.

Proof: Let $\mathrm{T} \subseteq \mathcal{N} + \dot{\mathrm{X}}$ where $\dot{\mathrm{X}} \leq_c \mathcal{F}$ to show that $\mathrm{T} \subseteq \dot{\mathrm{X}}$. Since $\mathcal{N} \leq \mathcal{L}$, then $\mathrm{T} \subseteq \mathcal{L} + \dot{\mathrm{X}}$. But $\mathcal{K} \leq_{Tc.ce} \mathcal{L}$, then by Definition 3.1, $\frac{\mathcal{L}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$ and $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\mathcal{L}}{\mathcal{K}} + \frac{\dot{\mathrm{X}}}{\mathcal{K}}$ with $\frac{\dot{\mathrm{X}}}{\mathcal{K}} \leq_c \frac{\mathcal{F}}{\dot{\mathrm{X}}}$. Thus, $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\dot{\mathrm{X}}}{\mathcal{K}}$ which implies $\mathrm{T} \subseteq \dot{\mathrm{X}}$ and hence $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$. ■

**Theorem 3.1** Consider the R-module $\mathcal{F}$ and $\mathrm{T} \leq \mathcal{F}$ with $\mathcal{N}, \mathcal{K}$ are submodules of $\mathcal{F}$ for which $\mathcal{K} \leq \mathcal{N}$. Then, $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ iff for any closed submodule $\dot{\mathrm{X}}$ of $\mathcal{F}$ for which $\mathrm{T} \subseteq \mathcal{N} + \dot{\mathrm{X}}$, implies $\mathrm{T} \subseteq \dot{\mathrm{X}}$.

Proof: Let $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ and $\mathrm{T} \subseteq \mathcal{N} + \dot{\mathrm{X}}$ where $\dot{\mathrm{X}}$ is closed submodule of $\mathcal{F}$ to show that $\mathrm{T} \subseteq \dot{\mathrm{X}}$. Since $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$, then by Definition 3.1, $\frac{\mathcal{N}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$. Now, $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\mathcal{N}+\dot{\mathrm{X}}}{\mathcal{K}} = \frac{\mathcal{N}}{\mathcal{K}} + \frac{\dot{\mathrm{X}}}{\mathcal{K}}$. Thus, $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\mathcal{N}}{\mathcal{K}} + \frac{\dot{\mathrm{X}}}{\mathcal{K}}$. Since $\frac{\mathcal{N}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$ then $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\dot{\mathrm{X}}}{\mathcal{K}}$ where $\frac{\dot{\mathrm{X}}}{\mathcal{K}} \leq_c \frac{\mathcal{F}}{\dot{\mathrm{X}}}$ which gives $\mathrm{T} \subseteq \dot{\mathrm{X}}$. Conversely, let for any closed submodule $\frac{\dot{\mathrm{X}}}{\mathcal{K}}$ of $\frac{\mathcal{F}}{\mathcal{K}}$ we have $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\dot{\mathrm{X}}}{\mathcal{K}}$. That is $\frac{\mathrm{T}}{\mathcal{K}} \subseteq \frac{\mathcal{N}}{\mathcal{K}} + \frac{\dot{\mathrm{X}}}{\mathcal{K}} = \frac{\mathcal{N}+\dot{\mathrm{X}}}{\mathcal{K}}$. This gives that $\frac{\mathcal{N}}{\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{K}}$ and by Definition 3.1, we get $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$. ■

**Definition 3.2** Suppose $\mathcal{F}$ be an R-module and T be a submodule of $\mathcal{F}$. The submodule $\mathcal{N}$ of $\mathcal{F}$ is termed T-closed-coclosed in $\mathcal{F}$ ($\mathcal{N} \leq_{Tc.cc} \mathcal{F}$), if $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ in $\mathcal{F}$ implies $\mathcal{K} = \mathcal{N}$ where $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$. Equivalently, $\mathcal{K}$ is known as T-closed-coclosed of $\mathcal{F}$, if it has no proper T-closed-coessential submodule.

**Remarks and Examples**

(1) Consider $Z_4$ as Z-module and let T = {0,2}, $\mathcal{K} = \{0\}$ and $\mathcal{N} = \{0,2\}$. Then, $\mathcal{N}$ is not T-c-coclosed in $Z_4$ since $\{0\} \neq \{0,2\}$.

(2) Consider $Z_8$ as Z-module and let T = {0,4}, $\mathcal{K} = \{0\}$ and $\mathcal{N} = \{0,2,4,6\}$. Then, $\mathcal{N}$ is not T-c-coclosed in $Z_8$ since $\{0\} \neq \{0,2,4,6\}$.

(3) Consider $Z_{24}$ as Z-module and let $\mathrm{T} =< 6 >$, $\mathcal{K} = \{0\}$ and $\mathcal{N} =< 2 >$. Then, $\mathcal{N}$ is not T-c-coclosed in $Z_{24}$ since $\{0\} \neq< 2 >$.

**Proposition 3.4** Consider the R-module $\mathcal{F}$ and $\mathrm{T} \leq \mathcal{F}$ with $\mathcal{N}, \mathcal{K}$ are submodules of $\mathcal{F}$ for which $\mathcal{K} \leq \mathcal{N} \leq \mathcal{F}$. If $\mathcal{N} \leq_{Tc.cc} \mathcal{F}$, then $\frac{\mathcal{N}}{\mathcal{K}} \leq_{Tc.cc} \frac{\mathcal{F}}{\mathcal{K}}$.

Proof: Let $\mathcal{L} \subseteq \mathcal{N}$, then $\frac{\mathcal{L}}{\mathcal{K}} \subseteq \frac{\mathcal{N}}{\mathcal{K}}$ and $\frac{\mathcal{L}}{\mathcal{K}}$ is T-closed-coessential of $\frac{\mathcal{N}}{\mathcal{K}}$ in $\frac{\mathcal{F}}{\mathcal{K}}$. That is $\frac{\mathcal{N}/\mathcal{K}}{\mathcal{L}/\mathcal{K}} \ll_{Tc} \frac{\mathcal{F}/\mathcal{K}}{\mathcal{L}/\mathcal{K}}$. Thus, $\frac{\mathcal{N}}{\mathcal{L}} \ll_{Tc} \frac{\mathcal{F}}{\mathcal{L}}$. That is mean $\mathcal{L} \leq_{Tc.ce} \mathcal{N}$ in $\mathcal{F}$. But $\mathcal{N} \leq_{Tc.cc} \mathcal{F}$, then we get $\mathcal{L} = \mathcal{N}$. Thus, $\frac{\mathcal{L}}{\mathcal{K}} = \frac{\mathcal{N}}{\mathcal{K}}$. Therefore, $\frac{\mathcal{N}}{\mathcal{K}} \leq_{Tc.cc} \frac{\mathcal{F}}{\mathcal{K}}$. ■

**Theorem 3.2** Consider the R-module $\mathcal{F}$ and $\mathrm{T} \leq \mathcal{F}$ with $\mathcal{N}, \mathcal{K}$ are submodules of $\mathcal{F}$ in which $\mathcal{K} \leq \mathcal{N}$. Then, $\mathcal{K} \leq_{Tc.cc} \mathcal{N}$ iff $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ with $\mathcal{N} = \mathcal{K}$.

Proof: Let $\mathcal{K} \leq_{Tc.cc} \mathcal{N}$ to show that $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ with $\mathcal{N} = \mathcal{K}$. Assume that $\mathcal{K} \nleq_{Tc.ce} \mathcal{N}$ and $\mathcal{N} \neq \mathcal{K}$, then by Definition 3.1, $\mathcal{K}$ is not T-c-coessential of $\mathcal{N}$ in $\mathcal{F}$ and $\mathcal{N} \neq \mathcal{K}$. Thus, by Definition 3.2, $\mathcal{K} \nleq_{Tc.cc} \mathcal{N}$ which is contradict our assumption. Conversely, let $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ with $\mathcal{N} = \mathcal{K}$ to prove that $\mathcal{K} \leq_{Tc.cc} \mathcal{N}$. Assume that $\mathcal{K} \nleq_{Tc.cc} \mathcal{N}$, then by Definition 3.2, we have a submodule

$\mathcal{K}$ of $\mathcal{N}$ which is either $\mathcal{K} \nleq_{Tc.ce} \mathcal{N}$ and $\mathcal{N} = \mathcal{K}$ or $\mathcal{K} \leq_{Tc.ce} \mathcal{N}$ with $\mathcal{N} \neq \mathcal{K}$ or both are not satisfied. So, according to Definition 3.2, all the possible cases leads to contradiction. Thus, $\mathcal{K} \leq_{Tc.cc} \mathcal{N}$. ■

## 4. Conclusions

In conclusion, new types of submodules have been introduced in this work. These submodules are named T-closed-co-essential and T-closed-co-closed submodules. Several features of the mentioned concepts have been proved.

**Acknowledgments:** The authors thank the reviewers for their valuable comments and suggestions that will improve the quality of our paper.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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