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On Minimal (m,n)-Ideal in an Ordered AG-Groupoid

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Abstract. The purpose of this paper is to study the definition of 0-minimal (m,n)-ideal in an ordered AG-groupoid and investigate its properties.

1. Introduction and Preliminaries

The concept of an Abel-Grassmann groupoid (AG-groupoid) [4] was first introduced by M. A. Kazim and M. Naseeruddin in 1972

Definition 1.1. *[2] A groupoid $(S,\cdot)$ is called an* AG-groupoid *or an* LA-semigroup*, if its satisfies left invertive law*

$$(a\cdot b)\cdot c=(c\cdot b)\cdot a,\quad \text{for all}\quad a,b,c\in S.$$

Definition 1.2. *[2] An AG-groupoid S is called a* locally associative *AG-groupoid if it satisfies*

$$(aa)a=a(aa),\quad \text{for all}\quad a\in S.$$

Theorem 1.1. *[2] Let S be a locally associative AG-groupoid then $a^1=a$ and $a^{n+1}=a^na$, for $n\geq 1$; for all $a\in S$.*

Theorem 1.2. *[2] Let S be a locally associative AG-groupoid with left identity then $a^ma^n=a^{m+n}$, $(a^m)^n=a^{mn}$ and $(ab)^n=a^nb^n$, for all $a,b\in S$ and m,n are positive integer.*

Theorem 1.3. *[2] If A and B are any subsets of a locally associative AG-groupoid S, then $(AB)^n=A^nB^n$, for $n\geq 1$.*

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Lemma 1.1. *[4] In an AG-groupoid S its satisfies the* medial law *if*

$$(ab)(cd) = (ac)(bd), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 1.3. *[8] An element $e \in S$ is called* left identity *if $ea = a$ for all $a \in S$.*

Lemma 1.2. *[2] If S is an AG-groupoid with left identity, then*

$$a(bc) = b(ac), \quad \text{for all} \quad a, b, c \in S.$$

Lemma 1.3. *[4] An AG-groupoid S with left identity its satisfies the* paramedial *if*

$$(ab)(cd) = (dc)(ba), \quad \text{for all} \quad a, b, c, d \in S.$$

Definition 1.4. *Let S be an AG-groupoid. A non-empty subset A of S is called an* AG-subgroupoid *of S if $AA \subseteq A$.*

Definition 1.5. *[3] A non-empty subset A of an AG-groupoid S is called a* left (right) ideal *of S if $SA \subseteq A(AS \subseteq S)$. As usual,*

A is called an ideal *if it is both left and right ideal.*

Definition 1.6. *[7] An AG-groupoid S is called regular if for each $a \in S$ there exists $x \in S$ such that $a = (ax)a$.*

The concept of an (m, n)-Dragica N. Krgovic introduced regular semigroup in 1975 [1].

Definition 1.7. *[1] Let S be a semigroup, and let m and n be positive integers. We say that S is called an* (m, n)-regular *if for every element $a \in S$ there exists an $x \in S$ such that $a = a^m x a^n$ (a^0 is defined as an operator element, so that $a^0 x = x a^0 = x$).*

The concept of an (m, n)-ideal and principal (m, n)-ideal in semigroup. was first introduced by S. Lajos in 1961

Definition 1.8. *[1] A non-empty subset A of a semigroup S is called an* (m, n)-ideal *if A satisfies of relation*

$$A^m S A^n \subseteq A$$

where m, n are non-negative integers.

Definition 1.9. *The principal (m, n)-ideal, generated by the element a, is*

$$[a]_{(m,n)} = \bigcup_{i=1}^{m+n} \{a^i\} \cup a^m S a^n.$$

The concept of an (m, n)-ideal in AG-groupoid was first introduced by M. Akram, N.Yaqood, and M.Khan [2] in 2013

Definition 1.10. *[1] A non-empty subset A of an AG-groupoid S is called an* $(m, 0)$-ideal$(0, n$-ideal$)$ *if $A^m S \subseteq A(SA^n \subseteq A)$, for $m, n \in \mathbb{N}$.*

Definition 1.11. *[1] Let S be an AG-groupoid. An AG-subgroupoid A of S is called an (m,n)-*ideal *of S if A satisfies the condition*

$$(A^m S)A^n \subseteq A$$

where m,n are non-negative integers (A^m is suppressed if $m=0$).

The concept of an ordered AG-groupoid was first introduced by T. Shah, I. Rehman, and R. Chinram [9] in 2010.

Definition 1.12. *[9] Let S be a nonempty set, $\cdot$ be a binary operation on S, and $\leq$ be a relation on S. Then $(S,\cdot,\leq)$ is called an* ordered AG-groupoid *if $(S,\cdot)$ is an AG-groupoid, $(S,\leq)$ is a partially ordered set, and for all $a,b,c \in S$, $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$.*

Theorem 1.4. *[9] An ordered AG-groupoid S is an ordered semigroup if and only if $a(bc)=(cb)a$ for all $a,b,c \in S$.*

For $H \subseteq S$, let $(H] = \{t \in S \mid t \leq h \quad \text{for some} \quad h \in H\}$. This lemma is similar to the case of ordered semigroups.

Lemma 1.4. *[9] Let S be an ordered AG-groupoid and A,B be subsets of S. The following statements hold:*

(1) $A \subseteq (A]$.
(2) *If $A \subseteq B$ then $(A] \subseteq (B]$.*
(3) $(A](B] \subseteq (AB]$.
(4) $((A](B]] = (AB]$.
(5) $(A \cup B] = (A] \cup (B]$.

Definition 1.13. *[9] A nonempty subset A of an ordered AG-groupoid S is called a* left ideal *of S if $(A] \subseteq A$ and $SA \subseteq A$ and called* right ideal *of S if $(A] \subseteq A$ and $AS \subseteq A$. A nonempty subset A of S is called an* ideal *if A is both left and right ideals of S.*

The concept of the minimal ideal of AG-groupoids was first introduced by M. Khan, KP. Shum and M. Faisal Iqba [5] in 2013.

Definition 1.14. *[5] Let S be an AG-groupoid and I be an ideal of S. S is said to be* minimal left (right) ideal *of S if I does not contain any other left (right) ideal other than itself.*

Definition 1.15. *[5] Let S be an AG-groupoid and I be an ideal of S. S is said to (m,n)-*minimal ideal *of S if it is minimal in the set of all nonzero ideals of S.*

2. On 0-Minimal $(0,2)$-Bi-Ideal in an AG-Groupoid

In section we study some properties of A is $(0,2)$-ideal and we defined 0-minimal $(0,2)$-bi-ideal in an AG-groupoid. Final we study relation of A is $(0,2)$-ideal and 0-minimal $(0,2)$-bi-ideal in an AG-groupoid.

Lemma 2.1. *Let S be a locally associative AG-groupoid with left identity and let A be an ideal of S. Then $S^2 = S, SA^2 = A^2S$ and $A \subseteq SA$ for all $A \subseteq S$.*

Proof. It is clear that $S^2 \subseteq S$. Let $x \in S$ then $x = ex \in SS$ so $S \subseteq SS = S^2$. That is $S = S^2$. Next show that $SA^2 = A^2S$ By Lemma **??** we have

$$SA^2 = S^2A^2 = (S \cdot S)(A \cdot A) = (A \cdot A)(S \cdot S) = A^2S^2 = A^2S.$$

Assume that $A \subseteq S$. To show that $A \subseteq SA$, let $a \in A$. Then $a = ea$ for all $e \in A \subseteq S$. Since $e \in S$ we have $ea \in SA$ so $A \subseteq SA$. □

Theorem 2.1. *Let S be an AG-groupoid with left identity and let A is an ideal of S. Then SA and SA^2 are ideal of S.*

Proof. Assume that S is an AG-groupoid with left identity and A is an ideal of S. Then

$$(S \cdot A)S \subseteq AS \subset A \subseteq SA \quad \text{and} \quad S(S \cdot A) \subseteq SA.$$

This shows that SA is an ideal of S. To show that SA^2 is an ideal of S. Now

$$(S \cdot A^2)S = (A^2 \cdot S)S = (S \cdot S)A^2 \subseteq SA^2$$

and

$$S(S \cdot A^2) = S(A^2 \cdot S) = A^2(S \cdot S) \subseteq A^2S = SA^2.$$

This shows that SA^2 is an ideal of S. □

Lemma 2.2. *Let S be an AG-groupoid with left identity. Then A is a $(0, 2)$-ideal of S if and only if A is an ideal of some left ideal of S.*

Proof. Let A be a $(0, 2)$-ideal of S, then $(S \cdot A)A = (A \cdot A)S = A^2S = SA^2 \subseteq A$ and $A(S \cdot A) = S(A \cdot A) = (S \cdot S)(A \cdot A) = SA^2 \subseteq A$. Hence, A is a left ideal SA of S.

Conversely, assume that A is a left ideal of a left ideal L of S, then

$$SA^2 = A^2S = (A \cdot A)S = (S \cdot A)A \subseteq (S \cdot L)A \subseteq LA \subseteq A;$$

and clearly A is an AG-subgroupoid of S, therefore A is a $(0, 2)$-ideal of S. □

Corollary 2.1. *Let S be an AG-groupoid with left identity. Then A is a $(0, 2)$-ideal of S if and only if A is a left ideal of some left ideal of S.*

Definition 2.1. *An AG-subgroupoid A of an AG-groupoid S is called a $(0, 2)$-bi-ideal of S if A is both a bi-ideal and a $(0, 2)$-ideal of S.*

Lemma 2.3. *Let S be an AG-groupoid with left identity. Then A is a $(0, 2)$-bi-ideal of S if and only if A is an ideal of some right ideal of S.*

Proof. Let A be a $(0,2)$-bi-ideal of S, then

$$\begin{aligned} SA^2 \cdot A &= A^2S \cdot A = AS \cdot A^2 = AS \cdot AA \\ &= AA \cdot S \cdot A \subseteq A \cdot SA = S \cdot AA = SA^2 \subseteq A. \end{aligned}$$

and

$$\begin{aligned} A \cdot SA^2 &= S \cdot AA^2 = SS \cdot AA^2 \\ &= A^2A \cdot SS = A^2A \cdot S = SA \cdot A^2 \\ &= (SS)A \cdot A^2 = (AS)S \cdot A^2 = A^2S \cdot AS \\ &= A^2A \cdot SS = A^2A \cdot S = A^2S \\ &= SA^2 \subseteq A. \end{aligned}$$

Hence A is an ideal of some right ideal SA^2 of S.

Conversely, assume that A is an ideal of a right ideal R of S, then

$$SA^2 = S \cdot AA = A \cdot SA = A \cdot (SS)A = A \cdot (AS)S \subseteq A \cdot (RS)S \subseteq AR \subseteq A.$$

and $(AS)A \subseteq (RS)A \subseteq RA \subseteq SA = (SS)A = (AS)S \subseteq AS \subseteq A$, which shows that A is a $(0,2)$-ideal of S. □

Theorem 2.2. *Let S be an AG-groupoid with left identity. Then the following statements are equivalent.*

(1) *A is a $(1,2)$-ideal of S,*
(2) *A is a left ideal of some bi-ideal of S,*
(3) *A is a bi-ideal of some ideal of S,*
(4) *A is a $(0,2)$-ideal of some right ideal of S,*
(5) *A is a left ideal of some $(0,2)$-ideal of S.*

Proof. $(1) \Rightarrow (2)$ To show that $SA^2 \cdot S$ is a bi-ideal of S, let $B := SA^2 \cdot S$ then

$$\begin{aligned} BS \cdot B &= [(SA^2 \cdot S)S] \cdot [SA^2 \cdot S] = [(S \cdot S)(SA^2)] \cdot [SA^2 \cdot S] \\ &= [(S \cdot S)(A^2S)] \cdot [SA^2 \cdot S] = [(S \cdot A^2)(SS)] \cdot [SA^2 \cdot S] \\ &= [(S \cdot A^2)S] \cdot [SA^2 \cdot S] = [(S \cdot A^2)(SA^2)] \cdot (S \cdot S) \\ &\subseteq SA^2 \cdot S = B. \end{aligned}$$

Similarly $[SA^2 \cdot S]^2 = (SA^2 \cdot S)(SA^2 \cdot S) = (SA^2 \cdot SA^2)(S \cdot S) = (SA^2)^2S^2 \subseteq SA^2 \cdot S$. Thus B is a bi-ideal of S. Let A be a $(1,2)$-ideal of S, then

$$\begin{aligned} (SA^2 \cdot S)A &= ((SA^2) \cdot (SS))A = ((SS) \cdot (A^2S))A = (S \cdot (A^2S))A \\ &= (A^2 \cdot (SS))A = (A^2 \cdot S)A = (A \cdot S)A^2 \subseteq AA^2 \subseteq A. \end{aligned}$$

which shows that A is a left ideal of a bi-ideal $(S \cdot A^2)S$ of S.

$(2) \Rightarrow (3)$ Let A be a left ideal of a bi-ideal B of S, then

$$\begin{aligned}
(A \cdot SA^2)A &= (S \cdot AA^2)A = (S \cdot (A(AA)))A \\
&\subseteq (S \cdot ((SA)(AA)))A = (S \cdot ((AA)(AS)))A \\
&= ((AA) \cdot S(AS))A = [((S(AS))A) \cdot A]A \\
&= [((A(SS))A) \cdot A]A = [((AS)A) \cdot A]A \\
&\subseteq [((BS)B) \cdot A]A \subseteq (B \cdot A)A \subseteq A.
\end{aligned}$$

which shows that A is a bi-ideal of an ideal SA^2 of S.

$(3) \Rightarrow (4)$ Let A be a bi-ideal of an ideal I of S, then

$$\begin{aligned}
SA^2 \cdot A^2 &= A^2A^2 \cdot S = A^2(AA) \cdot S = A(A^2A) \cdot S \\
&= A((AA)A) \cdot S \subseteq A((AI)A) \cdot S \subseteq AA \cdot S \\
&= SA \cdot A \subseteq SI \cdot S \subseteq IS \subseteq I.
\end{aligned}$$

which shows that A is a $(0,2)$-ideal of a right ideal SA^2 of S.

$(4) \Rightarrow (5)$ To show that SA^3 is a $(0,2)$-ideal of S, let $K := SA^3$ then

$$\begin{aligned}
SK^2 &= S(SA^3)^2 = S[(SA^3)(SA^3)] = S[(SS)(A^3A^3)] \\
&= (SS)[S(A^3A^3)] = S[A^3(SA^3)] \subseteq S(A^3A^3) \subseteq SA^3.
\end{aligned}$$

Similarly By. $(S \cdot A^3)^2 = (S \cdot A^3)(S \cdot A^3) = (S \cdot S)(A^3 \cdot A^3) \subseteq SA^3$. Thus SA^3 is a $(0,2)$-ideal of S. Let A be a $(0,2)$-ideal of a right ideal R of S then

$$\begin{aligned}
A \cdot SA^3 &= A \cdot (SS)(A^2A) = A \cdot (AA^2)(SS) = A \cdot (AA^2)S \\
&\subseteq A \cdot [(SA)(AA)S] = A \cdot [(AA)(AS)S] = (AA) \cdot [(A(AS))S] \\
&= [S(A(AS))] \cdot (AA) = [S(A(AS))] \cdot A^2 = [A(S(AS))] \cdot A^2 \\
&\subseteq [A(SA)] \cdot A^2 \subseteq [R(SS)] \cdot A^2 \subseteq RS \cdot A^2 \subseteq RA^2 \subseteq A,
\end{aligned}$$

which shows that A is a left ideal of a $(0,2)$-ideal SA^3 of S.

$(5) \Rightarrow (1)$ Let A be a left ideal of a $(0,2)$-ideal O of S, then

$$AS \cdot A^2 = A^2S \cdot A = (AA)(SS) \cdot A = (SS)(AA) \cdot A = SA^2 \cdot A \subseteq SO^2 \cdot A \subseteq OA \subseteq A.$$

which shows that A is a $(1,2)$-ideal of S. □

Definition 2.2. *An element $a \in S$ is called* idempotent *if $a = a^2$. if I is subset of S is called idempotent if every elements of I is idempotent.*

Lemma 2.4. *Let S be an AG-groupoid with left identity, and let A be an idempotent subset of S. Then A is a $(1,2)$-ideal of S if and only if there exist a left ideal L and a right ideal R of S such that $RL \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-ideal of S such that A is idempotent. Setting $L := SA$ and $R := SA^2$ then

$$\begin{aligned} RL &= SA^2 \cdot SA = A^2S \cdot SA = (SA \cdot S)A^2 \\ &= (SA \cdot SS)A^2 = (SS \cdot AS)A^2 = (SS \cdot A^2S)A^2 \\ &= (SA^2 \cdot SS)A^2 = (S(AA) \cdot SS)A^2 = (S(SS) \cdot (AA))A^2 \\ &= [(S\{(A \cdot (SS)A)\}]A^2 = [S(A \cdot SA)]A^2 = [A(S \cdot SA))]A^2 \\ &\subseteq [A(S(SS))]A^2 = [A(SS)]A^2 \subseteq (AS)A^2 \subseteq A. \end{aligned}$$

It is clear that $A = A^2 = AA \subseteq AA \cap AA \subseteq RA \cap AL \subseteq RS \cap SL \subseteq R \cap L$.

Conversely, let R be a right ideal and L be a left ideal of S such that $RL \subseteq A \subseteq R \cap L$, then $(AS) \cdot A^2 = AS \cdot AA \subseteq RS \cdot SL \subseteq RL \subseteq A$.

□

Assume that S is an AG-groupoid with a left identity with zero. Then it is easy to see that every left (right) ideal of S is a $(0,2)$-ideal of S. Hence if O is a 0-minimal $(0,2)$-ideal of S and A is a left (right) ideal of S contained in O, then either$A = \{0\}$ or $A = O$.

Definition 2.3. *An ideal I of S is called* 0-minimal ideal *if S has a zero 0, $I \neq \{0\}$, and the only ideal of S contained in I are $\{0\}$ and it self.*

Definition 2.4. *A $(0,2)$-bi-ideal A of an AG-groupoid S with zero element 0 will be said to be 0-minimal if $A \neq 0$ and $\{0\}$ is the only $(0,2)$-bi-ideal of S property contained in A.*

Lemma 2.5. *Let S be an AG-groupoid with left identity S with zero. Assume that A is a 0-minimal ideal of S and O is an AG-subgroupoid of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Proof. Let O be a $(0,2)$-ideal of S contained in a 0-minimal ideal A of S. Then $SO^2 \subseteq O \subseteq A$ Since SO^2 is an ideal of S therefore by minimality of A, $SO^2 = \{0\}$ or $SO^2 = A$. If $SO^2 = A$, then $A = SO^2 \subseteq O$ and therefore $O = A$. Let $SO^2 = \{0\}$, then $O^2S = SO^2 = \{0\} \subseteq O^2$, which shows that O^2 is a right ideal of S, and hence an ideal of S contained in A, therefore by minimality of A, we have $O^2 = \{0\}$ or $O^2 = A$. Now if $O^2 = A$ then $O = A$.

Conversely, let $O^2 = \{0\}$, then $SO^2 = O^2S = \{0\}S = \{0\} = O^2$. Now if $O = A$, then $SO^2 = SS \cdot OO = SO \cdot SO = SA \cdot SA \subseteq AA \subseteq A = O$ which shows that O is a $(0,2)$-ideal of S contained in A. □

Corollary 2.2. *Let S be an AG-groupoid with left identity S with zero. Assume that A is a 0-minimal left ideal of S and O is an AG-subgroupoid of A. Then O is a $(0,2)$-ideal of S contained in A if and only if $O^2 = \{0\}$ or $O = A$.*

Lemma 2.6. *Let S be an AG-groupoid with left identity S with zero. and O be a 0-minimal $(0,2)$-ideal of S. Then $O^2 = \{0\}$ or O is a 0-minimal right (left) ideal of S*

Proof. Let O be a 0-minimal $(0,2)$-ideal of S, then

$$S(O^2)^2 = SS \cdot O^2O^2 = O^2O^2 \cdot SS = O^2O^2 \cdot S = SO^2 \cdot O^2 \subseteq OO^2 \subseteq O^2$$

which shows that O^2 is a $(0,2)$-ideal of S contained in O, therefore by minimality of O. Then $O^2 = \{0\}$ or $O^2 = O$. Suppose that $O^2 = O$, then $OS = OO \cdot SS = SS \cdot OO = SO^2 \subseteq O$ which shows that O is a right ideal of S. Let R be a right ideal of S contained in O, then $R^2S = RR \cdot S \subseteq RS \cdot S \subseteq RS \subseteq R$ Thus, R is a $(0,2)$-ideal of S contained in O, and again by minimality of O, $R = \{0\}$ or $R = O$. □

The following corollary follows from Lemma 2.5 and Corollary 2.2.

Corollary 2.3. *Let S be an AG-groupoid with left identity. Then O is a minimal $(0,2)$-ideal of S if and only if O is a minimal left ideal of S.*

Definition 2.5. *Let S be an AG-groupoid. A bi-ideal B of S said to be a* minimal bi-ideal *of S if B does not contain any other proper bi-ideal of S.*

Theorem 2.3. *Let S be an AG-groupoid with left identity. Then A is a minimal $(2,1)$-ideal of S if and only if A is a minimal bi-ideal of S.*

Proof. Let A be a minimal $(2,1)$-ideal of S. Then

$$\begin{aligned}
[((A^2S \cdot A)^2S](A^2S \cdot A) &= [\{(A^2S \cdot A)(A^2S \cdot A)\}S](A^2S \cdot A) \subseteq [\{(AS \cdot A)(AS \cdot A)\}S](AS \cdot A)\\
&= [\{(AS)(AS)(A \cdot A)\}S](AS \cdot A) = [\{(AA)(SS)(A \cdot A)\}S](AS \cdot A)\\
&= [(A^2S \cdot AA)S](AS \cdot A) \subseteq [(A^2S \cdot AS)S](AS \cdot A)\\
&= [(AS \cdot AS)S](AS \cdot A) = [(AA \cdot SS)S](AS \cdot A)\\
&= (A^2S)S)(AS \cdot A) \subseteq (AS)S)(AS \cdot A)\\
&= ((AS \cdot AS))(S \cdot A) = (AA \cdot SS))(S \cdot A)\\
&= (A^2 \cdot SS)(S \cdot A) = (A^2S \cdot SA)\\
&= (AS \cdot SA^2) = (SA^2 \cdot S)A\\
&= (A^2S \cdot S)A = (AA)(S \cdot S)A\\
&= (SS)(A \cdot A)A = (SA^2)A = (A^2S)A.
\end{aligned}$$

and similarly we can show that $(A^2S \cdot A)^2 \subseteq A^2S \cdot A$. Now

$$\begin{aligned}
(A^2S \cdot A)^2 &= (A^2S \cdot A)(A^2S \cdot A) \subseteq (AS \cdot A)(AS \cdot A)\\
&= [AS \cdot (AS)](A \cdot A) = [AA \cdot (SS)](A \cdot A)\\
&= [(AA) \cdot (SS)]A^2 = (A^2 \cdot S)A^2 \subseteq (A^2 \cdot S)A.
\end{aligned}$$

Then $(A^2 \cdot S)A$ is AG-subgroupoid. Thus $(A^2 \cdot S)A$ is a $(2,1)$-ideal of S contained in A, therefore by minimality of A and $(A^2 \cdot S)A = A$. Now

$$\begin{aligned}
AS \cdot A &= (AS) \cdot (A^2 \cdot S)A = ((A^2 \cdot S)A)S \cdot A = (SA \cdot A^2S)A\\
&= [(A^2(SA \cdot S)]A \subseteq [(A^2(SS \cdot S)]A = [(A^2(S \cdot S)]A\\
&= (A^2 \cdot S)A = A.
\end{aligned}$$

It follows that A is a bi-ideal of S Suppose that there exists a bi-ideal B of S contained in A, then $(B^2 \cdot S)B \subseteq (B \cdot S)B \subseteq B$, so B is a $(2,1)$-ideal of S contained in A, therefore $B = A$.

Conversely, assume that A is a minimal bi-ideal of S, then $(A^2 \cdot S)A \subseteq (A \cdot S)A \subseteq A$ and $A^2 \subseteq A$ so A is a $(2,1)$-ideal of S. Let C be a $(2,1)$-ideal of S contained in A, then

$$\begin{aligned}
[(C^2S \cdot C)S](C^2S \cdot C) &= (SC \cdot C^2S)(C^2S \cdot C) \\
&= (SC^2 \cdot CS)(C^2S \cdot C) \\
&= [C(SC^2 \cdot S)](C^2S \cdot C) \\
&= [(C^2S \cdot C)(SC^2 \cdot S)]C \\
&= [(C^2S \cdot C)(SC^2 \cdot SS)]C \\
&= [(C^2S \cdot C)(SS \cdot C^2S)]C \\
&= [(C^2S \cdot C)(S \cdot C^2S)]C \\
&= [(C^2S \cdot C)(C^2 \cdot SS)]C \\
&= [(C^2S \cdot C)(C^2 \cdot S)]C \\
&= [C^2\{C^2S \cdot C)S\}]C \\
&= C^2(CS) \cdot C \subseteq C^2(SS) \cdot C \\
&= C^2S \cdot C.
\end{aligned}$$

This shows that $C^2S \cdot C$ is a bi-ideal of S, and by minimality of A and $C^2S \cdot C = A$. Thus $A = (C^2 \cdot S)C \subseteq C$, and therefore A is a minimal $(2,1)$-ideal of S. □

Definition 2.6. *Let S be an AG-groupoid. A bi-ideal B of S said to be a* 0-minimal bi-ideal *of S. If B does not contain any other proper non-zero bi-ideal of S.*

Theorem 2.4. *Let A be a* 0*-minimal* $(0,2)$*-bi-ideal of an AG-groupoid with left identity S with zero. Then exactly one of the following cases occurs:*

(1) $A = \{0, a\}, a^2 = 0,$
(2) $\forall a \in A \setminus \{0\}, SA^2 = A.$

Proof. Assume that A is a 0-minimal $(0,2)$-bi-ideal of S. Let $a \in A \setminus \{0\}$; then $Sa^2 \subseteq A$. Also Sa^2 is a $(0,2)$-bi-ideal of S, therefore $Sa^2 = \{0\}$ or $Sa^2 = A$.

Let $Sa^2 = \{0\}$. Since $a^2 \in A$ we have either $a^2 = a$ or $a^2 = 0$ or $a^2 \in A \setminus \{0, a^2\}$. If $a^2 = a$ then $a^3 = a^2a = a$, which is impossible because $a3 \in a^2S = Sa^2 = \{0\}$. Let $a^2 \in A \setminus \{0, a\}$ we have

$$S \cdot \{0, a^2\}\{0, a^2\} = SS \cdot a^2a^2 = Sa^2 \cdot Sa^2 = \{0\} \subseteq \{0, a^2\}$$

and

$$[\{0, a^2\}S]\{0, a^2\} = [\{0, a^2S\}\{0, a^2\} = a^2 \cdot Sa^2 \subseteq Sa^2 = \{0\} \subseteq \{0, a^2\}.$$

Therefore $\{0, a^2\}$ is a $(0,2)$-bi-ideal of S contained in A. We observe that $\{0, a^2\} \neq \{0\}$ and $\{0, a^2\} \neq A$. This is a contradiction to the fact that A is a 0-minimal $(0,2)$-bi-ideal of S. Therefore $a^2 = 0$ and $A = \{0, a\}$.

If $Sa^2 \neq \{0\}$, then $Sa^2 = A$. □

Corollary 2.4. *Let A be a 0-minimal $(0,2)$-bi-ideal of AG-groupoid with left identity S with zero such that $A^2 \neq 0$. Then $A = Sa^2$ for every $a \in A \setminus \{0\}$.*

Lemma 2.7. *Let S be an AG-groupoid with left identity. Then every right ideal of S is a $(0,2)$-bi-ideal of S.*

Proof. Assume that A is a right ideal of S. By Lemma 1.1 and **??** then

$$SA^2 = S \cdot AA = SS \cdot AA = AA \cdot SS = AS \cdot AS \subseteq AA \subseteq AS \subseteq A$$

and $(A \cdot S)A \subseteq A$. It is clearly $A^2 \subseteq A$ therefore A is a $(0,2)$-bi-ideal of S. □

Definition 2.7. *An AG-groupoid S with zero is said to be $0-(0,2)$-bi-simple if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-ideal of S*

Theorem 2.5. *Let S be an AG-groupoid with left identity S with zero. Then $S \cdot a^2 = S$ $\forall a \in S \setminus \{0\}$ if and only if S is a 0-$(0,2)$-bisimple if and only if S is right 0-simple.*

Proof. Assume that $Sa^2 = S$ for every $a \in S \setminus \{0\}$. Let A be a $(0,2)$-bi-ideal of S such that $A \neq \{0\}$. Let $a \in A \setminus \{0\}$, then $S = Sa^2 \subseteq SA^2 \subseteq A$. Therefore $S = A$. Since $S = Sa^2 \subseteq SS = S^2$, we have $S^2 = S \neq \{0\}$. Thus S is 0-$(0,2)$-bisimple. The converse statement follows from Corollary 2.4 Let R be a right ideal of a 0-$(0,2)$-bisimple S. Then by Lemma 2.7, R is a $(0,2)$-bi-ideal of S and so $R = \{0\}$ or $R = S$:

Conversely, assume that S is right 0-simple. Let $a \in S \setminus \{0\}$, then $Sa^2 = S$. Hence S is a 0-$(0,2)$-bisimple. □

Theorem 2.6. *Let A be a 0-minimal $(0,2)$-bi-ideal of AG-groupoid with left identity S with zero. Then either $A^2 = \{0\}$ or A is right 0-simple.*

Proof. Assume that A is 0-minimal $(0,2)$-bi-ideal of S such that $A^2 = \{0\}$. Then by using Corollary 2.4, $Sa^2 = A$ for every $a \in A \setminus \{0\}$. Since $a^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$. we have $a^4 = (a^2)^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$. Let $a \in A \setminus \{0\}$, then

$$\begin{aligned}
(Aa^2)S \cdot Aa^2 &= (a^2A)S \cdot (Aa^2) = [(S \cdot Aa^2)A]a^2 \subseteq [SA \cdot A]a^2 \\
&= [AA \cdot S]a^2 = [A^2SS]a^2 = A^2S \cdot a^2 \\
&= SA^2 \cdot a^2 = Aa^2.
\end{aligned}$$

and

$$\begin{aligned}
S(Aa^2)^2 &= S(Aa^2 \cdot Aa^2) = S(a^2A \cdot a^2A) = a^2(S \cdot (a^2A \cdot A)) \\
&= (aa)(S \cdot (a^2A \cdot A)) = ((a^2A \cdot A)S)(aa) \subseteq ((AA \cdot A)S)a^2 \subseteq (AA \cdot S)a^2 \\
&= A^2S \cdot a^2 = SA^2 \cdot a^2 = Aa^2.
\end{aligned}$$

which shows that Aa^2 is a $(0,2)$-bi-ideal of S contained in A. Hence $Aa^2 = \{0\}$ or $Aa^2 = A$. Since $a^4 \in Aa^2$ and $a^4 \in A \setminus \{0\}$, we get $Aa^2 = A$ Thus by using Theorem 2.5, A is right 0-simple. □

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