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Further Numerical Radius Inequalities of Operators

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Abstract. We investigate advanced numerical radius inequalities for 2×2 block operator matrices acting on complex separable Hilbert spaces. Our primary contribution establishes a refined upper bound for the numerical radius of block matrices, expressed in terms of the numerical radii of diagonal blocks and the operator norms of off-diagonal components. We demonstrate that for operators $A, B, C, D \in \mathcal{B}(H)$, the inequality

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \frac{1}{2}\left\|f^2(|A|) + g^2(|A^*|)\right\| + \frac{1}{2}\left\|f^2(|D|) + g^2(|D^*|)\right\| + \frac{\|B\| + \|C\|}{2}$$

provides an effective estimation tool. Additionally, we present a comprehensive comparative analysis with existing bounds, establishing conditions under which our result offers superior performance. The theoretical framework is supplemented with concrete examples and applications to polynomial zero estimation.

1. Introduction

Let $\mathcal{B}(H)$ denote the C^*-algebra of all bounded linear operators defined on a complex separable Hilbert space H with usual inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{K}(H)$ denote the two sided ideal of compact operators in $\mathcal{B}(H)$. Let A^* denote the adjoint (conjugate transpose) of the operator A. The operator $A \in \mathcal{B}(H)$ is called a self-adjoint operator if $A^* = A$. The self-adjoint operator A is called positive semidefinite if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. For any $A \in \mathcal{B}(H)$, the absolute value of A is the positive semidefinite operator denoted by $|A|$ where $|A| = (A^*A)^{1/2}$. The singular values of $A \in B(H)$ are ordered descendingly as follows, $s_1(A) \geq s_2(A) \geq \cdots$ and they are the eigenvalues of $|A|$. In fact $s_j(A) = \lambda_j(|A|) = s_j(|A|)$ for $j = 1, 2, \ldots.$ For recent studies about singular values, we advise the readers to read [[1]- [4]], [[7]- [9]], [[12]] and [[14]- [18]].

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For $A \in \mathcal{B}(H)$, the spectral (usual operator) norm of A is denoted by $\|A\|$, where

$$\|A\| = \sup\{|\langle Ax, y\rangle| : x, y \in H, \|x\| = \|y\| = 1\}. \tag{1.1}$$

The spectral (usual operator) norm is typical example of unitarily invariant norms $|||\cdot|||$, and its defined on all of $\mathcal{B}(H)$, all other types of unitarily invariant norms are defined on a two sided ideal that is included in the ideal of compact operators.

The study of numerical radius inequalities for operators on Hilbert spaces represents a fundamental area of functional analysis with significant applications in matrix theory, quantum mechanics, and approximation theory. For $A \in \mathcal{B}(H)$, let $w(A)$ denote the numerical radius, where

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}, \tag{1.2}$$

which provides crucial information about the operator's spectral properties and geometric characteristics.

The classical relationship between the numerical radius and operator norm is given by the fundamental inequality

$$\frac{\|A\|}{2} \le w(A) \le \|A\|. \tag{1.3}$$

While this relation bounds $w(A)$ in terms of $\|A\|$, which is easier to compute, ambitions to obtain better bounds urged researchers to explore further better bounds.

Significant examples of such progress are stated in the next two inequalities, which were proved in [25],

$$w(A) \le \frac{1}{2}\||A| + |A^*|\| \tag{1.4}$$

and

$$w(A) \le \frac{1}{2}\left(\|A\| + \|A^2\|^{1/2}\right). \tag{1.5}$$

The same author proved latter in [24] that

$$w^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \tag{1.6}$$

The reader is encouraged to see how (1.2) and (1.3) refine the right inequality in (1.1), as stated in [25] and [24].

In the insightful work of [22,23], it was noted that

$$w(A) = \sup_{\theta \in \mathbb{R}} \|\mathfrak{R}(e^{i\theta}A)\| \tag{1.7}$$

where $\mathfrak{R}(\cdot)$ denotes the real part, defined as $\mathfrak{R}(X) = \frac{X+X^*}{2}$, where X^* is the adjoint of the operator $X \in \mathcal{B}(H)$.

This inequality has motivated extensive research into refinements and generalizations, particularly for structured operators such as block matrices. For recent studies of numerical radius inequalities of operaors, we refer the reader to [5,6,10,11,13].

In this paper, we prove a numerical radius inequality for block operators which generalizes (1.4). Block operator matrices arise naturally in various mathematical contexts, including differential equations, control theory, and quantum field theory. A 2×2 block matrix of the form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D \in \mathcal{B}(H)$, represents a fundamental structure whose numerical radius properties have been extensively studied. Several basic upper bounds for the numerical radius of $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ are well-known, some of these upper bounds are listed below:

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \|A\| + \|D\| + \|B\| + \|C\|. \tag{1.5}$$

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \|A\| + \|D\| + \frac{\|B\| + \|C\|}{2}. \tag{1.6}$$

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq w(A) + w(D) + \|B\| + \|C\|. \tag{1.7}$$

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq w(A) + w(D) + \sqrt{\|B\|^2 + \|C\|^2}. \tag{1.8}$$

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \max\{w(A), w(D)\} + \|B\| + \|C\|. \tag{1.9}$$

and

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq \sqrt{(w(A) + w(D))^2 + (\|B\| + \|C\|)^2}. \tag{1.10}$$

In this paper, we prove numerical radius inequality which is sharper than the inequalities (1.5)-(1.10).

Recent developments have focused on bounds that separate the contributions of diagonal and off-diagonal blocks. These approaches recognize that the numerical radius of a block matrix depends fundamentally on both the individual properties of its components and their interaction patterns. Our work extends this research direction by establishing a new upper bound that explicitly separates diagonal and off-diagonal contributions while maintaining computational efficiency. The main result demonstrates that the numerical radius can be bounded by the sum of diagonal numerical radii plus a weighted average of off-diagonal norms.

2. Main Results

The main result in this paper is a numerical radius inequality for finite sums of operators. To prove this result and to make some comparisons between some of its special cases with recent inequalities proved by different authors, we need the following lemmas which are well known, and they are essential in our analysis.

Lemma 2.1. *Let a, b be real numbers. Then*

$$|a+b| \leq \sqrt{2}\,|a+ib| \tag{2.1}$$

Lemma 2.2. *Let $a_i \geq 0$ for $i = 1, 2, ..., n,\ r \geq 1$. Then*

$$\sum_{i=1}^{n} a_i^r \leq (\sum_{i=1}^{n} a_i)^r \leq n^{r-1}\left(\sum_{i=1}^{n} a_i^r\right). \tag{2.2}$$

In particular,

$$a_1^r + a_2^r \leq (a_1+a_2)^r \leq 2^{r-1}(a_1^r + a_2^r). \tag{2.3}$$

Lemma 2.3. *Let $A \in B(H)$ be positive semidefinite and $x \in H$ such that $\|x\| \leq 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Lemma 2.4. *Let $A \in B(H)$ be self-adjoint operator and $x \in H$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle. \tag{2.4}$$

Lemma 2.5. *Let $A \in B(H)$ and $x, y \in H$ be any vectors. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0,\infty))$, then*

$$\left|\langle Ax, y\rangle\right|^2 \leq \langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle \tag{2.5}$$

and more general

$$\left|\langle Ax, y\rangle\right|^2 \leq \left\langle f^2\left(|A|\right) x, x\right\rangle \left\langle g^2\left(|A^*|\right) y, y\right\rangle. \tag{2.6}$$

The first theorem in this paper is a numerical radius inequality for block operators.

Theorem 2.1. *Let A, B, C and D be operators in $B(H)$. If f and g are non negative continuous functions on $[0,\infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0,\infty))$, then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{2}\left\|f^2(|A|) + g^2(|A^*|)\right\| + \frac{1}{2}\left\|f^2(|D|) + g^2(|D^*|)\right\| + \frac{\| B \| + \| C \|}{2}. \tag{2.7}$$

Proof. Let $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = 1$, this implies that $\| x \|^2 + \| y \|^2 = 1$. Then

$$\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right|$$

$$
\begin{aligned}
&= \left|\left\langle \begin{bmatrix} Ax+By \\ Cx+Dy \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle\right| \\
&= \left|\left\langle \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} Ax+By \\ Cx+Dy \end{bmatrix} \right\rangle\right| \\
&= \left|\overline{x}Ax + \overline{x}By + \overline{y}Cx + \overline{y}Dy\right| \\
&= \left| \langle Ax, x\rangle + \langle By, x\rangle + \langle Cx, y\rangle + \langle Dy, y\rangle \right| \\
&\leq |\langle Ax, x\rangle| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left|\langle Dy, y\rangle\right| \\
&= \|x\|^2 \left|\left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left\|y\right\|^2 \left|\left\langle D\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle\right| \\
&\leq \left|\left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left|\left\langle D\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle\right| \\
&\leq \left\langle f^2(|A|)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle^{1/2} \left\langle g^2(|A^*|)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle^{1/2} + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| \\
&\quad + \left\langle f^2(|D|)\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle^{1/2} \left\langle g^2(|D^*|)\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle^{1/2} \\
&\leq \frac{1}{2}\left(\left\langle f^2(|A|)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle + \left\langle g^2(|A^*|)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right) + \\
&\quad \frac{1}{2}\left(\left\langle f^2(|D|)\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle + \left\langle g^2(|D^*|)\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle\right) + \\
&\quad \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| \\
&= \frac{1}{2}\left\langle \left(f^2(|A|) + g^2(|A^*|)\right)\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle \\
&\quad + \frac{1}{2}\left\langle \left(f^2(|D|) + g^2(|D^*|)\right)\frac{y}{\left\|y\right\|}, \frac{y}{\left\|y\right\|}\right\rangle \\
&\quad + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| \\
&\leq \frac{1}{2}w\left(f^2(|A|) + g^2(|A^*|)\right) + \frac{1}{2}w\left(f^2(|D|) + g^2(|D^*|)\right) \\
&\quad + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| \\
&\leq \frac{1}{2}w\left(f^2(|A|) + g^2(|A^*|)\right) + \frac{1}{2}w\left(f^2(|D|) + g^2(|D^*|)\right) \\
&\quad + \|B\|\,\|y\|\,\|x\| + \|C\|\,\|x\|\,\|y\| \\
&\leq \frac{1}{2}w\left(f^2(|A|) + g^2(|A^*|)\right) + \frac{1}{2}w\left(f^2(|D|) + g^2(|D^*|)\right) + \\
&\quad \|B\|\left(\frac{\|x\|^2 + \|y\|^2}{2}\right) + \|C\|\left(\frac{\|x\|^2 + \|y\|^2}{2}\right) \\
&= \frac{1}{2}w\left(f^2(|A|) + g^2(|A^*|)\right) + \frac{1}{2}w\left(f^2(|D|) + g^2(|D^*|)\right) + \frac{\| B \| + \| C \|}{2}
\end{aligned}
$$

$$= \frac{1}{2}\left\| f^2(|A|) + g^2(|A^*|)\right\| + \frac{1}{2}\left\| f^2(|D|) + g^2(|D^*|)\right\| + \frac{\| B \| + \| C \|}{2}.$$

Taking the supremum over all unit vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^n$, we give inequality (2.7). □

Corollary 2.1. *Let A, B, C and D be operators in $B(H)$. Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{2}\, \||A| + |A^*|\| + \frac{1}{2}\, \||D| + |D^*|\| + \frac{\| B \| + \| C \|}{2}. \tag{2.8}$$

Proof. Letting $f(t) = g(t) = \sqrt{t}$ in (2.7), we obtain (2.8). □

Corollary 2.2. *Let A, B be operators in $B(H)$. Then*

$$w\left(\begin{bmatrix} A & O \\ O & D \end{bmatrix}\right) \leq \frac{1}{2}\, \||A| + |A^*|\| + \frac{1}{2}\, \||D| + |D^*|\|\,. \tag{2.9}$$

Proof. Letting $B = C = O$ in (2.8), we obtain (2.9). □

Remark 2.1. *Letting $D = O$ in (2.9), we obtain (1.4).*

Theorem 2.2. *Let A, B, C, D be operators in $B(H)$. Then*

$$w\begin{bmatrix} A & B \\ C & D \end{bmatrix} \leq w(A) + w(D) + \frac{\| B \| + \| C \|}{2}. \tag{2.10}$$

Proof.

$$\begin{aligned}
& \left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right|, \text{ where } \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\| = 1, \| x \|^2 + \| y \|^2 = 1 \\
= & \left|\left\langle \begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right| \\
= & \left|\left\langle \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix}\begin{bmatrix} Ax + By \\ Cx + Dy \end{bmatrix}\right\rangle\right| \\
= & \left|\overline{x}Ax + \overline{x}By + \overline{y}Cx + \overline{y}Dy\right| \\
= & \left| \langle Ax, x\rangle + \langle By, x\rangle + \langle Cx, y\rangle + \langle Dy, y\rangle \right| \\
\leq & |\langle Ax, x\rangle| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left|\langle Dy, y\rangle\right| \\
= & \|x\|^2 \left|\left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left\|y\right\|^2 \left|\left\langle D\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle\right| \\
\leq & \left|\left\langle A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right\rangle\right| + \left|\langle By, x\rangle\right| + \left|\langle Cx, y\rangle\right| + \left|\left\langle D\frac{y}{\|y\|}, \frac{y}{\|y\|}\right\rangle\right| \\
\leq & w(A) + w(D) + \|By\|\,\|x\| + \|Cx\|\,\|y\| \\
\leq & w(A) + w(D) + \|B\|\,\|y\|\,\|x\| + \|C\|\,\|x\|\,\|y\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad w(A)+w(D)+\|B\|\left(\frac{\|x\|^2+\|y\|^2}{2}\right)+\|C\|\left(\frac{\|x\|^2+\|y\|^2}{2}\right)\\
&\qquad \text{(by arithmetic-geometric mean inequality)}\\
&= \quad w(A)+w(D)+\frac{\|B\|}{2}+\frac{\|C\|}{2}\\
&= \quad w(A)+w(D)+\frac{\| B \|+\| C \|}{2}.
\end{aligned}
$$

Taking the supremum over all unit vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^n$, we obtain inequality (2.10). □

Theorem 2.3. *Let $A_1, A_2, ..., A_n, X_1, X_2, ..., X_n \in B(H)$ be such that $A_1, A_2, ..., A_n \geq O$. Then*

$$
w\left(\left(\sum_{i=1}^{n-1} A_i X_i\right)+X_n A_n\right) \leq \sqrt{\left\|\sum_{i=1}^{n} A_i\right\|\left\|\left(\sum_{i=1}^{n-1} X_i^* A_i X_i\right)+X_n A_n X_n^*\right\|}. \tag{2.11}
$$

Proof. Let

$$
K=\begin{bmatrix}
A_1^{1/2} & A_2^{1/2} & \cdots & A_{n-1}^{1/2} & X_n A_n^{1/2}\\
O & O & O & \cdots & O\\
O & O & O & \cdots & O\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
O & O & O & \cdots & O
\end{bmatrix},
$$

$$
L^*=\begin{bmatrix}
A_1^{1/2}X_1 & O & O & \cdots & O\\
A_2^{1/2}X_2 & O & O & \cdots & O\\
\vdots & O & O & \cdots & O\\
A_{n-1}^{1/2}X_{n-1} & \vdots & \vdots & \vdots & \vdots\\
A_n^{1/2} & O & O & \cdots & O
\end{bmatrix},
$$

$$
M=\left\langle \begin{bmatrix}
X_1^* A_1^{1/2} & X_2^* A_2^{1/2} & \cdots & X_{n-1}^* A_{n-1}^{1/2} & A_n^{1/2}\\
O & O & O & \cdots & O\\
O & O & O & \cdots & O\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
O & O & O & \cdots & O
\end{bmatrix}
\begin{bmatrix}
A_1^{1/2}X_1 & O & O & \cdots & O\\
A_2^{1/2}X_2 & O & O & \cdots & O\\
\vdots & O & O & \cdots & O\\
A_{n-1}^{1/2}X_{n-1} & \vdots & \vdots & \vdots & \vdots\\
A_n^{1/2} & O & O & \cdots & O
\end{bmatrix} x, x\right\rangle
$$

and

$$
N=\left\langle \begin{bmatrix}
A_1^{1/2} & A_2^{1/2} & \cdots & A_{n-1}^{1/2} & X_n A_n^{1/2}\\
O & O & O & \cdots & O\\
O & O & O & \cdots & O\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
O & O & O & \cdots & O
\end{bmatrix}
\begin{bmatrix}
A_1^{1/2} & O & O & \cdots & O\\
A_2^{1/2} & O & O & \cdots & O\\
\vdots & O & O & \cdots & O\\
A_{n-1}^{1/2} & \vdots & \vdots & \vdots & \vdots\\
A_n^{1/2}X_n^* & O & O & \cdots & O
\end{bmatrix} x, x\right\rangle.
$$

Then

$$
\begin{aligned}
&\left|\left\langle\left(\left(\sum_{i=1}^{n-1}A_iX_i\right)+X_nA_n\right)x,x\right\rangle\right| \\
=\ & |\langle KL^*x,x\rangle| = |\langle L^*x,K^*x\rangle| \\
\le\ & \|L^*x\|\,\|K^*x\| = \langle L^*x,L^*x\rangle^{1/2}\langle K^*x,K^*x\rangle^{1/2} \\
\le\ & \frac{1}{2}\left(\langle L^*x,L^*x\rangle+\langle K^*x,K^*x\rangle\right) \\
=\ & \frac{1}{2}\left(\langle LL^*x,x\rangle+\langle KK^*x,x\rangle\right) \\
=\ & \frac{1}{2}(M+N) \\
=\ & \frac{1}{2}\left\langle\left(\left(\sum_{i=1}^{n-1}X_i^*A_iX_i\right)+A_n\right)x,x\right\rangle+\left\langle\left(\left(\sum_{i=1}^{n-1}A_i\right)+X_nA_nX_n^*\right)x,x\right\rangle \\
=\ & \frac{1}{2}\left\langle\left(\sum_{i=1}^{n}A_i+\left(\sum_{i=1}^{n-1}X_i^*A_iX_i\right)+X_nA_nX_n^*\right)x,x\right\rangle.
\end{aligned}
$$

Taking the supremum over all unit vectors $x\in H$, gives

$$
w\left(\left(\sum_{i=1}^{n-1}A_iX_i\right)+X_nA_n\right)\le\frac{1}{2}\left\|\left(\sum_{i=1}^{n}A_i+\left(\sum_{i=1}^{n-1}X_i^*A_iX_i\right)+X_nA_nX_n^*\right)\right\|. \tag{2.12}
$$

Applying the triangle inequality, gives

$$
\begin{aligned}
&w\left(\left(\sum_{i=1}^{n-1}A_iX_i\right)+X_nA_n\right) \\
\le\ & \frac{1}{2}\left\|\left(\sum_{i=1}^{n}A_i\right)\right\|+\frac{1}{2}\left\|\left(\left(\sum_{i=1}^{n-1}X_i^*A_iX_i\right)+X_nA_nX_n^*\right)\right\|.
\end{aligned} \tag{2.13}
$$

Now, replacing X_i by tX_i for $i=1,2,...,n$ in (2.13), where t is any complex number and taking the minimum over $t>0$, gives

$$
w\left(\left(\sum_{i=1}^{n-1}A_iX_i\right)+X_nA_n\right)\le\sqrt{\left\|\sum_{i=1}^{n}A_i\right\|\left\|\left(\sum_{i=1}^{n-1}X_i^*A_iX_i\right)+X_nA_nX_n^*\right\|}.
$$

□

Corollary 2.3. *Let $X_1,X_2,...,X_n\in B(H)$. Then*

$$
w\left(X_1+X_2+...+X_n\right)\le\sqrt{n}\sqrt{\left\|X_1^*X_1+X_2^*X_2+...+X_{n-1}^*X_{n-1}+X_nX_n^*\right\|}. \tag{2.14}
$$

Proof. Letting $A_i=I$ for $i=1,2,...,n$ in (2.11), gives (2.14). □

Remark 2.2. *Letting $X_1,X_2,...,X_n=I$ in (2.11), gives a generalization of the second inequality of (2).*

Corollary 2.4. *Let $A_1, A_2, X_1, X_2 \in B(H)$ be such that $A_1, A_2 \geq O$. Then*

$$w\left(A_1X_1 + X_2A_2\right) \leq \sqrt{\|A_1 + A_2\| \left\|X_1^*A_1X_1 + X_2A_2X_2^*\right\|}. \tag{2.15}$$

Proof. Letting $n = 2$ in (2.11), gives (2.15). □

The following inequality is special case of (2.15), which is a numerical radius inequality for anticommutator.

Corollary 2.5. *Let $A_1, X_1 \in B(H)$ be such that $A_1 \geq O$. Then*

$$w\left(A_1X_1 + X_1A_1\right) \leq \sqrt{2}\sqrt{\|A_1\| \left\|X_1^*A_1X_1 + X_1A_1X_1^*\right\|}. \tag{2.16}$$

Proof. Replacing X_2 by X_1 and A_2 by A_1 in (2.15), gives (2.16). □

Remark 2.3. *Letting $A_1 = A_2 = I$ in (2.15), gives*

$$w\left(X_1 + X_2\right) \leq \sqrt{2}\sqrt{\left\|X_1^*X_1 + X_2X_2^*\right\|} \tag{2.17}$$

Remark 2.4. *Letting $X_2 = X_1$ in (2.17), gives*

$$w\left(X_1\right) \leq \frac{1}{\sqrt{2}}\sqrt{\left\|X_1^*X_1 + X_1X_1^*\right\|}. \tag{2.18}$$

Squaring both sides of (2.18), gives (1.6). From this point of view, we conclude that (2.12) refines and generalize (1.6). Moreover, (2.11), (2.13), (2.14), (2.15), and (2.16) generalize (1.6).

Theorem 2.4. *Let $A_1, A_2, ..., A_n, X_1, X_2, ..., X_n \in B(H)$ be such that $A_1, A_2, ..., A_n \geq O$. Then*

$$w\left(\left(\sum_{i=1}^{n-1} A_iX_i\right) + X_nA_n\right) \leq \frac{1}{\sqrt{2}}\left\|\left(\sum_{i=1}^{n-1} X_i^*A_iX_i\right) + A_n\right\| + \frac{i}{\sqrt{2}}\left\|\sum_{i=1}^{n-1} A_i + X_nA_nX_n^*\right\|.$$

Proof. Let

$$K = \begin{bmatrix} A_1^{1/2} & A_2^{1/2} & \cdots & A_{n-1}^{1/2} & X_nA_n^{1/2} \\ O & O & O & \cdots & O \\ O & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O \end{bmatrix},$$

$$L^* = \begin{bmatrix} A_1^{1/2}X_1 & O & O & \cdots & O \\ A_2^{1/2}X_2 & O & O & \cdots & O \\ \vdots & O & O & \cdots & O \\ A_{n-1}^{1/2}X_{n-1} & \vdots & \vdots & \vdots & \vdots \\ A_n^{1/2} & O & O & \cdots & O \end{bmatrix},$$

$$M = \left\langle \begin{bmatrix} X_1^* A_1^{1/2} & X_2^* A_2^{1/2} & \cdots & X_{n-1}^* A_{n-1}^{1/2} & A_n^{1/2} \\ O & O & O & \cdots & O \\ O & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O \end{bmatrix} \begin{bmatrix} A_1^{1/2} X_1 & O & O & \cdots & O \\ A_2^{1/2} X_2 & O & O & \cdots & O \\ \vdots & O & O & \cdots & O \\ A_{n-1}^{1/2} X_{n-1} & \vdots & \vdots & \vdots & \vdots \\ A_n^{1/2} & O & O & \cdots & O \end{bmatrix} x, x \right\rangle$$

and

$$N = \left\langle \begin{bmatrix} A_1^{1/2} & A_2^{1/2} & \cdots & A_{n-1}^{1/2} & X_n A_n^{1/2} \\ O & O & O & \cdots & O \\ O & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & O & \cdots & O \end{bmatrix} \begin{bmatrix} A_1^{1/2} & O & O & \cdots & O \\ A_2^{1/2} & O & O & \cdots & O \\ \vdots & O & O & \cdots & O \\ A_{n-1}^{1/2} & \vdots & \vdots & \vdots & \vdots \\ A_n^{1/2} X_n^* & O & O & \cdots & O \end{bmatrix} x, x \right\rangle.$$

Then

$$\begin{aligned} & \left| \left\langle \left(\left(\sum_{i=1}^{n-1} A_i X_i \right) + X_n A_n \right) x, x \right\rangle \right| \\ = \;& |\langle K L^* x, x \rangle| = |\langle L^* x, K^* x \rangle| \\ \leq \;& \|L^* x\| \, \|K^* x\| = \langle L^* x, L^* x \rangle^{1/2} \langle K^* x, K^* x \rangle^{1/2} \\ \leq \;& \frac{1}{2} \left(\langle L^* x, L^* x \rangle + \langle K^* x, K^* x \rangle \right) \\ \leq \;& \frac{1}{\sqrt{2}} \left(\langle L^* x, L^* x \rangle + i \langle K^* x, K^* x \rangle \right) \\ & \text{(by applying inequality (2.1)} \\ = \;& \frac{1}{\sqrt{2}} \left\langle \left(\left(\sum_{i=1}^{n-1} X_i^* A_i X_i \right) + A_n \right) x, x \right\rangle + i \left\langle \left(\left(\sum_{i=1}^{n-1} A_i \right) + X_n A_n X_n^* \right) x, x \right\rangle. \end{aligned}$$

Taking the supremum over all unit vectors $x \in H$, gives

$$w\left(\left(\sum_{i=1}^{n-1} A_i X_i \right) + X_n A_n \right) \leq \frac{1}{\sqrt{2}} \left\| \left(\sum_{i=1}^{n-1} X_i^* A_i X_i \right) + A_n \right\| + \frac{i}{\sqrt{2}} \left\| \sum_{i=1}^{n-1} A_i + X_n A_n X_n^* \right\|.$$

□

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