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ON ENRICHED SUZUKI NONEXPANSIVE MAPPINGS IN P-UNIFORMLY CONVEX METRIC SPACES

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Abstract. This paper introduces and defines the concept of enriched Suzuki nonexpansive mappings $\mathcal{T}$ in p-uniformly convex metric spaces, thereby extending earlier results established in Hadamard spaces. We show that the τ-averaged mapping $\mathcal{T}_\tau$ preserves the fixed points of $\mathcal{T}$. In addition, we prove that $\mathcal{T}_\tau$ is quasi-nonexpansive and that the sequence generated by the Mann iteration converges to a fixed point of both $\mathcal{T}$ and its averaged counterpart $\mathcal{T}_\tau$. We further establish both Δ-convergence and strong convergence of the Mann iteration sequence for the τ-averaged mapping. Additionally, we present an illustrative example of an enriched Suzuki nonexpansive mapping within p-uniformly convex metric spaces.

1. Introduction

The Banach Contraction Principle is a foundational result in fixed point theory, stating that every contraction mapping on a complete metric space possesses a unique fixed point. This principle has sparked extensive research into more general types of mappings, particularly nonexpansive mappings, which, unlike contractions, do not necessarily decrease the distance between points but can still possess fixed points.

Formally, a map $\mathcal{T} : M \to M$ in a metric space (M, d) with a point $u \in M$ is called a fixed point of $\mathcal{T}$ if $u = \mathcal{T}(u)$ where $\mathcal{T}$ is a Banach contraction if there exists a fixed constant $k < 1$ such that $d(\mathcal{T}u, \mathcal{T}v) \leq kd(u, v)$ for all $u, v \in M$. One of the various generalizations of Banach contraction is the nonexpansive mappings (that is when the contraction $k = 1, d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v) \forall u, v \in M$) which has received several modifications including Suzuki nonexpansive mappings, which was

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introduced by Tomonari Suzuki [11] in normed spaces as an extension of an enriched nonexpansive techniques introduced by Berinde and Pacurar [15–18]. He focused on a class of mappings now known as Suzuki nonexpansive mappings, which generalize classical nonexpansive mappings by satisfying a specific condition known as condition (C), and established notable fixed point results within this framework.

The p-uniformly convex metric spaces, these spaces generalize the concept of uniform convexity found in Banach spaces [6] by introducing a parameter p which influences the degree of convexity. The p-uniformly convex spaces exhibit strong geometric properties, such as strict convexity and unique midpoints, which make them particularly suitable for the analysis of nonexpansive mappings and the establishment of fixed point theorems.

Very recently, Turcanu and Postaloche [13] introduced the notion of enriched Suzuki nonexpansive mappings in Hadamard space as follows:

Consider a nonempty subset C of a Hadamard space (M,d). A mapping $\mathcal{T} : C \to C$ is called an enriched Suzuki nonexpansive mapping if

$$d(\mathcal{T}_\tau u, \mathcal{T}_\tau v) \leq d(u,v),$$

for all $u,v \in C$ such that $d(u,\mathcal{T}_\tau u) \leq 2d(u,v)$.

They also showed that the fixed point set of enriched Suzuki nonexpansive mappings in Hadamard space is not empty. Furthermore, they proposed a simple Picard iteration

$$\begin{cases} u_o \in C \\ u_{n+1} = \mathcal{T}_\tau u_n, \quad n \geq 0. \end{cases} \tag{1.1}$$

They proved that the algorithm (1.1) converges to a fixed point of the enriched Suzuki nonexpansive mapping $\mathcal{T}$ in Hadamard spaces.

Over the years, extension of nonlinear problems to more general nonlinear spaces has become a common practice among researchers. For instance: Izuchukwu *et al.* [5] proposed proximal-type algorithms designed to solve split minimization problems within the setting of p-uniformly convex metric spaces.

They formulated the backward-backward algorithm, starting from an initial point $u_1 \in M$, as follows:

$$\begin{cases} v_n = J_{\lambda_n} u_n, \\ u_{n+1} = J_{\mu_n} v_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\mu_n\}$ denotes a sequence of positive real numbers, and $f, g : M \to (-\infty, \infty]$ are proper, convex, and lower semicontinuous functions. They also investigated the strong convergence of Algorithm (2) toward a solution of the associated split minimization problem:

$$\min \Psi(u,v) \text{ such that } (u,v) \in M \times M, \text{ where } \Psi(u,v) = f(u) + g(v) \ \forall\, u,v \in M.$$

Aremu *et al.* [2] developed and analyzed a multi-step iterative method involving a finite collection of asymptotically k_i-strictly pseudocontractive mappings with respect to p, along with a

p-resolvent operator associated to a proper, convex, and lower semicontinuous function, all within the framework of a p-uniformly convex metric space as follow:
Let M be a p-uniformly convex metric space. A mapping $\mathcal{T}: M \to M$ is said to be asymptotically k-strictly pseudocontractive with respect to p if there exist a constant $k \in [0,1)$ and a sequence $\{h_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$d(\mathcal{T}^n u, \mathcal{T}^n v)^p \leq h_n d(u,v)^p + k\Big(d(u,\mathcal{T}^n u) + d(v,\mathcal{T}^n v)\Big)^p, \ \forall\, u,v \in D,\ n \geq 1.$$

Let M be a p-uniformly convex metric space with $1 < p < \infty$ and convexity parameter $c \geq 2$. Suppose $f : M \to (-\infty,\infty]$ is a proper, convex, and lower semicontinuous function. Consider a finite family of mappings $\mathcal{T}_i : M \to M$, for $i = 1,2,\dots,m$, each of which is uniformly L_i-Lipschitz continuous and asymptotically k_i-strictly pseudocontractive with respect to p. Define $k := \max\{k_i : i = 1,2,\dots,m\}$ where each $k_i \in [0,1)$, and assume that for each i, there exists a sequence $\{h_{in}\}_{n=1}^{\infty} \subseteq [0,\infty)$. Let $\Gamma := \bigcap_{i=1}^{m} F(\mathcal{T}_i) \cap \arg\min_{u\in M} f(u)$, and assume $\Gamma \neq \emptyset$. Starting from an arbitrary initial point $u_1 \in M$, the iterative sequence $\{u_n\}$ is generated by the following multi-step algorithm:

$$\begin{cases} u_{n+1} = (1-\tau_{0,n})v_{1,n} \oplus \tau_{0,n}\mathcal{T}_1^n v_{1,n}, \\ v_{1,n} = (1-\tau_{1,n})v_{2,n} \oplus \tau_{1,n}\mathcal{T}_2^n v_{2,n}, \\ \vdots \\ v_{i,n} = (1-\tau_{i,n})v_{(i+1),n} \oplus \tau_{i,n}\mathcal{T}_{(i+1)}^n v_{(i+1),n}, \\ \vdots \\ v_{(m-2),n} = (1-\tau_{(m-2),n})v_{(m-1),n} \oplus \tau_{(m-2),n}\mathcal{T}_{(m-1)}^n v_{(m-1),n}, \\ v_{(m-1),n} = (1-\tau_{(m-1),n})w_n \oplus \tau_{(m-1),n}\mathcal{T}_m^n w_n, \\ w_n = \arg\min_{v\in M}\left[f(v) + \frac{1}{p\lambda_n^{p-1}} d(v,u_n)^p\right], \forall\, n \geq 1, \end{cases} \tag{1.3}$$

where $w_n = v_{m,n}$, $\forall n \geq 1$, and the following conditions are satisfied:

(C1) $0 < a \leq \tau_{in} \leq 1 - 2k_i$,
(C2) $\sum_{n=1}^{\infty}\left(\max_{1\leq i\leq m} h_{in} - 1\right) < \infty$.
(C3) $L = \max\{L_i, i = 1,2,\dots,m.\}$

Inspired by the works of Turcanu *et al.*, Izuchukwu *et al.*, and Aremu *et al.*, we extend the concept of enriched Suzuki nonexpansive mappings to the setting of p-uniformly convex metric spaces. In this context, we investigate key fixed point properties of such mappings and introduce an iterative method aimed at approximating their fixed points. Our findings contribute to and complement the existing body of research in fixed point theory within nonlinear metric frameworks.

2. Preliminaries

This section provides key definitions and preliminary results that form the foundation for our main theorems.

Following the definition by Naor and Silberman [1], a metric space M is said to be p-uniformly convex for $2 \leq p \leq \infty$ if and only if it is a geodesic space and

$$d^p(v^*, (1-t)u \oplus tv) \leq (1-t)d^p(v^*, u) + td^p(v^*, v) - \frac{c}{2}t(1-t)d^p(u,v). \tag{2.1}$$

Definition 2.1. *Let M be a metric space and $C \subseteq M$ a nonempty subset. A point $u \in C$ is called a fixed point of a mapping $\mathcal{T}$ if $\mathcal{T}(u) = u$. The collection of all fixed points of $\mathcal{T}$ is denoted by $F(\mathcal{T})$, that is,*

$$F(\mathcal{T}) = \{u \in C : u = \mathcal{T}u\}.$$

Let (M, d) be a metric space. We say that M is a geodesic space if, for any two points $u, v \in M$, there exists a geodesic path $c : [0, d(u,v)] \to M$ such that $c(0) = u$ and $c(d(u,v)) = v$. This map c is an isometry, and its image forms a geodesic segment joining u and v. Moreover, if every pair of points in M is connected by exactly one geodesic segment, then M is said to be uniquely geodesic. Let $\mathcal{T}$ be a mapping defined on a subset C of a Banach space E. The mapping $\mathcal{T}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|u - \mathcal{T}u\| \leq \|u - v\|,$$

which inplies that $\|\mathcal{T}u - \mathcal{T}v\| \leq \|u - v\| \ \forall \ u, v \in C$.

Definition 2.2. *[2] Let M be a complete convex metric space. A nonlinear mapping $\mathcal{T} : M \to M$ is said to be demiclosed at 0 if for any bounded sequence $\{u_n\}$ in M such that $\Delta - \lim_{n\to\infty} u_n = v^*$ and $\lim_{n\to\infty} d(u_n, \mathcal{T}u_n) = 0$, we have that $v^* \in F(\mathcal{T})$.*

Lemma 2.1. *Let $\{u_n\}$ be a bounded sequence in a metric space M and $r(\cdot, \{u_n\}) : X \to [0, \infty)$ be a continuous functional defined by $r(u, \{u_n\}) = \limsup_{n\to\infty} d(u, u_n)$. The asymptotic radius of $\{u_n\}$ is given by $r(\{u_n\}) := \inf\{r(u, \{u_n\}) : u \in M\}$, while the asymptotic center of $\{u_n\}$ is the set $A(\{u_n\}) = \{u \in M : r(u, \{u_n\}) = r(\{u_n\})\}$.*

Definition 2.3. *A sequence $\{u_n\}$ in M is said to be Δ-convergent to a point $u \in M$ if $A(\{u_{n_k}\}) = \{u\}$ for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$. In this case, we say that u is the Δ-limit of $\{u_n\}$ (see [9, 19]).*

Remark 2.1. *The notion of Δ-convergence in metric spaces was first introduced and explored by Lim [12]. It serves as an analogue to the notion of weak convergence in Banach spaces.*

Lemma 2.2. *[4, 8] Let M be a complete p-uniformly convex metric space. Then,*

(i) *Every bounded sequence in M has a unique asymptotic center,*
(ii) *every bounded sequence in M has a Δ-convergent subsequence.*

Lemma 2.3. *[7] Let C be a nonempty subset of a Hadamard space. If $\mathcal{T} : C \to C$ satisfies condition C, then*

$$d(u, \mathcal{T}v) \leq 3d(u, \mathcal{T}u) + d(u, v) \ \forall \ u, v \in C.$$

Proposition 2.1. *[7] Let C be a nonempty bounded closed convex subset of a Hadamard space. If $\mathcal{T} : C \to C$ satisfies condition C then $\mathcal{T}$ has a fixed point in C.*

3. Enriched Suzuki nonexpansive mapping in p-uniformly convex metric space

We begin by introducing the concept of enriched Suzuki nonexpansive mappings in p-uniformly convex metric space.

Definition 3.1. *Let C be a nonempty set in a p-uniformly convex metric space M. A mapping $\mathcal{T} : C \to C$ is said to be Suzuki nonexpansive mapping if*

$$d(\mathcal{T}u, \mathcal{T}v) \le d(u, v), \tag{3.1}$$

$\forall\ u, v \in C$ such that $\frac{1}{2}d(u, \mathcal{T}u) \le d(u, v)$.

Example 3.1 (Suzuki Nonexpansive Mapping in a Product Space)**.** *Let $C = \left[\frac{1}{n}, n\right] \times \mathbb{R}$ for some $n \ge 2$, and define the mapping $\mathcal{T} : C \to C$ by*

$$\mathcal{T}(u_1, u_2) = \left(\sqrt{u_1}, \tfrac{1}{2}u_2\right).$$

Consider the metric $d : C \times C \to \mathbb{R}$ defined by

$$d\left((u_1, u_2), (v_1, v_2)\right) = \sqrt{(u_1 - v_1)^2 + \left(u_2 - v_2 - u_1^2 + v_1^2\right)^2}.$$

We will show that $\mathcal{T}$ is a Suzuki nonexpansive mapping on (C, d).
To verify this, consider arbitrary $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in C. We check whether the condition

$$d(\mathcal{T}u, \mathcal{T}v) \le d(u, v).$$

$\forall\ u, v \in C$ such that $\frac{1}{2}d(u, \mathcal{T}u) \le d(u, v)$.
We compute:

$$\mathcal{T}u = \left(\sqrt{u_1}, \tfrac{1}{2}u_2\right), \quad \mathcal{T}v = \left(\sqrt{v_1}, \tfrac{1}{2}v_2\right),$$

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + \left(u_2 - v_2 - u_1^2 + v_1^2\right)^2},$$

$$d(\mathcal{T}u, \mathcal{T}v) = \sqrt{(\sqrt{u_1} - \sqrt{v_1})^2 + \left(\tfrac{1}{2}u_2 - \tfrac{1}{2}v_2 - u_1 + v_1\right)^2},$$

$$\begin{aligned} \frac{1}{2}d(u, \mathcal{T}u) &= \frac{1}{2}\sqrt{(u_1 - \sqrt{u_1})^2 + \left(u_2 - \tfrac{1}{2}u_2 - u_1^2 + u_1\right)^2}. \\ &= \frac{1}{2}\sqrt{(u_1 - \sqrt{u_1})^2 + \left(\tfrac{1}{2}u_2 - u_1^2 + u_1\right)^2}. \end{aligned} \tag{3.2}$$

Therefore, the mapping $\mathcal{T}$ satisfies the Suzuki nonexpansive condition:

$$d(\mathcal{T}u, \mathcal{T}v) \le d(u, v) \quad \Rightarrow \quad \frac{1}{2}d(u, \mathcal{T}u) \le d(u, v), \quad \forall \quad u, v \in C.$$

Definition 3.2. *Let C be a nonempty set in a p-uniformly convex metric space M. A mapping $\mathcal{T} : C \to C$ is said to be enriched Suzuki nonexpansive (ESN) mapping if there exist $b \in [0, \infty)$ such that*

$$d((bu \oplus \mathcal{T}u), (bv \oplus \mathcal{T}v)) \leq (b+1)d(u,v), \tag{3.3}$$

for all $u, v \in C$ satisfying

$$\frac{1}{2}d(u, \mathcal{T}u) \leq (b+1)d(u,v). \tag{3.4}$$

Remark 3.1. *From (3.3) we obtain*

$$d((bu \oplus \mathcal{T}u), (bv \oplus \mathcal{T}v)) \leq (b+1)d(u,v),$$

which implies that

$$\frac{1}{b+1}d((bu \oplus \mathcal{T}u), (bv \oplus \mathcal{T}v)) \leq d(u,v). \tag{3.5}$$

Note that (3.5) is equivalent to

$$d((\frac{b}{b+1}u \oplus \frac{1}{b+1}\mathcal{T}u), (\frac{b}{b+1}v \oplus \frac{1}{b+1}\mathcal{T}v)) \leq d(u,v)$$

and therefore,

$$d((1-\tau)u \oplus \tau\mathcal{T}u, (1-\tau)v \oplus \mathcal{T}v) \leq d(u,v), \tag{3.6}$$

where $\frac{1}{b+1} = \tau$, $\tau \in (0,1]$.
Since $p \geq 2$, it follows from (3.6) that the following inequality holds

$$d^p((1-\tau)u \oplus \tau\mathcal{T}u, (1-\tau)v \oplus \mathcal{T}v) \leq d^p(u,v). \tag{3.7}$$

Note that the LHS of (3.7) is an affine combination of the form

$$\mathcal{T}_\tau u = (1-\tau)u \oplus \tau\mathcal{T}u. \tag{3.8}$$

So, if we adopt (3.4) for $\mathcal{T}_\tau u$, we have

$$\begin{aligned} \frac{1}{2}d(u, \mathcal{T}u) &\leq (b+1)d(u,v) \text{ rewrites as} \\ \frac{1}{2}d(u, \mathcal{T}_\tau u) &\leq d(u,v), \end{aligned}$$

which implies that

$$d(u, \mathcal{T}_\tau u) \leq 2d(u,v).$$

Remark 3.2. *$\mathcal{T}_\tau$ shares the same set of fixed points as the original mapping $\mathcal{T}$, making it an intriguing object of study within the context of fixed-point theory.*

Lemma 3.1. *Let $\mathcal{T}$ be an enriched Suzuki nonexpansive mapping in p-uniformly convex metric space and $\mathcal{T}_\tau$ is the τ-averaged mapping of $\mathcal{T}$, $\tau \in (0,1]$. Then*

$$d^p(u, \mathcal{T}_\tau u) \leq \tau d^p(u, \mathcal{T}u).$$

Proof. By applying (2.1) and (3.8), we have

$$\begin{aligned} d^p(u,\mathcal{T}_\tau u) &= d^p(u,(1-\tau)u\oplus\tau\mathcal{T}u)\\ &\leq (1-\tau)d^p(u,u)+\tau d^p(u,\mathcal{T}u)-\frac{c}{2}\tau(1-\tau)d^p(u,\mathcal{T}u)\\ &\leq \tau d^p(u,\mathcal{T}u.). \end{aligned}$$

□

Lemma 3.2. *Let C be a nonempty subset of a p-uniformly convex metric space (M,d). A mapping $\mathcal{T}:C\to C$ is referred to as enriched Suzuki nonexpansive if*

$$d(\mathcal{T}_\tau u,\mathcal{T}_\tau v)\leq d(u,v),$$

for all $u,v\in C$ such that $d(u,\mathcal{T}_\tau u)\leq 2d(u,v)$.

Proof. Assume that $v\in F(\mathcal{T})$ then from (2.1) and (3.8) we have

$$\begin{aligned} d^p(\mathcal{T}_\tau u,\mathcal{T}_\tau v) &= d^p((1-\tau)u\oplus\tau\mathcal{T}u,\mathcal{T}_\tau v)\\ &\leq (1-\tau)d^p(\mathcal{T}_\tau v,u)+\tau d^p(\mathcal{T}_\tau v,\mathcal{T}u)-\frac{c}{2}\tau(1-\tau)d^p(u,\mathcal{T}u)\\ &\leq (1-\tau)d^p(\mathcal{T}_\tau v,u)+\tau d^p(\mathcal{T}_\tau v,\mathcal{T}u)\\ &= (1-\tau)d^p((1-\tau)v\oplus\tau\mathcal{T}v,u)+\tau d^p((1-\tau)v\oplus\tau\mathcal{T}v,\mathcal{T}u)\\ &\leq (1-\tau)[(1-\tau)d^p(u,v)+\tau d^p(u,\mathcal{T}v)-\frac{c}{2}\tau(1-\tau)d^p(v,\mathcal{T}v)]\\ &+ \tau[(1-\tau)d^p(\mathcal{T}u,v)+\tau d^p(\mathcal{T}u,\mathcal{T}v)-\frac{c}{2}\tau(1-\tau)d^p(v,\mathcal{T}v)]\\ &\leq (1-\tau)[(1-\tau)d^p(u,v)+\tau d^p(u,\mathcal{T}v)]+\tau[(1-\tau)d^p(\mathcal{T}u,v)+\tau d^p(\mathcal{T}u,\mathcal{T}v)]\\ &\leq (1-\tau)[(1-\tau)d^p(u,v)+\tau(d(u,v)+d(v,\mathcal{T}v))^p]\\ &+ \tau[(1-\tau)(d(\mathcal{T}u,\mathcal{T}v)+d(\mathcal{T}v,v))^p+\tau d^p(u,v)]\\ &\leq (1-\tau)[(1-\tau)d^p(u,v)+\tau d^p(u,v)]+\tau[(1-\tau)d^p(u,v)+\tau d^p(u,v)]\\ &\leq (1-\tau)[d^p(u,v)]+\tau[d^p(u,v)]\\ &\leq d^p(u,v). \end{aligned} \tag{3.9}$$

Therefore, we have from (3.9) that $d(\mathcal{T}_\tau u,\mathcal{T}_\tau v)\leq d(u,v)$ □

Remark 3.3. *The mapping $\mathcal{T}$ is said to be enriched Suzuki nonexpansive if and only if its associated averaged mapping $\mathcal{T}_\tau$ satisfies the Suzuki nonexpansive condition.*

Example 3.2. *Consider the metric space $M=\mathbb{R}$ with the usual metric $d(u,v)=|u-v|$. Let $C=[0,\infty)$ be the set of non-negative real numbers. Define the mapping $\mathcal{T}:C\to C$ by*

$$\mathcal{T}u=\frac{u}{2}.$$

Let's verify that $\mathcal{T}$ is an enriched Suzuki nonexpansive mapping.

For $\tau = 1$, we have:

$$\begin{aligned}\mathcal{T}u &= (1-\tau)u + \tau\mathcal{T}u \\ &= (1-1)u + \mathcal{T}u \\ &= \mathcal{T}u \\ &= \frac{u}{2}.\end{aligned}$$

For $\tau = 1$, we also get:

$$\begin{aligned}\mathcal{T}v &= (1-\tau)v + \tau\mathcal{T}v \\ &= (1-1)v + \mathcal{T}v \\ &= \mathcal{T}v \\ &= \frac{v}{2}.\end{aligned}$$

Check the condition: $d(u, \mathcal{T}_\tau u) \leq 2d(u, v)$

$$\begin{aligned}d(u, \mathcal{T}_\tau u) &\leq 2d(u, v) \\ |u - \mathcal{T}_\tau u| &\leq 2|u - v| \\ |u - \frac{u}{2}| &\leq 2|u - v| \\ |\frac{u}{2}| &\leq 2|u - v|.\end{aligned}$$

Check the condition: $d(\mathcal{T}_\tau u, \mathcal{T}_\tau v) \leq d(u, v)$

$$\begin{aligned}d(\mathcal{T}_\tau u, \mathcal{T}_\tau v) &\leq d(u, v) \\ |\mathcal{T}_\tau u - \mathcal{T}_\tau v| &\leq |u - v| \\ |\frac{u}{2} - \frac{v}{2}| &\leq |u - v| \\ |\frac{u - v}{2}| &\leq |u - v|.\end{aligned}$$

Since both conditions are satisfied, the mapping $\mathcal{T}u = \frac{u}{2}$ is an enriched Suzuki nonexpansive mapping in the p-uniformly convex metric space. Also it is obvious to see that $F(\mathcal{T}) = \{0\}$ and thus $F(\mathcal{T}) \neq \emptyset$.

In the following sections, we investigate convergence theorems related to enriched Suzuki nonexpansive mappings within the framework of p-uniformly convex metric spaces.

4. Main Result

We begin with some useful lemmas

Lemma 4.1. *[13] Let C be a nonempty, bounded, closed, and convex subset of a p-uniformly convex metric space. If $\mathcal{T} : C \to C$ is an enriched Suzuki nonexpansive mapping, then the fixed point set $F(\mathcal{T})$ is nonempty, closed, convex, and therefore contractible.*

Lemma 4.2. *Let C be a nonempty bounded closed convex subset of a p-uniformly convex metric space. If $\mathcal{T} : C \to C$ is an enriched Suzuki nonexpansive mapping, then $\mathcal{T}$ has a fixed point in C.*

Proof. As previously noted, the enriched Suzuki nonexpansiveness of $\mathcal{T}$ is equivalent to the Suzuki nonexpansiveness of its averaged mapping $\mathcal{T}_\tau$ for any $\tau \in (0,1]$. By Proposition 2.1, $\mathcal{T}_\tau$ admits a fixed point $v^* \in C$. Moreover, from (3.7), it follows that v^* is also a fixed point of $\mathcal{T}$. □

Before presenting the results on Δ-convergence and strong convergence, we first establish several technical lemmas essential to our analysis.

Theorem 4.1. *Let C be a nonempty, bounded, closed, and convex subset of a p-uniformly convex metric space with $1 < p < \infty$ and convexity parameter $c \geq 2$. Assume that $\mathcal{T} : C \to C$ is an enriched Suzuki nonexpansive mapping. Consider the sequence $\{u_n\}$ defined for $n \geq 0$ as follows:*

$$\begin{cases} u_o \in C \\ u_{n+1} = (1-\tau_n)u_n \oplus \tau_n \mathcal{T} u_n, \quad n \geq 0. \end{cases} \tag{4.1}$$

Then, the $\lim_{n\to\infty} d^p(u_n, v^*)$ exist for any $v^* \in F(\mathcal{T})$.

Proof. Since the mapping $\mathcal{T}_\tau$ satisfies the Suzuki nonexpansiveness condition, it is also quasi-nonexpansive. This, in turn, implies that

$$\begin{aligned} d^p(u_{n+1}, v^*) &= d^p((1-\tau_n)u_n \oplus \tau_n \mathcal{T} u_n, v^*) \\ &\leq (1-\tau_n)d^p(u_n, v^*) + \tau_n d^p(\mathcal{T} u_n, v^*) - \frac{c}{2}\tau_n(1-\tau_n)d^p(u_n, \mathcal{T} u_n) \\ &\leq (1-\tau_n)d^p(u_n, v^*) + \tau_n d^p(u_n, v^*) - \frac{c}{2}\tau_n(1-\tau_n)d^p(u_n, \mathcal{T} u_n) \\ &= d^p(u_n, v^*) - \frac{c}{2}\tau_n(1-\tau_n)d^p(u_n, \mathcal{T} u_n) \\ &\leq d^p(u_n, v^*). \end{aligned} \tag{4.2}$$

For any $v^* \in C$, the sequence is non-increasing, bounded and hence convergent. □

The following result confirms that enriched Suzuki nonexpansive mappings adhere to the demi-closeness principle.

Lemma 4.3. *Let C be a nonempty, bounded, closed, and convex subset of a p-uniformly convex metric space (M, d), and let $\mathcal{T} : C \to C$ be an enriched Suzuki nonexpansive mapping. Suppose $\{u_n\}$ is a sequence in C such that $\lim_{n\to\infty} d^p(u_n, \mathcal{T} u_n) = 0$ and $u_n \to v^* \in M$. Then it follows that $v^* \in C$ and $\mathcal{T} v^* = v^*$.*

Proof. $\mathcal{T} : C \to C$ being an ESN mapping is equivalent to the τ - averaged $\mathcal{T}_\tau$ satisfying condition C. On the other hand according to lemma (2.4)

$$d(u_n, \mathcal{T}_\tau v^*) \leq 3d(u_n, \mathcal{T}_\tau u_n) + d(u_n, v^*),$$

which by taking the limsup gives

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, \mathcal{T}_\tau v^*) &\le 3\limsup_{n\to\infty} d(u_n, \mathcal{T}_\tau u_n) + \limsup_{n\to\infty} d(u_n, v^*)\\ &\le \limsup_{n\to\infty} d(u_n, v^*).\end{aligned}$$

Since $p \geq 2$, then

$$\limsup_{n\to\infty} d^p(u_n, \mathcal{T}_\tau v^*) \le \limsup_{n\to\infty} d^p(u_n, v^*).$$

By the uniqueness of asymptotic centers, it follows that $\mathcal{T}_\tau v^* = v^*$, hence $\mathcal{T} v^* = v^*$ as $F(\mathcal{T}) = F(\mathcal{T}_\tau)$. □

Lemma 4.4. *Let (M, d) be a p-uniformly convex metric space, and let C be a nonempty, closed, and convex subset. If $\mathcal{T} : C \to C$ is a quasi-nonexpansive mapping with $Fix(\mathcal{T}) = \{0\}$, then its associated τ-averaged mapping $\mathcal{T}_\tau$, for any $\tau \in (0, 1]$, is asymptotically regular.*

Proof. Let $u_0 \in C$ be arbitrary and consider the sequence of Mann iterations given by $(1-\tau_n)u_n \oplus \tau_n \mathcal{T} u_n$, for $n \geq 0$. For any $v^* \in \mathrm{Fix}(\mathcal{T})$, it follows from the fundamental inequality (2.1) and the quasi-nonexpansiveness of $\mathcal{T}$.

$$d^p(u_{n+1}, v^*) \le d^p(u_n, v^*),$$

which means that the sequence $\{d^p(u_n, v^*)\}$ is nonincreasing and we also have from (4.2) above that

$$\begin{aligned} d^p(u_{n+1}, v^*) &\le d^p(u_n, v^*) - \frac{c}{2}\tau_n(1-\tau_n)d^p(u_n, \mathcal{T}u_n)\\ \frac{c}{2}\tau_n(1-\tau_n)d^p(u_n, \mathcal{T}u_n) &\le d^p(u_n, v^*) - d^p(u_{n+1}, v^*)\\ \tau_n(1-\tau_n)d^p(u_n, \mathcal{T}u_n) &\le \frac{2}{c}[d^p(u_n, v^*) - d^p(u_{n+1}, v^*)]\\ d^p(u_n, \mathcal{T}u_n) &\le \frac{2}{c(\tau_n(1-\tau_n))}[d^p(u_n, v^*) - d^p(u_{n+1}, v^*)]. \end{aligned} \tag{4.3}$$

Taking the limit of both sides of (4.3),

$$\lim_{n\to\infty} d^p(u_n, \mathcal{T}u_n) \le \frac{2}{c(\tau_n(1-\tau_n))}[\lim_{n\to\infty} d^p(u_n, v^*) - \lim_{n\to\infty} d^p(u_{n+1}, v^*)]$$

which implies that

$$\lim_{n\to\infty} d^p(u_n, \mathcal{T}u_n) = 0. \tag{4.4}$$

Which leads to the desired result. □

The above proof suggest an important fact, which is underline in the following.

Remark 4.1. *The Mann iteration sequence provides an approximate fixed point sequence for both the original mapping $\mathcal{T}$ and its τ-averaged counterpart $\mathcal{T}_\tau$; that is,*

$$\lim_{n\to\infty} d^p(u_n, \mathcal{T}u_n) = 0 \text{ and } \lim_{n\to\infty} d^p(u_n, \mathcal{T}_\tau u_n) = 0. \tag{4.5}$$

Theorem 4.2. *Let C be a nonempty, bounded, closed, and convex subset of a p-uniformly convex metric space (M, d). If $\mathcal{T} : C \to C$ is an enriched Suzuki nonexpansive mapping, then the Mann iteration sequence $\{u_n\}$ defined by (4.1) Δ-converges to a fixed point of $\mathcal{T}$.*

Proof. Let the set of all asymptotic centers be denoted by $\omega_A(\{u_n\}) := \bigcup A(\{H_n\})$, where the union is taken over all subsequences of $\{u_n\}$. Now, let $h \in \omega_A(\{u_n\})$, and consider the sequence $\{h_n\}$ such that $A(\{h_n\}) = h$. Since $\{h_n\}$ is bounded, it has a subsequence $\{h_{nk}\}$ that Δ-converges to some $h' \in C$. By Lemma 4.5, $\lim_{n\to\infty} d^p(h_{nk}, \mathcal{T}_\tau h_{nk}) = 0$, which implies $h' \in F(\mathcal{T}) = F(\mathcal{T}_\tau)$. Furthermore, according to Lemma 4.4, the limit $\lim_{n\to\infty} d^p(u_n, h')$ exists. We now show that $h = h'$. Assume the contrary. In this case, the following inequalities arise, utilizing the uniqueness properties of the asymptotic center.

$$\begin{aligned} \limsup_{n\to\infty} d^p(h_{nk}, h') &< \limsup_{n\to\infty} d^p(h_{nk}, h) \\ &\leq \limsup_{n\to\infty} d^p(h_n, h) \\ &< \limsup_{n\to\infty} d^p(h_n, h') \\ &= \limsup_{n\to\infty} d^p(h_{nk}, h'), \end{aligned}$$

Which leads to a contradiction, confirming that $h = h' \in \mathrm{Fix}(\mathcal{T})$. Now, let $A(\{u_n\}) = v^*$. By Lemma 4.4, the limit $\lim_{n\to\infty} d^p(u_n, u)$ exists, which implies $v^* = h$. Therefore, the sequence $\{u_n\}$ is Δ-convergent to $v^* \in \mathrm{Fix}(\mathcal{T})$. □

Theorem 4.3. *Let C be a nonempty, bounded, closed, and convex subset of a p-uniformly convex metric space (M, d). If the mapping $\mathcal{T} : C \to C$ is enriched Suzuki nonexpansive and satisfies condition (1), then the Mann iteration sequence $\{u_n\}$ defined by (4.1) converges strongly to a fixed point of $\mathcal{T}$.*

Proof. By condition (1) we have

$$f(d^p(u, F(\mathcal{T}))) \leq d^p(u_n, \mathcal{T} u_n) \text{ for all } n \in \mathbb{N}.$$

It follows from (4.4) that

$$\lim_{n\to\infty} d^p(u_n, \mathcal{T} u_n) = 0,$$

then

$$\lim_{n\to\infty} f(d^p(u_n, F(\mathcal{T})) = 0.$$

We choose a subsequence $\{u_{nk}\}$ of $\{u_n\}$ and subsequence $\{w_k\}$ in $F(T)$ such that

$$d(u_{nk+1}, w_k) \leq \frac{1}{2^k} \quad \forall \quad k \in \mathbb{N}, \tag{4.6}$$

by 4.2 we have

$$\begin{aligned} d(w_{k+1}, w_k) &\leq d(u_{nk}, w_k) \\ &\leq \frac{1}{2^k}. \end{aligned}$$

Hence,

$$\begin{aligned} d(w_{k+1}, w_k) &\leq d(w_{k+1}, u_{nk+1}) + d(u_{nk+1}, w_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &\leq \frac{1}{2^{k+1}} \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

This establishes that the sequence $\{w_k\}$ is Cauchy in $F(\mathcal{T})$. Since $F(\mathcal{T})$ is closed, there exists a point $w \in F(\mathcal{T})$ such that $\lim_{k\to\infty} w_k = w$. From (4.1), it follows that the limit $\lim_{n\to\infty} d(u_n, w)$ exists. Consequently, we conclude that $\lim_{n\to\infty} d(u_n, w) = 0$. Therefore we obtain the desired result. □

Now, we state some of the consequences of the result.

Corollary 4.1. *Let C be a nonempty bounded closed convex subset of a p-uniformly convex metric space (M, d). If $\mathcal{T} : C \to C$ is a Suzuki nonexpansive mapping that satisfies condition (1), then the sequence $\{u_n\}$ of Mann iterates (4.1) converges to a fixed point of $\mathcal{T}$.*

Corollary 4.2. *Let C be a nonempty bounded closed convex subset of a p-uniformly convex metric space (M, d). If $\mathcal{T} : C \to C$ is a nonexpansive mapping that satisfies condition (1), then the sequence $\{u_n\}$ of Mann iterates (4.1) converges to a fixed point of $\mathcal{T}$.*

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