Int. J. Anal. Appl. (2025), 23:284

International Journal of Analysis and Applications

Output Feedback Stabilization of a Class of Coupled ODE-PDE Systems and Its Applications

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Abstract. This paper investigates the output feedback stabilization problem of a class of coupled ODE-PDE cascade systems. A state feedback controller was designed based on a backstepping transformation. Then, using the unique measurement signal, a state observer is designed to realize the real-time estimation of the system state. Furthermore, an observer based output feedback control was established to achieve exponential stability of the system. Finally, we presented the application of our results in reliability systems and distributed reactor systems.

1. Introduction

In this paper, we consider the following ODE-PDE system

$$\begin{cases} v_t(x,t) = -v_x(x,t) + b(x)v(1,t), \ 0 < x < 1, t > 0, \\ v(0,t) = \sigma\omega(t), \ t \geq 0, \\ \dot{\omega}(t) = \beta_1\omega(t) + \beta_2 v(1,t) + U(t), \ t \geq 0, \\ y(t) = v(1,t), \ t \geq 0, \\ v(x,0) = v_0(x), \omega(0) = \omega_0, \ 0 \leq x \leq 1, \end{cases} \tag{1.1}$$

where v_x and v_t represent the partial derivatives of v with respect to x and t, respectively, $\dot{\omega}(t)$ denotes the derivative of ω with respect to t, $b(\cdot) \in C^1[0,1]$, and $\sigma, \beta_1, \beta_2 \in \mathbb{R}$ are system parameters and $\sigma \neq 0$. In addition, $U(t)$ is the input and $y(t)$ is the output of this system. Our goal is to design a controller using a unique measurement signal $y(t)$ to make the closed-loop system of system (1.1) exponentially stable.

The system (1.1) includes the heat exchanger equation [1], the centralized and distributed reactor system [2] and the reliability system ([3], [4]). For the case where $\omega = 0$ and b is a bounded

Received: Aug. 10, 2025.

2020 *Mathematics Subject Classification.* 93D15, 35M33, 93C20.

Key words and phrases. hyperbolic systems; coupled ODE-PDE systems; output feedback; stabilization.

https://doi.org/10.28924/2291-8639-23-2025-284

ISSN: 2291-8639

exponential function, the exponential stability and differentiability of system (1.1) were studied in references ([1], [5]), and the output feedback regulation problem was studied in reference [6]. For the case where $\sigma = 1, \beta_1 > 0, \beta_2 \in [0,1]$ and b is bounded exponential function, in [2] designed a discrete regulator for system (1.1).

The boundary feedback stabilization problem of first-order hyperbolic unstable partial differential equations (PDEs) is discussed in [7], and its application in finite dimensional systems with actuator and sensor delays is studied. Anfinsen and Aamo [8] incorporates first-order actuator and sensor dynamics into the PDE in [7], and designs the controller and observer for the system. They designed two types of observers and combined them with state feedback to form output feedback control laws, which made the closed-loop system exponentially stable. Wang and Jin [9] extends the results in [8] from the boundary to non-local terms related to any intermediate point. Recently, observers and output feedback controllers were designed for hyperbolic ODE-PDE coupled systems in literature ([10], [11], [12], [13], [14]). The state feedback and output feedback controllers for one-dimensional wave equations were proposed in [15], where the non-local terms are boundary dependent. The finite-time exponential stability of one-dimensional wave equations was studied in [16] by designing a state observer and an output feedback controller. Recently, Hu et al. [4] designed an adaptive observer for multi state repairable systems, the mathematical model of this system is governed by coupled transport and integro-differential equations.

This paper is inspired by ([8], [10]) and [16] to investigate the output feedback stabilization problem of system (1.1). This article has the following innovative points: (i) A state feedback controller was designed based on backstepping transformation. (ii) Using the unique measurement signal $v(1,t)$, a state observer was designed to make the observation error system exponentially decay. (iii) Designed an observer based output feedback controller to ensure finite-time exponential stability of the closed-loop system (1.1).

The arrangement of this article is as follows: In sections 2, a state feedback controller is designed and the finite-time stability of the closed-loop system is proved. A state observer was designed in sections 3. In Sections 4, an output feedback controller was designed and the finite-time stability of the output feedback closed-loop system was demonstrated. Finally, we presented some applications of our result.

2. State feedback control

Taking inspiration from ([10], [17]), we introduce the following backstepping integral transformation

$$\begin{aligned} h(x,t) &= [(I+\mathbb{P})v](x,t) \\ &= v(x,t) - \int_x^1 k(x,y)v(y,t)dy, \end{aligned} \tag{2.1}$$

where $k(\cdot,\cdot)$ is the undetermined kernel function. By taking partial derivatives in space and time for system (2.1), we obtain

$$h_x(x,t) = v_x(x,t) + k(x,x)v(x,t) - \int_x^1 k_x(x,y)v(y,t)dy, \tag{2.2a}$$

$$\begin{aligned} h_t(x,t) &= -v_x(x,t) + b(x)v(1,t) + k(x,1)v(1,t) - k(x,x)v(x,t) \\ &\quad - \int_x^1 k_y(x,y)v(y,t)dy - \int_x^1 k(x,y)b(y)v(1,t)dy. \end{aligned} \tag{2.2b}$$

Adding Eqs. (2.2a) and (2.2b) yields

$$\begin{aligned} h_t(x,t) + h_x(x,t) &= -\int_x^1 \left[k_x(x,y) + k_y(x,y)\right] v(y,t)dy \\ &\quad + \left[b(x) + k(x,1) - \int_x^1 k(x,y)b(y)dy\right] v(1,t). \end{aligned} \tag{2.3}$$

We choose the kernel function $k(\cdot,\cdot)$ as

$$\begin{cases} k_x(x,y) + k_y(x,y) = 0, \\ k(x,1) = \int_x^1 k(x,y)b(y)dy - b(x). \end{cases} \tag{2.4}$$

Then, using Eq. (2.3) and system (2.4) we have

$$h_t(x,t) = -h_x(x,t). \tag{2.5}$$

The following Lemma 2.1 indicates that system (2.4) has a unique solution:

Lemma 2.1. *The system (2.4) has a unique solution* $k(x,y) \in C^1([0,1] \times [0,1])$ *such that*

$$\sup_{0 \le x,y \le 1} |k(x,y)| \leqslant M_1 e^{M_1(x-y)}, \tag{2.6}$$

where $M_1 = \max_{x\in[0,1]} |b(x)|$.

Proof. By using a proof similar to [7, Theorem 1], it can be concluded that the result of this lemma holds true. □

The inverse transformation of transformation (2.1) is defined as follows

$$\begin{aligned} v(x,t) &= [(I + \mathbb{P})^{-1}h](x,t) \\ &= h(x,t) + \int_x^1 m(x,y)h(y,t)dy, \end{aligned} \tag{2.7}$$

where $m(\cdot,\cdot)$ is an undetermined kernel function. By taking the partial derivatives of the transformation (2.7) in space and time, we obtain

$$v_x(x,t) = h_x(x,t) - m(x,x)h(x,t) + \int_x^1 m_x(x,y)h(y,t)dy, \tag{2.8a}$$

$$v_t(x,t) = -h_x(x,t) - m(x,1)h(1,t) + m(x,x)h(x,t) + \int_x^1 m_y(x,y)h(y,t)dy. \tag{2.8b}$$

From Eqs. (2.8a)-(2.8b), we obtain

$$\begin{aligned} 0 &= v_t(x,t) + v_x(x,t) - b(x)v(1,t) \\ &= \int_x^1 \Big[m_x(x,y) + m_y(x,y)\Big] h(y,t)dy - [m(x,1) + b(x)]h(1,t). \end{aligned} \tag{2.9}$$

Based on Eq. (2.9), we select the kernel function $m(\cdot,\cdot)$ that satisfies

$$\begin{cases} m_x(x,y) + m_y(x,y) = 0, \\ m(x,1) = -b(x). \end{cases} \tag{2.10}$$

Through a simple calculation, we obtain that there exists a unique continuous differentiable solution for system (2.10):

$$m(x,y) = m(x-y) = -b(x-y+1).$$

From the transformation (2.1) and the system (1.1), we define

$$h(0,t) \quad = \sigma\omega(t) - \int_0^1 k(0,y)v(y,t)dy := p(t). \tag{2.11}$$

Consequently, system h satisfies

$$\begin{cases} h_t(x,t) = -h_x(x,t), \\ h(0,t) = p(t). \end{cases} \tag{2.12}$$

Take the derivative of function $p(t)$, and use $v(0,t) = \sigma\omega(t)$ and $k(0,1) = \int_0^1 k(0,y)b(y)dy - b(0)$, from systems (2.12) and (1.1) we obtain

$$\begin{aligned} \dot{p}(t) &= \sigma\dot{\omega}(t) - \int_0^l k(0,y)v_t(y,t)dy \\ &= \sigma\left[\beta_1\omega(t) + \beta_2 v(1,t) + U(t)\right] + k(0,1)v(1,t) - k(0,0)v(0,t) \\ &\quad - \int_0^1 k_y(0,y)v(y,t)dy - v(1,t)\int_0^1 k(0,y)b(y)dy \\ &= \sigma(\beta_1 - k(0,0))\omega(t) + \sigma U(t) - (b(0) - \sigma\beta_2)v(1,t) \\ &\quad - \int_0^1 k_y(0,y)v(y,t)dy. \end{aligned} \tag{2.13}$$

Design the following full state feedback controller

$$\begin{aligned} U(t) = \quad & (k(0,0) - \gamma_1)\omega(t) + \frac{b(0) - \sigma\beta_2}{\sigma} v(1,t) \\ & + \frac{1}{\sigma}\int_0^1 \Big[(\gamma_1 - \beta_1)k(0,y) + k_y(0,y)\Big] v(y,t)dy, \end{aligned} \tag{2.14}$$

where $\gamma_1 > \beta_1$ is the regulation constant. Hence, by Eqs. (2.12)-(2.14) we see that system "(h,p)" satisfies

$$\begin{cases} h_t(x,t) = -h_x(x,t),\ 0 < x < 1, t > 0, \\ h(0,t) = p(t),\ t \geq 0, \\ \dot{p}(t) = (\beta_1 - \gamma_1)p(t),\ t \geq 0, \\ h(x,0) = h_0(x),\ p(0) = p_0,\ 0 \leq x \leq 1. \end{cases} \tag{2.15}$$

Therefore, under the controller (2.14), we obtain the following closed-loop system for system (1.1):

$$\begin{cases} v_t(x,t) = -v_x(x,t) + b(x)v(1,t), \\ v(0,t) = \sigma\omega(t), \\ \dot{\omega}(t) = [k(0,0) - \gamma_1 + \beta_1]\,\omega(t) + \dfrac{1}{\sigma}b(0)v(1,t) \\ \qquad + \dfrac{1}{\sigma}\displaystyle\int_0^1 \left[(\gamma_1 - \beta_1)k(0,y) + k_y(0,y)\right] v(y,t)dy, \\ v(x,0) = v_0(x),\ \omega(0) = \omega_0. \end{cases} \tag{2.16}$$

Now, we consider system (2.16) on space $H = L^2(0,1) \times \mathbb{R}$, and the norm on H is as follows

$$\|(f,g)\|_H^2 = \int_0^1 |f(x)|^2 dx + |g|^2, \quad \forall (f,g) \in H.$$

Theorem 2.1. *For any initial value $(v_0(\cdot), \omega_0) \in H$, system (2.16) has a unique solution $(v(\cdot,t), \omega(t)) \in C(0,\infty; H)$ that satisfies*

$$\|(v(\cdot,t), \omega(t))\|_H \leq M_2 e^{(\beta_1 - \gamma_1)t},\ t > 1, \tag{2.17}$$

where $M_2 > 0$ is a constant and $\gamma_1 > \beta_1$.

Proof. Solving system (2.15), we obtain $p(t) = p_0 e^{(\beta_1 - \gamma_1)t}, t \geq 0$ and

$$h(x,t) = \begin{cases} h_0(x-t), & t \leq x \leq 1, \\ p_0 e^{(\beta_1 - \gamma_1)t}, & x < t. \end{cases} \tag{2.18}$$

This indicates that $h(x,t)$ and $p(t)$ are exponentially stable when $t > 1$ and $\gamma_1 > \beta_1$.

When $t \leq x \leq 1$, using the Eqs. (2.1), (2.7)-(2.10) and (2.18), we obtain

$$\begin{aligned} v(x,t) &= h_0(x-t) + \int_x^1 m(x,y)h_0(y-t)dy \\ &= v_0(x-t) - \int_0^1 k(x,y)v_0(y-t)dy \\ &\quad - \int_x^1 b(x-y+1)\left(v_0(y-t) - \int_0^1 k(y,\xi)v_0(\xi - t)d\xi\right)dy. \end{aligned} \tag{2.19}$$

When $x < t$, applying the Eqs. (2.7), (2.18) and $p_0 = \sigma\omega_0 - \int_0^1 k(0,y)v_0(y)dy$, we have

$$\begin{aligned} v(x,t) &= p_0 e^{(\beta_1-\gamma_1)t} + \int_x^1 m(x,y) p_0 e^{(\beta_1-\gamma_1)t} dy \\ &= e^{(\beta_1-\gamma_1)t}\left(\sigma\omega_0 - \int_0^1 k(0,y)v_0(y)dy\right)\left(1 - \int_x^1 b(x-y+1)dy\right). \end{aligned} \tag{2.20}$$

Then, using Eqs. (2.11), (2.15), (2.18) and (2.20), we obtain

$$\begin{aligned} \omega(t) &= \frac{1}{\sigma}p(t) + \frac{1}{\sigma}\int_0^1 k(0,y)v(y,t)dy \\ &= \frac{1}{\sigma}e^{(\beta_1-\gamma_1)t}\left(\sigma\omega_0 - \int_0^1 k(0,y)v_0(y)dy\right) + \frac{1}{\sigma}\int_0^1 k(0,y)v(y,t)dy. \end{aligned} \tag{2.21}$$

Therefore, $(v(\cdot,t),\omega(t)) \in C(0,\infty;H)$ is a solution of system (2.16).

When $t > 1$, since $(v_0,\omega_0) \in H$, Lemma 2.1 and $b \in C^1[0,1]$, kernel function $k(\cdot,\cdot)$ and function $b(\cdot)$ are bounded, that is,

$$\sup_{0\le x,y\le 1} |k(x,y)| \le L_1, \ \sup_{0\le x\le 1} |b(x)| \le L_2. \tag{2.22}$$

where $L_1, L_2 > 0$ are constants. Then, using Eqs. (2.20)-(2.22) and Hölder inequality, we obtain

$$\|v(x,t)\|_{L^2(0,1)} \le e^{(\beta_1-\gamma_1)t}\left(|\sigma||\omega_0| + L_1\|v_0\|\right)\left(1+L_2\right), \ t > 1, \tag{2.23a}$$

$$\begin{aligned} |\omega(t)| &\le \frac{1}{|\sigma|}e^{(\beta_1-\gamma_1)t}\left(|\sigma||\omega_0| + L_1\|v_0\|\right) + \frac{L_1}{|\sigma|}\int_0^1 |v(y,t)|dy \\ &\le \frac{1}{|\sigma|}e^{(\beta_1-\gamma_1)t}\left(|\sigma||\omega_0| + L_1\|v_0\|\right) + \frac{L_1}{|\sigma|}\|v(\cdot,t)\|_{L^2(0,1)} \\ &\le M_2 e^{(\beta_1-\gamma_1)t}, \ t > 1. \end{aligned} \tag{2.23b}$$

where $M_2 > 0$ is a constant. Hence, combining the inequalities (2.23a) with (2.22), it can be concluded that Eq. (2.17) is established. □

Remark 2.1. *From system (2.15), it can be seen that increasing the regulation constant γ_1 can make the convergence speed of the target system faster.*

3. State observer

Using the unique measurement signal $v(1,t)$, we design the following state observer

$$\begin{cases} \hat{v}_t(x,t) = -\hat{v}_x(x,t) + b(x)\hat{v}(1,t) + L(x)[v(1,t) - \hat{v}(1,t)], \\ \hat{v}(0,t) = \sigma\hat{\omega}(t), \\ \dot{\hat{\omega}}(t) = \beta_1\hat{\omega}(t) + \beta_2\hat{v}(1,t) + U(t) + \gamma_2[v(1,t) - \hat{v}(1,t)], \\ \hat{v}(x,0) = \hat{v}_0(x), \quad \hat{\omega}(0) = \hat{\omega}_0, \end{cases} \tag{3.1}$$

where γ_2 is the regulation constant and $L(x)$ is the regulation function such that

$$\begin{cases} \gamma_2 > \beta_2 + \dfrac{\beta_1}{\sigma} e^{\beta_1}, & \text{if } \sigma > 0, \\ \gamma_2 < \beta_2 + \dfrac{\beta_1}{\sigma} e^{\beta_1}, & \text{if } \sigma < 0, \end{cases} \tag{3.2}$$

and

$$L(x) = b(x) + \sigma(\gamma_2 - \beta_2) e^{-\beta_1 x}. \tag{3.3}$$

To demonstrate the asymptotic convergence of the observer mentioned above, we introduce observer error $\tilde{v} = v - \hat{v}, \tilde{\omega} = \omega - \hat{\omega}$. Then, system $(\tilde{v}, \tilde{\omega})$ satisfies

$$\begin{cases} \tilde{v}_t(x,t) = -\tilde{v}_x(x,t) + [b(x) - L(x)]\tilde{v}(1,t), \\ \tilde{v}(0,t) = \sigma\tilde{\omega}(t), \\ \dot{\tilde{\omega}}(t) = \beta_1\tilde{\omega}(t) + (\beta_2 - \gamma_2)\tilde{v}(1,t), \\ \tilde{v}(x,0) = \tilde{v}_0(x), \quad \tilde{\omega}(0) = \tilde{\omega}_0. \end{cases} \tag{3.4}$$

Theorem 3.1. *For any initial value $(\tilde{v}_0(\cdot), \tilde{\omega}_0) \in H$, system (3.4) has a unique solution $(\tilde{v}(\cdot,t), \tilde{\omega}(t)) \in C(0,\infty; H)$ that satisfies*

$$\|(\tilde{v}(\cdot,t), \tilde{\omega}(t))\| \le M_3 e^{\beta_3 t} |\tilde{\omega}_0|, \quad \forall t > 1, \tag{3.5}$$

where $M_3 > 0$ and $\beta_3 = \beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1} < 0$.

Proof. We introduce the following transformation

$$z(x,t) = \tilde{v}(x,t) - \sigma e^{-\beta_1 x}\tilde{\omega}(t). \tag{3.6}$$

Then, under the transformation (3.6), the error system (3.4) can be written as

$$\begin{cases} z_t(x,t) = -z_x(x,t), \\ z(0,t) = 0, \\ \dot{\tilde{\omega}}(t) = \left[\beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1}\right]\tilde{\omega}(t) + (\beta_2 - \gamma_2) z(1,t), \\ z(x,0) = z_0(x), \tilde{\omega}(0) = \tilde{\omega}_0. \end{cases} \tag{3.7}$$

System (3.7) can be divided into two parts: $z-$subsystem and $\tilde{w}-$subsystem. Solving $z-$subsystem, we obtain

$$z(x,t) = \begin{cases} z_0(x-t), & t \le x \le 1, \\ 0, & x < t. \end{cases} \tag{3.8}$$

Hence,

$$z(\cdot,t) \equiv 0, \ \forall\ t > 1. \tag{3.9}$$

If we take $\beta_3 = \beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1}$, then from Eq. (3.2) we deduce that $\beta_3 < 0$. By solve $\tilde{w}-$subsystem, we have

$$\tilde{\omega}(t) = e^{\beta_3 t}\tilde{\omega}_0 + (\beta_2 - \gamma_2)\int_0^t e^{\beta_3(t-s)} z(1,s) ds. \tag{3.10}$$

Therefore, using Eqs. (3.8), (3.10) and (3.6), it can be conclude that $(\tilde{v}(\cdot,t), \tilde{\omega}(t)) \in C(0,\infty; H)$ is a solution of system (3.4).

Finally, from Eqs. (3.9), (3.7) and (3.6) we obtain that $\tilde{\omega}(t) = e^{\beta_3 t}\tilde{\omega}_0$ and $\tilde{v}(x,t) = \sigma e^{-\beta_1 x}e^{\beta_3 t}\tilde{\omega}_0$ when $t > 1$. This indicates that Eq. (3.5) is established □

4. Output feedback control and main results

We consider the following output feedback controller

$$\begin{aligned} U(t) &= (k(0,0)-\gamma_1)\hat{\omega}(t) + \frac{b(0)-\sigma\beta_2}{\sigma}\hat{v}(1,t) \\ &\quad + \frac{1}{\sigma}\int_0^1 \Big[(\gamma_1-\beta_1)k(0,y) + k_y(0,y)\Big]\hat{v}(y,t)dy. \end{aligned} \tag{4.1}$$

Then, under the output feedback controller (4.1), we can obtain the following closed-loop system for system (1.1)

$$\begin{cases} v_t(x,t) = -v_x(x,t) + b(x)v(1,t), \\ v(0,t) = \sigma\omega(t), \\ \dot{\omega}(t) = [k(0,0)-\gamma_1+\beta_1]\,\omega(t) + \dfrac{1}{\sigma}b(0)v(1,t) \\ \qquad + \dfrac{1}{\sigma}\displaystyle\int_0^1 \Big[(\gamma_1-\beta_1)k(0,y)+k_y(0,y)\Big]v(y,t)dy, \\ \hat{v}_t(x,t) = -\hat{v}_x(x,t) + b(x)\hat{v}(1,t) + L(x)[v(1,t)-\hat{v}(1,t)], \\ \hat{v}(0,t) = \sigma\hat{\omega}(t), \\ \dot{\hat{\omega}}(t) = (k(0,0)-\gamma_1+\beta_1)\hat{\omega}(t) + \dfrac{1}{\sigma}b(0)\hat{v}(1,t) \\ \qquad + \dfrac{1}{\sigma}\displaystyle\int_0^1 \Big[(\gamma_1-\beta_1)k(0,y)+k_y(0,y)\Big]\hat{v}(y,t)dy + \gamma_2[v(1,t)-\hat{v}(1,t)], \\ v(x,0) = v_0(x), \omega(0) = \omega_0, \hat{v}(x,0) = \hat{v}_0(x), \hat{\omega}(0) = \hat{\omega}_0. \end{cases} \tag{4.2}$$

The main result of this paper is the following Theorem 4.1:

Theorem 4.1. *For any initial value $(v_0(\cdot), \omega_0, \hat{v}_0(\cdot), \hat{\omega}_0) \in H \times H$, the closed-loop system (4.2) has a unique (weak) solution $(v(\cdot,t), \omega(t), \hat{v}(\cdot,t), \hat{\omega}(t)) \in C(0,\infty; H\times H)$ such that*

$$\|(v(\cdot,t),\omega(t),\hat{v}(\cdot,t),\hat{\omega}(t))\|_{H\times H} \leqslant M_4 \|\,(v_0(\cdot),\omega_0,\hat{v}_0(\cdot),\hat{\omega}_0)\,\| e^{-\varepsilon t},\ \forall t > 1, \tag{4.3}$$

where $M_4 > 0$ and $\varepsilon = \min\{\gamma_1-\beta_1, -(\beta_1+\sigma(\beta_2-\gamma_2)e^{-\beta_1})\} > 0$.

To prove Theorem 4.1, we consider the following equivalent transformation

$$\begin{pmatrix} v \\ \omega \\ \hat{v} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} (I+\mathbb{P})^{-1} & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma}(I+\mathbb{P})^{-1} & 0 & 0 \\ (I+\mathbb{P})^{-1} & 0 & -I & 0 \\ 0 & \frac{1}{\sigma}(I+\mathbb{P})^{-1} & 0 & -I \end{pmatrix} \begin{pmatrix} h \\ p \\ \tilde{v} \\ \tilde{\omega} \end{pmatrix}. \tag{4.4}$$

Then, the system (4.2) is equivalent to

$$\begin{cases} h_t(x,t) = -h_x(x,t), \\ h(0,t) = p(t), \\ \dot{p}(t) = (\beta_1 - \gamma_1)p(t) + \sigma\zeta(t), \\ \tilde{v}_t(x,t) = -\tilde{v}_x(x,t) + [b(x) - L(x)]\tilde{v}(1,t), \\ \tilde{v}(0,t) = \sigma\tilde{\omega}(t), \\ \dot{\tilde{\omega}}(t) = \beta_1\tilde{\omega}(t) + (\beta_2 - \gamma_2)\tilde{v}(1,t), \\ h(x,0) = h_0(x), p(0) = p_0, \tilde{v}(x,0) = \tilde{v}_0(x), \tilde{\omega}(0) = \tilde{\omega}_0. \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \sigma\zeta(t) &= -\sigma(k(0,0) - \gamma_1)\tilde{\omega}(t) - (b(0) - \sigma\beta_2)\tilde{v}(1,t) \\ &\quad - \int_0^1 \left[(\gamma_1 - \beta_1)k(0,y) + k_y(0,y)\right]\tilde{v}(y,t)dy. \end{aligned} \tag{4.6}$$

The system (4.5) can be divided into two parts $(\tilde{v},\tilde{\omega})-$subsystem and $(h,p)-$subsystem. Since Theorem 3.1 we know that subsystem $(\tilde{v},\tilde{\omega})$ has a unique solution on $C(0,\infty;H)$. In the following, we need to prove (h,p) has a unique solution on $C(0,\infty;H)$.

Lemma 4.1. *For any initial value $(h_0(\cdot),p_0) \in H$, the subsystem (h,p) of system (4.5) has a unique solution $(h(\cdot,t),p(t)) \in C(0,\infty;H)$ that satisfies*

$$\|(h(\cdot,t),p(t))\|_H \leq M_5 e^{-\varepsilon t}, \quad \forall t > 1, \tag{4.7}$$

where $M_5 > 0$ and $\varepsilon = \min\{\gamma_1 - \beta_1, -(\beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1})\} > 0$.

Proof. According to (4.6), Theorem 3.1 and Lemma 2.1, there exist constants $M_6 > 0$ and $\beta_3 < 0$ such that

$$\begin{aligned} |\sigma\zeta(t)| &\leq |\sigma||k(0,0) - \gamma_1||\tilde{\omega}(t)| + |b(0) - \sigma\beta_2||\tilde{v}(1,t)| \\ &\quad + \int_0^1 \left|(\gamma_1 - \beta_1)k(0,y) + k_y(0,y)\right||\tilde{v}(y,t)|dy \\ &\leq M_6|\tilde{\omega}_0|e^{\beta_3 t}, \quad t > 1. \end{aligned} \tag{4.8}$$

The subsystem (h,p) of system (4.5) can be written as

$$\begin{cases} h_t(x,t) = -h_x(x,t), \\ h(0,t) = p(t), \\ \dot{p}(t) = (\beta_1 - \gamma_1)p(t) + \sigma\zeta(t), \\ h(x,0) = h_0(x), p(0) = p_0. \end{cases} \tag{4.9}$$

The system (4.9) can be divided into two parts $h-$subsystem and $p-$subsystem.

First, we solve the $p-$subsystem of system (4.9). Using $p_0 = \sigma\omega_0 - \int_0^1 k(0,y)v_0(y)dy$, from system (4.9) we obtain

$$p(t) = e^{(\beta_1-\gamma_1)t}\left(\sigma\omega_0 - \int_0^1 k(0,y)v_0(y)dy\right) + \int_0^t e^{(\beta_1-\gamma_1)(t-s)}\sigma\zeta(s)ds. \tag{4.10}$$

Hence, using Eqs. (4.8), (2.22) and the Cauchy-Schwarz inequality, there exist constants $M_7 > 0$ and $\varepsilon = \min\{\gamma_1 - \beta_1, -(\beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1})\} > 0$ such that

$$\begin{aligned} |p(t)| &\leq e^{(\beta_1-\gamma_1)t}(|\sigma||\omega_0| + L_1\|v_0\|) + M_6|\tilde{\omega}_0| \int_0^t e^{(\beta_1-\gamma_1)(t-s)+\beta_3 s} ds \\ &= e^{(\beta_1-\gamma_1)t}(|\sigma||\omega_0| + L_1\|v_0\|) + \frac{M_6|\tilde{\omega}_0|}{\beta_1-\gamma_1-\beta_3}\left(e^{(\beta_1-\gamma_1)t} - e^{\beta_3 t}\right) \\ &\leq M_7(|\sigma||\omega_0| + L_1\|v_0\| + |\tilde{\omega}_0|)e^{-\varepsilon t}, \quad t > 1. \end{aligned} \tag{4.11}$$

By solve the $h-$subsystem of system (4.9), we obtain

$$h(x,t) = \begin{cases} h_0(x-t), & t \leq x \leq 1, \\ e^{(\beta_1-\gamma_1)t}p_0 + \int_0^t e^{(\beta_1-\gamma_1)(t-s)}\sigma\zeta(s)ds, & x < t. \end{cases} \tag{4.12}$$

Therefore, using Eqs. (4.8), (2.22) and the Cauchy-Schwarz inequality, we obtain the establishment of Eq. (4.7). □

Using Theorem 3.1 and Lemma 4.1, we obtain the following result:

Lemma 4.2. *For any initial value $(h_0(\cdot), p_0, \tilde{v}_0(\cdot), \tilde{\omega}_0) \in H \times H$, system (4.5) has a unique solution $(h(\cdot,t), p(t), \tilde{v}(\cdot,t), \tilde{\omega}(t)) \in C(0,\infty; H \times H)$ such that*

$$\|(h(\cdot,t), p(t), \tilde{v}(\cdot,t), \tilde{\omega}(t))\|_{H\times H} \leqslant M_8 \|\left(h_0(\cdot), p_0, \tilde{v}_0(\cdot), \tilde{\omega}_0\right)\| e^{-\varepsilon t}, \ \forall t > 1, \tag{4.13}$$

where $M_8 > 0$ and $\varepsilon = \min\{\gamma_1 - \beta_1, -(\beta_1 + \sigma(\beta_2 - \gamma_2)e^{-\beta_1})\} > 0$.

Proof of Theorem 4.1. Since the initial value $(h_0(\cdot), p_0, \tilde{v}_0(\cdot), \tilde{\omega}_0)$ of system 4.5 in $H \times H$, by Theorem 3.1 and Lemma 4.1, system (4.5) has a unique solution $(h(\cdot,t), p(t), \tilde{v}(\cdot,t), \tilde{\omega}(t)) \in C(0,\infty; H \times H)$ that makes Eq. (4.13) hold true. Therefore, using the transform (4.4), we obtain that $(v(\cdot,t), \omega(t), \hat{v}(\cdot,t), \hat{\omega}(t)) \in C(0,\infty; H \times H)$ is a solution of closed-loop system (4.2). In addition, since Eqs. (4.4) and (4.13), it can be conclude that Eq. (4.3) is established. □

5. Applications

In this section, we present our results for application in reliability systems and centralized and distributed reactor systems.

1. Lumped and distributed reactors system. We consider the unstable coupled ODE-PDE cascade system with cyclic flow [2]

$$\begin{cases} v_t(x,t) = -v_x(x,t) + b_0 e^{b(1-x)} v(1,t), \ 0 < x < 1, t > 0, \\ v(0,t) = \omega(t), \ t \geq 0, \\ \dot{\omega}(t) = \beta_1\omega(t) + \beta_2 v(1,t) + U(t), \ t \geq 0, \\ y(t) = v(1,t), \ t \geq 0, \\ v(x,0) = v_0(x), \omega(0) = \omega_0, \ 0 \leq x \leq 1, \end{cases} \tag{5.1}$$

where $\omega(t)$ denotes the dynamics of the scalar property $\omega \in \mathbb{R}$, $v(x,t)$ represents the transport of a scalar property $v(\cdot,t) \in L^2(0,1)$ through a reactor, $\beta_1 > 0$ is the constant term responsible for the generation of ω, and $\beta_2 \in [0,1]$ is the recycle in the system input stream are system parameters and $b_0, b \in \mathbb{R}$. In addition, $U(t)$ is the input and $y(t)$ is the output of system (5.1). When $\omega = 0$, the exponential stability and differentiability of system (1.1) were studied by ([1], [5]), and the output feedback regulation problem was studied in [6]. Cassol and Dubljevic [2] designed a discrete regulator for system (5.1) by using the Cayley-Tustin time discretization transformation [18]. Moreover, in this article, we designed an output feedback controller for system (1.1), which includes system (5.1), in the continuous time domain.

Hence, the result of Theorem 4.1 indicates the output feedback stabilization problem of the ODE-PDE cascade system (5.1).

2. Reliability system. We consider the following simple reliable system ([3], [4])

$$\begin{cases} \dot{p}_0(t) = -\lambda p_0(t) + \int_0^1 \mu(x) p_1(x,t)dx + U(t), \quad t > 0, \\ \partial_t p_1(x,t) = -\partial_x p_1(x,t) - \mu(x)p_1(x,t), \quad 0 < x < 1,\ t > 0, \\ p_1(0,t) = \lambda p_0(t), \quad t \geq 0, \\ y(t) = e^{\int_0^1 \mu(\xi)d\xi} p(1,t), \quad t \geq 0, \\ p_0(0) = p_{0,0}, \quad p_1(x,0) = p_{1,0}, \quad 0 \leq x \leq 1. \end{cases} \tag{5.2}$$

The above system describes a simple device which transfer its state between good state 0 and failure mode 1. Here λ represents the constant failure rate of the device for failure mode; $\mu(x)$ represents the time-dependent repair rate when the device is in state 1 and has an elapsed repair time of $x \in [0,1]$; $p_0(t)$ represents the probability that the device is in state 0 , i.e., the good state, at time t; $p_1(x,t)$ represents the probability density (with respect to repair time) that the failed device is in state 1 and has an elapsed repair time of x at time t. $U(t)$ is the input and $y(t)$ is the output of system (5.1). When $U(t) = 0$ and $y(t) = 0$, the well-posedness and asymptotic behavior of system (5.2) have been well addressed using C_0-semigroup theory in the existing literature ([19], [20]). Recently, Hu et al. [4] designed an adaptive observer for system (5.2). For more explanation of system (5.2), we suggest that readers refer to references ([3], [4], [19], [20]).

We define the following state transformations

$$v(x,t) = e^{\int_0^x \mu(\xi)d\xi} p_1(x,t), \quad \omega(t) = p_0(t) \tag{5.3}$$

and $\sigma = \lambda, \beta_1 = -\lambda, \beta_2 v(1,t) = \int_0^1 \mu(x) e^{-\int_0^x \mu(\xi)d\xi} v(x,t)dx$, then system (5.2) can be written as system (1.1) as the following form

$$\begin{cases} \dot{\omega}(t) = \beta_1 \omega(t) + \beta_2 v(1,t) + U(t), \\ v_t(x,t) = -v_x(x,t) \\ v(0,t) = \sigma\omega(t), \\ y(t) = v(1,t), \quad t \geq 0, \\ \omega(0) = \omega_0, \quad v(x,0) = v_0, \quad 0 \leq x \leq 1. \end{cases} \tag{5.4}$$

Hence, the result of Theorem 4.1 indicates the output feedback stabilization problem of reliable system (5.2).

6. Conclusion

This paper investigates the output feedback stabilization problem of a class of coupled ODE-PDE systems. A state feedback controller for the system was designed based on the novel backstepping transformation. Then, a state observer was designed using the unique measurement signal $v(1,\cdot)$ to achieve real-time estimation of the system state. Furthermore, an observer based output feedback control was established to achieve exponential stability of the system. Our result shows that the output feedback control can make the closed-loop system exponentially stable. Finally, we demonstrated the application of our results in some simple reliability systems and distributed reactor systems.

Funding: This work was supported by the National Natural Science Foundation of China (No: 12301150) and Natural Science Foundation of Xinjiang Uygur Autonomous Region (No: 2024D01C229) , and Talent Project of Tianchi Doctoral Program in Xinjiang Uygur Autonomous Region (Grant No. 51052501847).

Acknowledgements: We are grateful to the anonymous referees, who read carefully the manuscript and made valuable comments and suggestions.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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