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Solving Integral Equation via Fixed Point Theorem on Neutrosophic Bipolar Cone Metric Spaces

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Abstract. In this article, we introduce the notion of neutrosophic bipolar cone metric space and prove fixed point theorems. The authors demonstrate various satisfying contraction mapping results using non-trivial examples. Finally, we prove existence and uniqueness results of the integral equation to strengthen our obtained result.

1. Introduction

The basic idea of fixed-point theory consists of the notion of metric spaces and the Banach contraction principle. On metric spaces, numerous generalizations have been made so far [1–4]. This illustrates the style, attractiveness and development of the idea of metric spaces.

Zadeh presented the idea of fuzzy sets [5]. The term "fuzzy" is frequently encountered and widely utilized in modern research concerning the logical and set-theoretical foundations of mathematics. We believe that the primary explanation for this rapid development is simple to understand. The environment we live in, the notion we use, and the data derived from our findings and metric are all, for the most part, imprecise and inaccurate, which leaves us with a world full with

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clarity. Fuzzy sets,fuzzy languages, fuzzy orderings, and so on allow us to investigate and deal with the previously specified degree of uncertainty in a strictly formal and mathematical way.

Fuzzy sets have been able to successfully move many mathematical structures within its idea. In [6], Schweizer and Sklar established the notion of continuous t-norms. The notion of fuzzy metric spaces was first presented by Kramosil and Michalek [7]. The notion of fuzzy was applied to classical concepts of metric and metric spaces through continuous t-norms. The resulting notions were compared with those derived from various statistical generalizations of metric spaces, namely probabilistic ones. Garbiec [8] provided a fuzzy version of the Banach contraction principle in fuzzy metric spaces. Some $\alpha - \phi$-fuzzy cone contraction findings were demonstrated by Ur-Reham et al. [9] using integral type application.

Park et al. [10] created an intuitionistic fuzzy metric space. Konwar [11] established multiple fixed-point theorems and presented the idea of an intuitionistic fuzzy b-metric space. The concept of neutrosophic metric spaces was first proposed by Kirişci and Simsek [12]. Some remarkable fixed-point findings were demonstrated in the context of neutrosophic metric spaces by Simsek and Kirişci [13]. A few fixed point results were established in the context of neutrosophic metric spaces by Sowndrarajan et al. [14]. Bipolar metric spaces were first proposed by Mutlu and Gurdal [15], who also demonstrated fixed point theorems. In the context of bipolar metric spaces, numerous researchers have recently established a variety of fixed point findings utilizing a variety of contractions ([16]- [30]).

Huang and Zhang [31] introduced cone metric spaces and proved several fixed point theorems for contractive mappings. In a cone b-metric space, Mani et al. [32] have demonstrated a fixed point result. Several fixed point theorems in a partial cone metric space have been proven by Dey and Saha [33]. A common fixed point theorem in a partial cone metric space has been proven by Shateri [34]. In a cone metric space, Arif et al. [35] established an ordered implicit relation and demonstrated a few fixed point theorems. In a cone A-metric space, Arif et al. [36] established an ordered implicit relation and demonstrated a few fixed point theorems.

In this paper, we introduce the notion of neutrosophic bipolar cone metric space and prove fixed point theorems. The main objectives of this paper are as follows:

- To introduce the notion of neutrosophic bipolar cone metric space
- To prove several fixed-point theorems for contraction mappings
- To find the existence and uniqueness results of an integral equation.

2. Preliminaries

In this section, the authors present definitions that will help readers to understand the main section. Let $\mathcal{A}$ be a real Banach space and $\mathcal{R} \subseteq \mathcal{A}$. $\mathcal{R}$ is called a cone iff

- $\mathcal{R}$ is closed, nonempty, and $\mathcal{R} \neq \{0\}$;
- If $\overline{a}, \overline{c} \in \mathbb{R}$ and $\overline{a}, \overline{c} \geq 0$, then $\overline{a}\vartheta + \overline{c}\rho \in \mathcal{R}, \ \forall \ \vartheta, \rho \in \mathcal{R}$;
- $\vartheta \in \mathcal{R}$ and $-\vartheta \in \mathcal{R} \implies \vartheta = 0$.

$\vartheta \leq \rho$ iff $\rho - \vartheta \in \mathcal{R}$ denotes a partial ordering $\leq$ with respect to $\mathcal{R}$, where $\mathcal{R}$ is a cone $\mathcal{R} \subset \mathcal{A}$. While $\vartheta \ll \rho$ stands for $\rho - \vartheta \in int(\mathcal{R})$, where $int(\mathcal{R})$ indicates the interior of $\mathcal{R}$. Also, we write $\vartheta < \rho$ to show that $\vartheta \leq \rho$ and $\vartheta \neq \rho$.

The cone $\mathcal{R}$ is called normal if there is a number $N > 0$ such that for all $\vartheta, \rho \in \mathcal{A}$,

$$0 \leq \vartheta \leq \rho \text{ implies } \|\vartheta\| \leq N\|\rho\|.$$

The least positive number satisfying above is called the normal constant of $\mathcal{W}$.

If every increasing sequence which is bounded above is convergent, then the cone $\mathcal{W}$ is said to be regular. That is, if $\{\vartheta_\nu\}$ is a sequence such that

$$\vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_\nu \leq \cdots \leq \rho$$

for some $\rho \in \mathcal{A}$, then there is $\vartheta \in \mathcal{A}$ such that $\|\vartheta_\nu - \vartheta\| \to 0 (\nu \to \infty)$.

Comparatively, if every decreasing sequence that is bounded from below be convergent, then the cone $\mathcal{R}$ is regular. A regular cone is a normal cone. In the following, we assume that $\mathcal{A}$ is a Banach space, $\leq$ is a partial ordering with respect to $\mathcal{R}$ and that $\mathcal{R}$ is a cone in $\mathcal{A}$ with $int\mathcal{R} \neq \emptyset$. Here, con-t-nm means continuous triangle norm and con-t-co-nm means continuous triangle co-norm

Definition 2.1. *([10]) Let* $*: [0,1] \times [0,1] \to [0,1]$ *be a binary operation is said to be con-t-nm if:*

(1) $\breve{u} * \breve{v} = \breve{v} * \breve{u}, (\forall) \breve{u}, \breve{v} \in [0,1]$;
(2) $*$ *is continuous;*
(3) $\breve{u} * 1 = \breve{u}, (\forall) \breve{u} \in [0,1]$;
(4) $(\breve{u} * \breve{v}) * \breve{c} = \breve{u} * (\breve{v} * \breve{c}), \forall\, \breve{u}, \breve{v}, \breve{c} \in [0,1]$;
(5) If $\breve{u} \leq \breve{c}$ *and* $\breve{v} \leq \breve{d}$, $\forall\, \breve{u}, \breve{v}, \breve{c}, \breve{d} \in [0,1]$, *then* $\breve{u} * \breve{v} \leq \breve{c} * \breve{d}$.

Definition 2.2. *([10]) Let* $\circ: [0,1] \times [0,1] \to [0,1]$ *be a binary operation is said to be con-t-co-nm if:*

(1) $\breve{u} \circ \breve{v} = \breve{v} \circ \breve{u}, \forall\, \breve{u}, \breve{v} \in [0,1]$;
(2) $\circ$ *is continuous;*
(3) $\breve{u} \circ 0 = 0, \forall\, \breve{u} \in [0,1]$;
(4) $(\breve{u} \circ \breve{v}) \circ \breve{c} = \breve{u} \circ (\breve{v} \circ \breve{c}), \forall\, \breve{u}, \breve{v}, \breve{c} \in [0,1]$;
(5) If $\breve{u} \leq \breve{c}$ *and* $\breve{c} \leq \breve{d}$, $\forall\, \breve{u}, \breve{v}, \breve{c}, \breve{d} \in [0,1]$, *then* $\breve{u} \circ \breve{v} \leq \breve{c} \circ \breve{d}$.

Definition 2.3. *([11]) Take* $\mathcal{P} \neq \emptyset$. *Let* $*$ *be a con-t-nm,* $\circ$ *be a con-t-co-nm,* $\mathfrak{b} \geq 1$ *and* $\mathcal{E}, \mathcal{F}$ *be fuzzy sets on* $\mathcal{P} \times \mathcal{P} \times (0, +\infty)$. *If* $(\mathcal{P}, \mathcal{E}, \mathcal{F}, *, \circ)$ *fullfils all* $\vartheta, \rho \in \mathcal{P}$ *and* $\check{s}, \pi > 0$:

(I) $\mathcal{E}(\vartheta, \rho, \pi) + \mathcal{F}(\vartheta, \rho, \pi) \leq 1$;
(II) $\mathcal{E}(\vartheta, \rho, \pi) > 0$;
(III) $\mathcal{E}(\vartheta, \rho, \pi) = 1 \Leftrightarrow \vartheta = \rho$;
(IV) $\mathcal{E}(\vartheta, \rho, \pi) = \mathcal{E}(\rho, \vartheta, \pi)$;
(V) $\mathcal{E}(\vartheta, \lambda, \mathfrak{b}(\pi + \check{s})) \geq \mathcal{E}(\vartheta, \rho, \pi) * \mathcal{E}(\rho, \lambda, \check{s})$;
(VI) $\mathcal{E}(\vartheta, \rho, \cdot)$ *is a non-decreasing function of* $\mathbb{R}^+$ *and* $\lim_{\pi \to +\infty} \mathcal{E}(\vartheta, \rho, \pi) = 1$;
(VII) $\mathcal{F}(\vartheta, \rho, \pi) > 0$;

(VIII) $\mathcal{F}(\vartheta,\rho,\pi)=0 \Leftrightarrow \vartheta=\rho$;

(IX) $\mathcal{F}(\vartheta,\rho,\pi)=\mathcal{F}(\rho,\vartheta,\pi)$;

(X) $\mathcal{F}(\vartheta,\lambda,\mathfrak{b}(\pi+\check{s}))\leq\mathcal{F}(\vartheta,\rho,\pi)\circ\mathcal{F}(\rho,\lambda,\check{s})$;

(XI) $\mathcal{F}(\vartheta,\rho,\cdot)$ *is a non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{\pi\to+\infty}\mathcal{F}(\vartheta,\rho,\pi)=0$,

Then, $(\mathcal{P},\mathcal{E},\mathcal{F},*,\circ)$ *is called an intuitionistic fuzzy* $\mathfrak{b}$*-metric space.*

Definition 2.4. *([12]) Let* $\mathcal{P}\neq\emptyset, *$ *is a con-t-nm,* $\circ$ *be a con-t-co-nm, and* $\mathcal{E},\mathcal{F},\mathcal{S}$ *are neutrosophic sets on* $\mathcal{P}\times\mathcal{P}\times(0,+\infty)$ *is called a neutosophic metric on* $\mathcal{P}$*, if for all* $\vartheta,\rho,\lambda\in\mathcal{P}$*, then the axioms are satisfied:*

(1) $\mathcal{E}(\vartheta,\rho,\pi)+\mathcal{F}(\vartheta,\rho,\pi)+\mathcal{S}(\vartheta,\rho,\pi)\leq 3$;

(2) $\mathcal{E}(\vartheta,\rho,\pi)>0$;

(3) $\mathcal{E}(\vartheta,\rho,\pi)=1$ *for all* $\pi>0$*, iff* $\vartheta=\rho$;

(4) $\mathcal{E}(\vartheta,\rho,\pi)=\mathcal{E}(\rho,\vartheta,\pi)$;

(5) $\mathcal{E}(\vartheta,\lambda,\pi+\check{s})\geq\mathcal{E}(\vartheta,\rho,\pi)*\mathcal{E}(\rho,\lambda,\check{s})$;

(6) $\mathcal{E}(\vartheta,\rho,\cdot)\colon(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{E}(\vartheta,\rho,\pi)=1$;

(7) $\mathcal{F}(\vartheta,\rho,\pi)<1$;

(8) $\mathcal{F}(\vartheta,\rho,\pi)=0$ *for all* $\pi>0$*, iff* $\vartheta=\rho$;

(9) $\mathcal{F}(\vartheta,\rho,\pi)=\mathcal{F}(\rho,\vartheta,\pi)$;

(10) $\mathcal{F}(\vartheta,\lambda,\pi+\check{s})\leq\mathcal{F}(\vartheta,\rho,\pi)\circ\mathcal{F}(\rho,\lambda,\check{s})$;

(11) $\mathcal{F}(\vartheta,\rho,\cdot)\colon(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{F}(\vartheta,\rho,\pi)=0$;

(12) $\mathcal{S}(\vartheta,\rho,\pi)<1$;

(13) $\mathcal{S}(\vartheta,\rho,\pi)=0$ *for all* $\pi>0$*, iff* $\vartheta=\rho$;

(14) $\mathcal{S}(\vartheta,\rho,\pi)=\mathcal{S}(\rho,\vartheta,\pi)$;

(15) $\mathcal{S}(\vartheta,\lambda,\pi+\check{s})\leq\mathcal{S}(\vartheta,\rho,\pi)\circ\mathcal{S}(\rho,\lambda,\check{s})$;

(16) $\mathcal{S}(\vartheta,\rho,\cdot)\colon(0,+\infty)\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{S}(\vartheta,\rho,\pi)=0$;

(17) If $\pi\leq 0$*, then* $\mathcal{E}(\vartheta,\rho,\pi)=0,\mathcal{F}(\vartheta,\rho,\pi)=0$;

Then, $(\mathcal{P},\mathcal{E},\mathcal{F},\mathcal{S},*,\circ)$ *is called a neutrosophic metric space.*

Definition 2.5. *[15] Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be non-empty sets and* $\varrho:\mathcal{P}\times\mathcal{Q}\to[0,+\infty)$ *be a function, implies that*

(a) $\varrho(\vartheta,\rho)=0$ *iff* $\vartheta=\rho$, $\forall\ (\vartheta,\rho)\in\mathcal{P}\times\mathcal{Q}$

(b) $\varrho(\vartheta,\rho)=\varrho(\vartheta,\rho)$, $\forall\ (\vartheta,\rho)\in\mathcal{P}\cap\mathcal{Q}$

(c) $\varrho(\vartheta,\rho)\leq\varrho(\vartheta,\gamma)+\varrho(\vartheta_1,\gamma)+\varrho(\vartheta_1,\rho)$*, for all* $\vartheta,\vartheta_1\in\mathcal{P}$ *and* $\gamma,\rho\in\mathcal{Q}$.

The pair $(\mathcal{P},\mathcal{Q},\varrho)$ *is called a bipolar metric space.*

3. Main results

In this section, we present neutrosophic bipolar cone metric space and shows a few fixed-point results Here $\mathcal{UFP}$ means unique fixed point.

Definition 3.1. *Let* $\mathcal{P}\neq\emptyset$*,* $\mathcal{Q}\neq\emptyset$ *be two sets and* $*$ *be a con-t-nm,* $\circ$ *be a con-t-co-nm,* $\mathcal{R}$ *is a closed cone and* $\mathcal{E},\mathcal{F},\mathcal{G}$ *be neutrosophic sets on* $\mathcal{P}\times\mathcal{Q}\times int(\mathcal{R})$ *is called a neutrosophic bipolar cone metric on* $\mathcal{P}\times\mathcal{Q}$*, if for all* $\vartheta,\mathfrak{x}\in\mathcal{P}$*,* $\rho,\lambda\in\mathcal{Q}$ *and* $\pi,\hat{s},\hat{w}\in int(\mathcal{R})$*, then the axioms satisfied:*

(i) $\mathcal{E}(\vartheta,\rho,\pi)+\mathcal{F}(\vartheta,\rho,\pi)+\mathcal{G}(\vartheta,\rho,\pi)\leq 3$;
(ii) $\mathcal{E}(\vartheta,\rho,\pi)>0$;
(iii) $\mathcal{E}(\vartheta,\rho,\pi)=1$ *for all* $\pi\in int(\mathcal{R})$, *iff* $\vartheta=\rho$;
(iv) $\mathcal{E}(\vartheta,\rho,\pi)=\mathcal{E}(\rho,\vartheta,\pi)$;
(v) $\mathcal{E}(\vartheta,\lambda,\pi+\check{s}+\check{w})\geq\mathcal{E}\Big(\vartheta,\rho,\pi\Big)*\mathcal{E}\Big(\mathfrak{x},\rho,\check{s}\Big)*\mathcal{E}\Big(\mathfrak{x},\lambda,\check{w}\Big)$;
(vi) $\mathcal{E}(\vartheta,\rho,\cdot)\colon int(\mathcal{R})\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{E}(\vartheta,\rho,\pi)=1$;
(vii) $\mathcal{E}(\vartheta,\rho,\cdot)$ *is nondecreasing;*
(viii) $\mathcal{F}(\vartheta,\rho,\pi)<1$;
(ix) $\mathcal{F}(\vartheta,\rho,\pi)=0$ *for all* $\pi\in int(\mathcal{R})$, *iff* $\vartheta=\rho$;
(x) $\mathcal{F}(\vartheta,\rho,\pi)=\mathcal{F}(\rho,\vartheta,\pi)$;
(xi) $\mathcal{F}(\vartheta,\lambda,\pi+\check{s}+\check{w})\leq\mathcal{F}\Big(\vartheta,\rho,\pi\Big)\circ\mathcal{F}\Big(\mathfrak{x},\rho,\check{s}\Big)\circ\mathcal{F}\Big(\mathfrak{x},\lambda,\check{w}\Big)$;
(xii) $\mathcal{F}(\vartheta,\rho,\cdot)\colon int(\mathcal{R})\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{F}(\vartheta,\rho,\pi)=0$;
(xiii) $\mathcal{F}(\vartheta,\rho,\cdot)$ *is nondecreasing;*
(xiv) $\mathcal{G}(\vartheta,\rho,\pi)<1$;
(xv) $\mathcal{G}(\vartheta,\rho,\pi)=0$ *for all* $\pi\in int(\mathcal{R})$, *iff* $\vartheta=\rho$;
(xvi) $\mathcal{G}(\vartheta,\rho,\pi)=\mathcal{G}(\rho,\vartheta,\pi)$;
(xvii) $\mathcal{G}(\vartheta,\lambda,\pi+\check{s}+\check{w})\leq\mathcal{G}\Big(\vartheta,\rho,\pi\Big)\circ\mathcal{G}\Big(\mathfrak{x},\rho,\check{s}\Big)\circ\mathcal{G}\Big(\mathfrak{x},\lambda,\check{w}\Big)$;
(xviii) $\mathcal{G}(\vartheta,\rho,\cdot)\colon int(\mathcal{R})\to[0,1]$ *is continuous and* $\lim_{\pi\to+\infty}\mathcal{G}(\vartheta,\rho,\pi)=0$;
(xix) $\mathcal{G}(\vartheta,\rho,\cdot)$ *is nondecreasing;*
(xx) *If* $\pi\leq 0$, *then* $\mathcal{E}(\vartheta,\rho,\pi)=0,\mathcal{F}(\vartheta,\rho,\pi)=1$ *and* $\mathcal{S}(\vartheta,\rho,\pi)=1$.

Then, $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ *is called a neutrosophic bipolar cone metric space.*

Definition 3.2. *Let* $\check{p}:\mathcal{P}_1\cup\mathcal{Q}_1\to\mathcal{P}_2\cup\mathcal{Q}_2$ *be a mapping, where* $(\mathcal{P}_1,\mathcal{Q}_1)$ *and* $(\mathcal{P}_2,\mathcal{Q}_2)$ *pairs of sets.*

(H1) *If* $\check{p}(\mathcal{P}_1)\subseteq\mathcal{P}_2$ *and* $\check{p}(\mathcal{Q}_1)\subseteq\mathcal{Q}_2$, *then* $\check{p}$ *is said to be a covariant map* $(\mathcal{P}_1,\mathcal{Q}_1,\mathcal{E}_1,\mathcal{F}_1,\mathcal{G}_1,*,\circ,\mathcal{R})$ *to* $(\mathcal{P}_2,\mathcal{Q}_2,\mathcal{E}_2,\mathcal{F}_2,\mathcal{G}_2,*,\circ,\mathcal{R})$ *and this is written as* $\check{p}:(\mathcal{P}_1,\mathcal{Q}_1,\mathcal{E}_1,\mathcal{F}_1,\mathcal{G}_1,*,\circ,\mathcal{R})\rightrightarrows(\mathcal{P}_2,\mathcal{Q}_2,\mathcal{E}_2,\mathcal{F}_2,\mathcal{G}_2,*,\circ,\mathcal{R})$.
(H2) *If* $\check{p}(\mathcal{P}_1)\subseteq\mathcal{Q}_2$ *and* $\check{p}(\mathcal{Q}_1)\subseteq\mathcal{P}_2$, *then* $\check{p}$ *is called a contravariant map from* $(\mathcal{P}_1,\mathcal{Q}_1,\mathcal{E}_1,\mathcal{F}_1,\mathcal{G}_1,*,\circ,\mathcal{R})$ *to* $(\mathcal{P}_2,\mathcal{Q}_2,\mathcal{E}_2,\mathcal{F}_2,\mathcal{G}_2,*,\circ,\mathcal{R})$ *and this is denoted as* $\check{p}:(\mathcal{P}_1,\mathcal{Q}_1,\mathcal{E}_1,\mathcal{F}_1,\mathcal{G}_1,*,\circ,\mathcal{R})\leftrightarrows(\mathcal{P}_2,\mathcal{Q}_2,\mathcal{E}_2,\mathcal{F}_2,\mathcal{G}_2,*,\circ,\mathcal{R})$.

Definition 3.3. *Let* $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ *is a neutrosophic bipolar cone metric space, an open ball is then defined* $\mathcal{G}(\vartheta,\mathfrak{r},\pi)$ *with center* ϑ, *radius* $\mathfrak{r},0<\mathfrak{r}<1$ *and* $\pi\in int(\mathcal{R})$ *is defined by:*

$$\mathcal{G}(\vartheta,\mathfrak{r},\pi)=\{\rho\in\mathcal{P}\colon\mathcal{E}(\vartheta,\rho,\pi)>1-\mathfrak{r},\mathcal{F}(\vartheta,\rho,\pi)<\mathfrak{r},\mathcal{G}(\vartheta,\rho,\pi)<\mathfrak{r}\}.$$

Definition 3.4. *Let* $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ *is a neutrosophic bipolar cone metric space.*

(B1) *A point* $\vartheta\in\mathcal{P}\cup\mathcal{Q}$ *is called a left point if* $\vartheta\in\mathcal{P}$, *a right point if* $\vartheta\in\mathcal{Q}$ *and a central point if it holds.*
(B2) *If a sequence* $\{\vartheta_{\check{n}}\}\subset\mathcal{P}$ *is said to be a left sequence and* $\{\beta_n\}\subset\mathcal{Q}$ *is said to be a right sequence.*

(B3) If a sequence $\{\vartheta_{\check{n}}\} \subset \mathcal{P} \cup \mathcal{Q}$ *is called converge to a point* ϑ *iff* $\{\vartheta_{\check{n}}\}$ *is a left sequence,* ϑ *is a right point and*

$$\lim_{\check{n}\to+\infty} \mathcal{E}(\vartheta_{\check{n}}, \vartheta, \pi) = 1, \lim_{\check{n}\to+\infty} \mathcal{F}(\vartheta_{\check{n}}, \vartheta, \pi) = 0,$$
$$\lim_{\check{n}\to+\infty} \mathcal{G}(\vartheta_{\check{n}}, \vartheta, \pi) = 0 \quad \textit{for all} \quad \pi \in int(\mathcal{R})$$

or $\{\vartheta_{\check{n}}\}$ *is a right sequence,* ϑ *is a left point and*

$$\lim_{\check{n}\to+\infty} \mathcal{E}(\vartheta, \vartheta_{\check{n}}, \pi) = 1, \lim_{\check{n}\to+\infty} \mathcal{F}(\vartheta, \vartheta_{\check{n}}, \pi) = 0,$$
$$\lim_{\check{n}\to+\infty} \mathcal{G}(\vartheta, \vartheta_{\check{n}}, \pi) = 0 \quad \textit{for all} \quad \pi \in int(\mathcal{R}).$$

(B4) A sequence $\{(\vartheta_{\check{n}}, \beta_{\check{n}})\} \subset \mathcal{P} \times \mathcal{Q}$ *is said to be a bisequence. If* $\{\vartheta_{\check{n}}\}$ *and* $\{\beta_{\check{n}}\}$ *are converge then* $\{(\vartheta_{\check{n}}, \beta_{\check{n}})\}$ *be bisequence is said to be convergent in* $\mathcal{P} \times \mathcal{Q}$.

(B5) If $\{\vartheta_{\check{n}}\}$ *and* $\{\beta_{\check{n}}\}$ *are converge to a point* $\beta \in \mathcal{P} \cap \mathcal{Q}$ *then* $\{(\vartheta_{\check{n}}, \beta_{\check{n}})\}$ *be a bisequence is said to be biconvergent. A sequence* $\{(\vartheta_{\check{n}}, \beta_{\check{n}})\}$ *is a Cauchy bisequence if*

$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{E}(\vartheta_{\check{n}}, \beta_{\check{m}}, \pi) = 1, \lim_{\check{n},\check{m}\to+\infty} \mathcal{F}(\vartheta_{\check{n}}, \beta_{\check{m}}, \pi) = 0, \lim_{\check{n},\check{m}\to+\infty} \mathcal{G}(\vartheta_{\check{n}}, \beta_{\check{m}}, \pi) = 0,$$

for all $\pi \in int(\mathcal{R})$.

(B6) A neutrosophic bipolar cone metric space is said to be complete if every Cauchy bisequence is convergent.

Lemma 3.1. *Let* $\{\vartheta_{\check{n}}\}$ *be a Cauchy sequence in neutrosophic bipolar cone metric space* $(\mathcal{P}, \mathcal{Q}, \mathcal{E}, \mathcal{F}, \mathcal{G}, *, \circ, \mathcal{R})$ *implies that* $\vartheta_{\check{n}} \neq \vartheta_{\check{m}}$ *if* $\check{m}, \check{n} \in \mathbb{N}$ *with* $\check{n} \neq \check{m}$. *Then sequence* $\{\vartheta_{\check{n}}\}$ *can only reach a single limit point.*

Proof. Alternatively, suppose that $\vartheta_{\check{n}} \to \vartheta \in \mathcal{Q}$ and $\vartheta_{\check{n}} \to \rho \in \mathcal{P} \cap \mathcal{Q}$, for $\vartheta \neq \rho$. Then, $\lim_{\check{n}\to+\infty} \mathcal{E}(\vartheta_{checkn}, \vartheta, \pi) = 1, \lim_{\check{n}\to+\infty} \mathcal{F}(\vartheta_{\check{n}}, \vartheta, \pi) = 0, \lim_{\check{n}\to+\infty} \mathcal{G}(\vartheta_{\check{n}}, \vartheta, \pi) = 0$, and $\lim_{\check{n}\to+\infty} \mathcal{E}(\vartheta_{\check{n}}, \rho, \pi) = 1, \lim_{\check{n}\to+\infty} \mathcal{F}(\vartheta_{\check{n}}, \rho, \pi) = 0, \lim_{\check{n}\to+\infty} \mathcal{G}(\vartheta_{\check{n}}, \rho, \pi) = 0$, for all $\pi \in int(\mathcal{R})$. Suppose

$$\begin{aligned}
\mathcal{E}(\vartheta, \rho, \pi) &\geq \mathcal{E}\left(\vartheta, \vartheta_{\check{n}}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}+1}, \rho, \frac{\pi}{3}\right) \\
&\to 1 * 1 * 1, \quad \text{as} \quad \check{n}, \to +\infty, \\
\mathcal{F}(\vartheta, \rho, \pi) &\leq \mathcal{F}\left(\vartheta, \vartheta_{\check{n}}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}+1}, \rho, \frac{\pi}{3}\right) \\
&\to 0 \circ 0 \circ 0, \quad \text{as} \quad \check{n}, \to +\infty, \\
\mathcal{G}(\vartheta, \rho, \pi) &\leq \mathcal{G}\left(\vartheta, \vartheta_{\check{n}}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}+1}, \rho, \frac{\pi}{3}\right) \\
&\to 0 \circ 0 \circ 0, \quad \text{as} \quad \check{n} \to +\infty.
\end{aligned}$$

That is $\mathcal{E}(\vartheta, \rho, \pi) \geq 1 * 1 * 1 = 1, \mathcal{F}(\vartheta, \rho, \pi) \leq 0 \circ 0 \circ 0 = 0$ and $\mathcal{G}(\vartheta, \rho, \pi) \leq 0 \circ 0 \circ 0 = 0$. Thus $\vartheta = \rho$, this means that the sequence converges to no more than one limit point. □

Lemma 3.2. *Let $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a neutrosophic bipolar cone metric space. If $0<\theta<1$ and for all $\vartheta,\rho\in\mathcal{P},\pi\in int(\mathcal{R})$,*

$$\mathcal{E}(\vartheta,\rho,\pi)\geq\mathcal{E}\left(\vartheta,\rho,\frac{\pi}{\theta}\right),\mathcal{F}(\vartheta,\rho,\pi)\leq\mathcal{F}\left(\vartheta,\rho,\frac{\pi}{\theta}\right),\mathcal{G}(\vartheta,\rho,\pi)\leq\mathcal{G}\left(\vartheta,\rho,\frac{\pi}{\theta}\right) \tag{3.1}$$

then $\vartheta=\rho$.

Proof. By (3.1) such that

$$\mathcal{E}(\vartheta,\rho,\pi)\geq\mathcal{E}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right),\mathcal{F}(\vartheta,\rho,\pi)\leq\mathcal{F}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right),$$
$$\mathcal{G}(\vartheta,\rho,\pi)\leq\mathcal{G}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right),\check{n}\in\mathbb{N},\pi\in int(\mathcal{R}).$$

Now

$$\mathcal{E}(\vartheta,\rho,\pi)\geq\lim_{\check{n}\to+\infty}\mathcal{E}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right)=1,$$
$$\mathcal{F}(\vartheta,\rho,\pi)\leq\lim_{\check{n}\to+\infty}\mathcal{F}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right)=0,$$
$$\mathcal{G}(\vartheta,\rho,\pi)\leq\lim_{\check{n}\to+\infty}\mathcal{G}\left(\vartheta,\rho,\frac{\pi}{\theta^{\check{n}}}\right)=0,\pi\in int(\mathcal{R}).$$

Also, by definition of ((iii)), ((ix)), ((xv)), that is, $\vartheta=\rho$. □

Theorem 3.1. *Suppose $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete neutrosophic bipolar cone metric space with $0<\theta<1$ and if*

$$\lim_{\pi\to+\infty}\mathcal{E}(\vartheta,\rho,\pi)=1,\lim_{\pi\to+\infty}\mathcal{F}(\vartheta,\rho,\pi)=0 \quad and \quad \lim_{\pi\to+\infty}\mathcal{G}(\vartheta,\rho,\pi)=0 \tag{3.2}$$

for all $\vartheta\in\mathcal{P}$, $\rho\in\mathcal{Q}$ and $\pi\in int\mathcal{R}$. Let $\check{p}:\mathcal{P}\cup\mathcal{Q}\to\mathcal{P}\cup\mathcal{Q}$ be a map it holds:

(i) $\check{p}(\mathcal{P})\subseteq\mathcal{P}$ *and* $\check{p}(\mathcal{Q})\subseteq\mathcal{Q}$;

(ii)

$$\mathcal{E}(\check{p}\vartheta,\check{p}\rho,\theta\pi)\geq\mathcal{E}(\vartheta,\rho,\pi),$$
$$\mathcal{F}(\check{p}\vartheta,\check{p}\rho,\theta\pi)\leq\mathcal{F}(\vartheta,\rho,\pi) \quad and \quad \mathcal{G}(\check{p}\vartheta,\check{p}\rho,\theta\pi)\leq\mathcal{G}(\vartheta,\rho,\pi) \tag{3.3}$$

for all $\vartheta\in\mathcal{P}$, $\rho\in\mathcal{Q}$ and $\pi\in int\mathcal{R}$.

Then $\check{p}$ has a $\mathcal{UFP}$.

Proof. Let $\vartheta_0\in\mathcal{P}$ and $\rho_0\in\mathcal{Q}$ and assume that $\check{p}(\vartheta_{\check{n}})=\vartheta_{\check{n}+1}$ and $\check{p}(\rho_{\check{n}})=\rho_{\check{n}+1}$ for all $\check{n}\in\mathbb{N}\cup\{0\}$. Then we get $(\vartheta_{\check{n}},\mathcal{Q}_{\check{n}})$as a bisequence on neutrosophic bipolar cone metric space $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$. Now, we have

$$\mathcal{E}(\vartheta_1,\rho_1,\pi)=\mathcal{E}(\check{p}\vartheta_0,\check{p}\rho_0,\pi)\geq\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{\theta}),$$

$$\mathcal{F}(\vartheta_1,\rho_1,\pi)=\mathcal{F}(\check{p}\vartheta_0,\check{p}\rho_0,\pi)\leq\mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{\theta})$$

and

$$\mathcal{G}(\vartheta_1, \rho_1, \pi) = \mathcal{G}(\check{p}\vartheta_0, \check{p}\rho_0, \pi) \leq \mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{\theta}),$$

for all $\pi \in int\mathcal{R}$ and $\check{n} \in \mathbb{N}$. By simple induction, we get

$$\begin{aligned}
\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) &= \mathcal{E}(\check{p}\vartheta_{\check{n}-1}, \check{p}\rho_{\check{n}-1}, \pi) \geq \mathcal{E}(\vartheta_{\check{n}-1}, \rho_{\check{n}-1}, \frac{\pi}{\theta}) \geq \mathcal{E}\left(\vartheta_{\check{n}-2}, \rho_{\check{n}-2}, \frac{\pi}{\theta^2}\right) \\
&\geq \mathcal{E}\left(\vartheta_{\check{n}-3}, \rho_{\check{n}-3}, \frac{\pi}{\theta^3}\right) \geq \cdots \geq \mathcal{E}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right), \\
\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) &= \mathcal{F}(\check{p}\vartheta_{\check{n}-1}, \check{p}\rho_{\check{n}-1}, \pi) \leq \mathcal{F}(\vartheta_{\check{n}-1}, \rho_{\check{n}-1}, \frac{\pi}{\theta}) \leq \mathcal{F}\left(\vartheta_{\check{n}-2}, \rho_{\check{n}-2}, \frac{\pi}{\theta^2}\right) \\
&\leq \mathcal{F}\left(\vartheta_{\check{n}-3}, \rho_{\check{n}-3}, \frac{\pi}{\theta^3}\right) \leq \cdots \leq \mathcal{F}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) &= \mathcal{G}(\check{p}\vartheta_{\check{n}-1}, \check{p}\rho_{\check{n}-1}, \pi) \leq \mathcal{G}(\vartheta_{\check{n}-1}, \rho_{\check{n}-1}, \frac{\pi}{\theta}) \leq \mathcal{G}\left(\vartheta_{\check{n}-2}, \rho_{\check{n}-2}, \frac{\pi}{\theta^2}\right) \\
&\leq \mathcal{G}\left(\vartheta_{\check{n}-3}, \rho_{\check{n}-3}, \frac{\pi}{\theta^3}\right) \leq \cdots \leq \mathcal{G}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right).
\end{aligned}$$

We obtain

$$\begin{gathered}
\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) \geq \mathcal{E}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right), \\
\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) \leq \mathcal{F}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right), \mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) \leq \mathcal{G}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right).
\end{gathered} \tag{3.4}$$

and

$$\begin{gathered}
\mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \pi) \geq \mathcal{E}\left(\vartheta_1, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right), \\
\mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \pi) \leq \mathcal{F}\left(\vartheta_1, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right), \mathcal{G}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \pi) \leq \mathcal{G}\left(\vartheta_1, \rho_0, \frac{\pi}{\theta^{\check{n}}}\right).
\end{gathered} \tag{3.5}$$

Letting $\check{n} < \check{m}$, for $\check{n}, \check{m} \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\geq \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{m}}, \frac{\pi}{3}) \\
&\vdots \\
&\geq \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) * \cdots * \mathcal{E}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \\
&* \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) * \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}}), \\
\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\leq \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{m}}, \frac{\pi}{3}) \\
&\vdots \\
&\leq \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \cdots \circ \mathcal{F}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})
\end{aligned}$$

$$\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}}),$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\leq\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{m}},\frac{\pi}{3})\\&\vdots\\&\leq\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\cdots\circ\mathcal{G}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&\circ\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}}).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\geq\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})*\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})*\cdots*\mathcal{E}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\&\geq\mathcal{E}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})*\mathcal{E}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})*\cdots*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\\&*\mathcal{E}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\leq\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\cdots\circ\mathcal{F}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\&\leq\mathcal{F}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})\circ\mathcal{F}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})\circ\cdots\circ\mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\\&\circ\mathcal{F}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\circ\mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\leq\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\cdots\circ\mathcal{G}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&*\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\&\leq\mathcal{G}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})\circ\mathcal{G}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})\circ\cdots\circ\mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\\&\circ\mathcal{G}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\circ\mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}).\end{aligned}$$

Which implies that,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\geq\mathcal{E}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})*\mathcal{E}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})*\cdots*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\\&*\mathcal{E}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)&\leq\mathcal{F}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})\circ\mathcal{F}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})\circ\cdots\circ\mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\\&\circ\mathcal{F}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\circ\mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}),\end{aligned}$$

and

$$\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)\leq\mathcal{G}(\vartheta,\rho,\frac{\pi}{3\theta^{\check{n}}})\circ\mathcal{G}(\vartheta_1,\rho_0,\frac{\pi}{3\theta^{\check{n}}})\circ\cdots\circ\mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})$$
$$\circ\,\mathcal{G}(\vartheta_1,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}-1}})\circ\mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{\check{m}}}).$$

By using (3.2), for each case $\check{n},\check{m}\to+\infty$, we obtain

$$\lim_{\check{n},\check{m}\to+\infty}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)=1*1*\cdots*1=1,$$
$$\lim_{\check{n},\check{m}\to+\infty}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)=0\circ0\circ\cdots\circ0=0$$

and

$$\lim_{\check{n},\check{m}\to+\infty}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)=0\circ0\circ\cdots\circ0=0.$$

Which implies that bisequence $(\vartheta_{\check{n}},\rho_{\check{n}})$ is a Cauchy bisequence. From $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete neutrosophic bipolar cone metric space. Then, $\{\vartheta_{\check{n}}\}\to\vartheta$ and $\{\rho_{\check{n}}\}\to\vartheta$, where $\vartheta\in\mathcal{P}\cap\mathcal{Q}$. Using (v), (xi), (xvii) and (3.2), we get

$$\begin{aligned}\mathcal{E}(\vartheta,\check{p}\vartheta,\pi)&\geq\mathcal{E}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)*\mathcal{E}\left(\vartheta_{\check{n}+1},\vartheta_{\check{n}+1},\frac{\pi}{3}\right)*\mathcal{E}\left(\vartheta_{\check{n}+1},\check{p}\vartheta,\frac{\pi}{3}\right)\\&=\mathcal{E}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)*\mathcal{E}\left(\check{p}\vartheta_{\check{n}},\check{p}\vartheta_{\check{n}},\frac{\pi}{3}\right)*\mathcal{E}\left(\check{p}\vartheta_{\check{n}},\check{p}\vartheta,\frac{\pi}{3}\right)\\&\to1*1*1=1\quad\text{as}\quad\check{n}\to+\infty,\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta,\check{p}\vartheta,\pi)&\leq\mathcal{F}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{F}\left(\vartheta_{\check{n}+1},\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{F}\left(\vartheta_{\check{n}+1},\check{p}\vartheta,\frac{\pi}{3}\right)\\&=\mathcal{F}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{F}\left(\check{p}\vartheta_{\check{n}+1},\check{p}\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{F}\left(\check{p}\vartheta_{\check{n}+1},\check{p}\vartheta,\frac{\pi}{3}\right)\\&\to0\circ0\circ0=0\quad\text{as}\quad\check{n}\to+\infty\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta,\check{p}\vartheta,\pi)&\leq\mathcal{G}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{G}\left(\vartheta_{\check{n}+1},\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{G}\left(\vartheta_{\check{n}+1},\check{p}\vartheta,\frac{\pi}{3}\right)\\&=\mathcal{G}\left(\vartheta,\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{G}\left(\check{p}\vartheta_{\check{n}+1},\check{p}\vartheta_{\check{n}+1},\frac{\pi}{3}\right)\circ\mathcal{G}\left(\check{p}\vartheta_{\check{n}+1},\check{p}\vartheta,\frac{\pi}{3}\right)\\&\to0\circ0\circ0=0\quad\text{as}\quad\check{n}\to+\infty\end{aligned}$$

Hence, $\check{p}\vartheta=\vartheta$.

Now, we examine the uniqueness. Let $\check{p}\check{c}=\check{c}$ for some $\check{c}\in\mathcal{P}\cap\mathcal{Q}$, then

$$\begin{aligned}1\geq\mathcal{E}(\check{c},\vartheta,\pi)=\mathcal{E}(\check{p}\check{c},\check{p}\vartheta,\pi)&\geq\mathcal{E}\left(\check{c},\vartheta,\frac{\pi}{\theta}\right)=\mathcal{E}\left(\check{p}\check{c},\check{p}\vartheta,\frac{\pi}{\theta}\right)\\&\geq\mathcal{E}\left(\check{c},\vartheta,\frac{\pi}{\theta^2}\right)\geq\cdots\geq\mathcal{E}\left(\check{c},\vartheta,\frac{\pi}{\theta^{\check{n}}}\right)\to1\quad\text{as}\quad\check{n}\to+\infty,\end{aligned}$$

$$0 \le \mathcal{F}(\check{c}, \vartheta, \pi) = \mathcal{F}(\check{p}\check{c}, \check{p}\vartheta, \pi) \le \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta}\right) = \mathcal{F}\left(\check{p}\check{c}, \check{p}\vartheta, \frac{\pi}{\theta}\right)$$
$$\le \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta^2}\right) \le \cdots \le \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta^{\check{n}}}\right) \to 0 \quad \text{as} \quad \check{n} \to +\infty,$$

and

$$0 \le \mathcal{G}(\check{c}, \vartheta, \pi) = \mathcal{G}(\check{p}\check{c}, \check{p}\vartheta, \pi) \le \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta}\right) = \mathcal{G}\left(\check{p}\check{c}, \check{p}\vartheta, \frac{\pi}{\theta}\right)$$
$$\le \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta^2}\right) \le \cdots \le \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta^{\check{n}}}\right) \to 0 \quad \text{as} \quad \check{n} \to +\infty,$$

by using (iii), (ix) and (xv), $\vartheta = \check{c}$. □

Theorem 3.2. *Suppose $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete neutrosophic bipolar cone metric space with $0 < \theta < 1$ and if*

$$\lim_{\pi\to+\infty} \mathcal{E}(\vartheta, \rho, \pi) = 1, \lim_{\pi\to+\infty} \mathcal{F}(\vartheta, \rho, \pi) = 0 \quad \textit{and} \quad \lim_{\pi\to+\infty} \mathcal{G}(\vartheta, \rho, \pi) = 0 \tag{3.6}$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi \in int(\mathcal{R})$. Let $\check{p}: \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ be a map it holds:

(i) $\check{p}(\mathcal{P}) \subseteq \mathcal{Q}$ *and* $\check{p}(\mathcal{Q}) \subseteq \mathcal{P}$;

(ii)

$$\mathcal{E}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \ge \mathcal{E}(\rho, \vartheta, \pi),$$
$$\mathcal{F}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \le \mathcal{F}(\rho, \vartheta, \pi) \quad \textit{and} \quad \mathcal{G}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \le \mathcal{G}(\rho, \vartheta, \pi) \tag{3.7}$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi \in int(\mathcal{R})$.

Then $\check{p}$ has a $\mathcal{UFP}$.

Proof. Let $\vartheta_0 \in \mathcal{P}$ and $\rho_0 \in \mathcal{Q}$ and assume that $\check{p}(\vartheta_{\check{n}}) = \rho_{\check{n}}$ and $\check{p}(\rho_{\check{n}}) = \vartheta_{\check{n}+1}$ for all $\check{n} \in \mathbb{N} \cup \{0\}$. Then we get $(\vartheta_{\check{n}}, \mathcal{Q}_{\check{n}})$as a bisequence on neutrosophic bipolar cone metric space $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$. Now, we have

$$\mathcal{E}(\vartheta_1, \rho_0, \pi) = \mathcal{E}(\check{p}\rho_0, \check{p}\vartheta_0, \pi) \ge \mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{\theta}),$$

$$\mathcal{F}(\vartheta_1, \rho_0, \pi) = \mathcal{F}(\check{p}\rho_0, \check{p}\vartheta_0, \pi) \le \mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{\theta})$$

and

$$\mathcal{G}(\vartheta_1, \rho_0, \pi) = \mathcal{G}(\check{p}\rho_0, \check{p}\vartheta_0, \pi) \le \mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{\theta}),$$

for all $\pi > 0$ and $\check{n} \in \mathbb{N}$. By simple induction, we get

$$\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) = \mathcal{E}(\check{p}\rho_{\check{n}-1}, \check{p}\vartheta_{\check{n}}, \pi) \ge \mathcal{E}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{2\check{n}}}\right),$$

$$\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \pi) = \mathcal{F}(\check{p}\rho_{\check{n}-1}, \check{p}\vartheta_{\check{n}}, \pi) \le \mathcal{F}\left(\vartheta_0, \rho_0, \frac{\pi}{\theta^{2\check{n}}}\right),$$

$$\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\pi)=\mathcal{G}(\check{p}\rho_{\check{n}-1},\check{p}\vartheta_{\check{n}},\pi)\leq\mathcal{G}\bigg(\vartheta_0,\rho_0,\frac{\pi}{\theta^{2\check{n}}}\bigg)$$

and

$$\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi)=\mathcal{E}(\check{p}\rho_{\check{n}},\check{p}\vartheta_{\check{n}},\pi)\geq\mathcal{E}\bigg(\vartheta_0,\rho_0,\frac{\pi}{\theta^{2\check{n}+1}}\bigg),$$

$$\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi)=\mathcal{F}(\check{p}\rho_{\check{n}},\check{p}\vartheta_{\check{n}},\pi)\leq\mathcal{F}\bigg(\vartheta_0,\rho_0,\frac{\pi}{\theta^{2\check{n}+1}}\bigg).$$

$$\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi)=\mathcal{G}(\check{p}\rho_{\check{n}},\check{p}\vartheta_{\check{n}},\pi)\leq\mathcal{G}\bigg(\vartheta_0,\rho_0,\frac{\pi}{\theta^{2\check{n}+1}}\bigg).$$

Letting $\check{n}<\check{m}$, for $\check{n},\check{m}\in\mathbb{N}$. Then,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)\geq{}&\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})*\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})*\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{m}},\frac{\pi}{3})\\&\vdots\\&\geq\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})*\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})*\cdots*\mathcal{E}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)\leq{}&\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{m}},\frac{\pi}{3})\\&\vdots\\&\leq\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\cdots\circ\mathcal{F}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)\leq{}&\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{m}},\frac{\pi}{3})\\&\vdots\\&\leq\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})\circ\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})\circ\cdots\circ\mathcal{G}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&\circ\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\circ\mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}}).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi)\geq{}&\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3})*\mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3})*\cdots*\mathcal{E}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\&*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})*\mathcal{E}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\&\geq\mathcal{E}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}})*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}})*\cdots*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\&*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}})*\mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) \leq{}& \mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3}) \circ \cdots \circ \mathcal{F}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\ &\circ \mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{F}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\ &\leq \mathcal{F}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}}) \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}}) \circ \cdots \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\ &\circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}}) \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) \leq{}& \mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\frac{\pi}{3}) \circ \mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\frac{\pi}{3}) \circ \cdots \circ \mathcal{G}(\vartheta_{\check{m}-1},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}})\\ &\circ \mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}-1},\frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{G}(\vartheta_{\check{m}},\rho_{\check{m}},\frac{\pi}{3^{\check{m}-1}})\\ &\leq \mathcal{G}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}}) \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}}) \circ \cdots \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\ &\circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}}) \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}).\end{aligned}$$

Which implies that,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) \geq{}& \mathcal{E}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}}) * \mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}}) * \cdots * \mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\ &* \mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}}) * \mathcal{E}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) \leq{}& \mathcal{F}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}}) \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}}) \circ \cdots \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\ &\circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}}) \circ \mathcal{F}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) \leq{}& \mathcal{G}(\vartheta,\rho,\frac{\pi}{3\theta^{2\check{n}}}) \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3\theta^{2\check{n}+1}}) \circ \cdots \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-2}})\\ &\circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}-1}}) \circ \mathcal{G}(\vartheta_0,\rho_0,\frac{\pi}{3^{\check{m}-1}\theta^{2\check{m}}}).\end{aligned}$$

Using (3.6), for each case $\check{n},\check{m} \to +\infty$, we deduce

$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{E}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) = 1 * 1 * \cdots * 1 = 1,$$
$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{F}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) = 0 \circ 0 \circ \cdots \circ 0 = 0$$

and

$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{G}(\vartheta_{\check{n}},\rho_{\check{m}},\pi) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Which implies that bisequence $(\vartheta_{\check{n}},\rho_{\check{n}})$ is a Cauchy bisequence. From $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete neutrosophic bipolar cone metric space. Then, $\{\vartheta_{\check{n}}\} \to \vartheta$ and $\{\rho_{\check{n}}\} \to \vartheta$, where $\vartheta \in \mathcal{P} \cap \mathcal{Q}$.

Using (v), (xi), (xvii) and (3.6), we get

$$\begin{aligned}\mathcal{E}(\vartheta, \check{p}\vartheta, \pi) &\geq \mathcal{E}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &= \mathcal{E}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\check{p}\vartheta_{\check{n}}, \check{p}\vartheta_{\check{n}}, \frac{\pi}{3}\right) * \mathcal{E}\left(\check{p}\vartheta_{\check{n}}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &\to 1 * 1 * 1 = 1 \quad \text{as} \quad \check{n} \to +\infty,\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta, \check{p}\vartheta, \pi) &\leq \mathcal{F}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &= \mathcal{F}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \check{n} \to +\infty\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta, \check{p}\vartheta, \pi) &\leq \mathcal{G}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &= \mathcal{G}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \check{n} \to +\infty\end{aligned}$$

Hence, $\check{p}\vartheta = \vartheta$.
Let $\check{p}\check{c} = \check{c}$ for some $\check{c} \in \mathcal{P} \cap \mathcal{Q}$, then

$$\begin{aligned}1 \geq \mathcal{E}(\check{c}, \vartheta, \pi) &= \mathcal{E}(\check{p}\vartheta, \check{p}\check{c}, \pi) \geq \mathcal{E}\left(\check{c}, \vartheta, \frac{\pi}{\theta}\right) = \mathcal{E}\left(\check{p}\vartheta, \check{p}\check{c}, \frac{\pi}{\theta}\right) \\ &\geq \mathcal{E}\left(\check{c}, \vartheta, \frac{\pi}{\theta^2}\right) \geq \cdots \geq \mathcal{E}\left(\check{c}, \vartheta, \frac{\pi}{\theta^{\check{n}}}\right) \to 1 \quad \text{as} \quad \check{n} \to +\infty,\\ 0 \leq \mathcal{F}(\check{c}, \vartheta, \pi) &= \mathcal{F}(\check{p}\vartheta, \check{p}\check{c}, \pi) \leq \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta}\right) = \mathcal{F}\left(\check{p}\vartheta, \check{p}\check{c}, \frac{\pi}{\theta}\right) \\ &\leq \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta^2}\right) \leq \cdots \leq \mathcal{F}\left(\check{c}, \vartheta, \frac{\pi}{\theta^{\check{n}}}\right) \to 0 \quad \text{as} \quad \check{n} \to +\infty,\end{aligned}$$

and

$$\begin{aligned}0 \leq \mathcal{G}(\check{c}, \vartheta, \pi) &= \mathcal{G}(\check{p}\vartheta, \check{p}\check{c}, \pi) \leq \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta}\right) = \mathcal{G}\left(\check{p}\vartheta, \check{p}\check{c}, \frac{\pi}{\theta}\right) \\ &\leq \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta^2}\right) \leq \cdots \leq \mathcal{G}\left(\check{c}, \vartheta, \frac{\pi}{\theta^{\check{n}}}\right) \to 0 \quad \text{as} \quad \check{n} \to +\infty,\end{aligned}$$

by using ((iii)), ((ix)) and ((xv)), $\vartheta = \check{c}$. □

Corollary 3.1. *Suppose $(\mathcal{P}, \mathcal{Q}, \mathcal{E}, \mathcal{F}, \mathcal{G}, *, \circ, \mathcal{R})$ is a complete neutrosophic bipolar cone metric space with $0 < \theta < 1$ and if*

$$\lim_{\pi \to +\infty} \mathcal{E}(\vartheta, \rho, \pi) = 1, \lim_{\pi \to +\infty} \mathcal{F}(\vartheta, \rho, \pi) = 0 \quad \textit{and} \quad \lim_{\pi \to +\infty} \mathcal{G}(\vartheta, \rho, \pi) = 0 \tag{3.8}$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi > 0$. Let $\check{p}\colon \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ be a map it holds:

(i) $\check{p}(\mathcal{P}) \subseteq \mathcal{P}$ *and* $\check{p}(\mathcal{Q}) \subseteq \mathcal{Q}$;

(ii)

$$\mathcal{E}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \geq \min\{\mathcal{E}(\vartheta, \rho, \pi), \mathcal{E}(\vartheta, \check{p}\vartheta, \pi), \mathcal{E}(\rho, \check{p}\rho, \pi)\},$$
$$\mathcal{F}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \leq \min\{\mathcal{F}(\vartheta, \rho, \pi), \mathcal{E}(\vartheta, \check{p}\vartheta, \pi), \mathcal{E}(\rho, \check{p}\rho, \pi)\},$$

and

$$\mathcal{G}(\check{p}\vartheta, \check{p}\rho, \theta\pi) \leq \min\{\mathcal{G}(\vartheta, \rho, \pi), \mathcal{E}(\vartheta, \check{p}\vartheta, \pi), \mathcal{G}(\rho, \check{p}\rho, \pi)\},$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi > 0$. Then, $\check{p}$ has a $\mathcal{UFP}$.

Proof. From 3.1 and Lemma 3.2 the proof can be easily proved □

Definition 3.5. *Let $(\mathcal{P}, \mathcal{Q}, \mathcal{E}, \mathcal{F}, \mathcal{G}, *, \circ, \mathcal{R})$ be a neutrosophic bipolar cone metric space. A map $\check{p}\colon \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ is an NBC(neutrosophic bipolar cone)-contraction if we can find that $0 < \theta < 1$, implies that*

$$\frac{1}{\mathcal{E}(\check{p}\vartheta, \check{p}\rho, \pi)} - 1 \leq \theta\left[\frac{1}{\mathcal{E}(\vartheta, \rho, \pi)} - 1\right] \tag{3.9}$$
$$\mathcal{F}(\check{p}\vartheta, \check{p}\rho, \pi) \leq \theta\mathcal{F}(\vartheta, \rho, \pi), \tag{3.10}$$

and

$$\mathcal{G}(\check{p}\vartheta, \check{p}\rho, \pi) \leq \theta\mathcal{G}(\vartheta, \rho, \pi), \tag{3.11}$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi \in int(\mathcal{R})$.

Now, we prove the theorem for NBC contraction.

Theorem 3.3. *Let $(\mathcal{P}, \mathcal{Q}, \mathcal{E}, \mathcal{F}, \mathcal{G}, *, \circ, \mathcal{R})$ be a complete **neutrosophic bipolar cone metric space** and if*

$$\lim_{\pi\to+\infty} \mathcal{E}(\vartheta, \rho, \pi) = 1, \lim_{\pi\to+\infty} \mathcal{F}(\vartheta, \rho, \pi) = 0 \quad and \quad \lim_{\pi\to+\infty} \mathcal{G}(\vartheta, \rho, \pi) = 0 \tag{3.12}$$

for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi \in int(\mathcal{R})$. Let Let $\check{p}\colon \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ be a map it holds:

(i) $\check{p}(\mathcal{P}) \subseteq \mathcal{P}$ *and* $\check{p}(\mathcal{Q}) \subseteq \mathcal{Q}$;

(ii) $\check{p}$ *is NBC-contraction, for all $\vartheta \in \mathcal{P}$, $\rho \in \mathcal{Q}$ and $\pi \in int(\mathcal{R})$.*

Then, $\check{p}$ has a $\mathcal{UFP}$.

Proof. Let $\vartheta_0 \in \mathcal{P}$ and $\rho_0 \in \mathcal{Q}$ and assume that $\check{p}(\vartheta_{\check{n}}) = \vartheta_{\check{n}+1}$ and $\check{p}(\rho_{\check{n}}) = \rho_{\check{n}+1}$ for all $\check{n} \in \mathbb{N} \cup \{0\}$. Then we get $(\vartheta_{\check{n}}, \mathcal{Q}_{\check{n}})$as a bisequence on neutrosophic bipolar cone metric space

$(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$. By using (3.9), (3.10) and (3.11) for all $\pi \in int(\mathcal{R})$, we deduce

$$\begin{aligned}\frac{1}{\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\pi)}-1 &= \frac{1}{\mathcal{E}(\check{p}\vartheta_{\check{n}-1},\check{p}\rho_{\check{n}-1},\pi)}-1 \leq \theta\left[\frac{1}{\mathcal{E}(\vartheta_{\check{n}-1},\rho_{\check{n}-1},\pi)}\right] \\ &= \frac{\theta}{\mathcal{E}(\vartheta_{\check{n}-1},\rho_{\check{n}-1},\pi)}-\theta \\ \Rightarrow \frac{1}{\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\pi)} &\leq \frac{\theta}{\mathcal{E}(\vartheta_{\check{n}-1},\rho_{\check{n}-1},\pi)}+(1-\theta) \leq \frac{\theta^2}{\mathcal{E}(\vartheta_{\check{n}-2},\rho_{\check{n}-2},\pi)} \\ &\quad +\theta(1-\theta)+(1-\theta)\end{aligned}$$

Continuing like this, we obtain

$$\begin{aligned}\frac{1}{\mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\pi)} &\leq \frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0,\rho_0,\pi)}+\theta^{\check{n}-1}(1-\theta)+\theta^{\check{n}-2}(1-\theta) \\ &\quad +\cdots+\theta(1-\theta)+(1-\theta) \\ &\leq \frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0,\rho_0,\pi)}+(\theta^{\check{n}-1}+\theta^{\check{n}-2}+\cdots+1)(1-\theta) \\ &\leq \frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0,\rho_0,\pi)}+(1-\theta^{\check{n}})\end{aligned}$$

We obtain

$$\frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0,\rho_0,\pi)}+(1-\theta^{\check{n}})} \leq \mathcal{E}(\vartheta_{\check{n}},\rho_{\check{n}},\pi) \tag{3.13}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}},\pi) &= \mathcal{F}(\check{p}\vartheta_{\check{n}-1},\check{p}\rho_{\check{n}-1},\pi) \leq \theta\mathcal{F}(\vartheta_{\check{n}-1},\rho_{\check{n}-1},\pi) = \mathcal{F}(\check{p}\vartheta_{\check{n}-2},\check{p}\rho_{\check{n}-2},\pi) \\ &\leq \theta^2\mathcal{F}(\vartheta_{\check{n}-2},\vartheta_{\rho-2},\pi) \leq \cdots \leq \theta^{\check{n}}\mathcal{F}(\vartheta_0,\rho_0,\pi),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}},\pi) &= \mathcal{G}(\check{p}\vartheta_{\check{n}-1},\check{p}\rho_{\check{n}-1},\pi) \leq \theta\mathcal{G}(\vartheta_{\check{n}-1},\rho_{\check{n}-1},\pi) = \mathcal{G}(\check{p}\vartheta_{\check{n}-2},\check{p}\rho_{\check{n}-2},\pi) \\ &\leq \theta^2\mathcal{G}(\vartheta_{\check{n}-2},\rho_{\check{n}-2},\pi) \leq \cdots \leq \theta^{\check{n}}\mathcal{G}(\vartheta_0,\rho_0,\pi)\end{aligned} \tag{3.15}$$

and

$$\frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_1,\rho_0,\pi)}+(1-\theta^{\check{n}})} \leq \mathcal{E}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi) \tag{3.16}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi) &= \mathcal{F}(\check{p}\vartheta_{\check{n}},\check{p}\rho_{\check{n}-1},\pi) \leq \theta\mathcal{F}(\vartheta_{\check{n}},\rho_{\check{n}-1},\pi) = \mathcal{F}(\check{p}\vartheta_{\check{n}-1},\check{p}\rho_{\check{n}-2},\pi) \\ &\leq \theta^2\mathcal{F}(\vartheta_{\check{n}-1},\vartheta_{\rho-2},\pi) \leq \cdots \leq \theta^{\check{n}}\mathcal{F}(\vartheta_1,\rho_0,\pi),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}+1},\rho_{\check{n}},\pi) &= \mathcal{G}(\check{p}\vartheta_{\check{n}},\check{p}\rho_{\check{n}-1},\pi) \leq \theta\mathcal{G}(\vartheta_{\check{n}},\rho_{\check{n}-1},\pi) = \mathcal{G}(\check{p}\vartheta_{\check{n}-1},\check{p}\rho_{\check{n}-2},\pi) \\ &\leq \theta^2\mathcal{G}(\vartheta_{\check{n}-1},\rho_{\check{n}-2},\pi) \leq \cdots \leq \theta^{\check{n}}\mathcal{G}(\vartheta_1,\rho_0,\pi).\end{aligned} \tag{3.18}$$

Letting $\check{n} < \check{m}$, for $\check{n}, \check{m} \in \mathbb{N}$. Then,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\geq \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{m}}, \frac{\pi}{3})\\ &\vdots\\ &\geq \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) * \cdots * \mathcal{E}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})\\ &* \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) * \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\leq \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{m}}, \frac{\pi}{3})\\ &\vdots\\ &\leq \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \cdots \circ \mathcal{F}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})\\ &\circ \mathcal{F}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{F}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\leq \mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{G}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{G}(\vartheta_{\check{n}+1}, \rho_{\check{m}}, \frac{\pi}{3})\\ &\vdots\\ &\leq \mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{G}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \cdots \circ \mathcal{G}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})\\ &\circ \mathcal{G}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{G}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}}).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\geq \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) * \mathcal{E}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) * \cdots * \mathcal{E}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})\\ &* \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) * \mathcal{E}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}})\\ &\geq \frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3})} + (1-\theta^{\check{n}})} * \frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_1, \rho_0, \frac{\pi}{3})} + (1-\theta^{\check{n}})}\\ &* \cdots * \frac{1}{\frac{\theta^{\check{m}-1}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}-1})}\\ &* \frac{1}{\frac{\theta^{\check{m}-1}}{\mathcal{E}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}-1})} * \frac{1}{\frac{\theta^{\check{m}}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}})},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\leq \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{F}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \cdots \circ \mathcal{F}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}})\\ &\circ \mathcal{F}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{F}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}})\\ &\leq \theta^{\check{n}}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \theta^{\check{n}}\mathcal{F}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \cdots \circ \theta^{\check{m}-1}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}})\\ &\circ \theta^{\check{m}-1}\mathcal{F}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \theta^{\check{m}}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) &\leq \mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \mathcal{G}(\vartheta_{\check{n}+1}, \rho_{\check{n}}, \frac{\pi}{3}) \circ \cdots \circ \mathcal{G}(\vartheta_{\check{m}-1}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \\ &\circ \mathcal{G}(\vartheta_{\check{m}}, \rho_{\check{m}-1}, \frac{\pi}{3^{\check{m}-1}}) \circ \mathcal{G}(\vartheta_{\check{m}}, \rho_{\check{m}}, \frac{\pi}{3^{\check{m}-1}}) \\ &\leq \theta^{\check{n}}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \theta^{\check{n}}\mathcal{G}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \cdots \circ \theta^{\check{m}-1}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \\ &\circ \theta^{\check{m}-1}\mathcal{G}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \theta^{\check{m}}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}).\end{aligned}$$

Which implies that,

$$\begin{aligned}\mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) \geq &\frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3})} + (1-\theta^{\check{n}})} * \frac{1}{\frac{\theta^{\check{n}}}{\mathcal{E}(\vartheta_1, \rho_0, \frac{\pi}{3})} + (1-\theta^{\check{n}})} \\ &* \cdots * \frac{1}{\frac{\theta^{\check{m}-1}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}-1})} \\ &* \frac{1}{\frac{\theta^{\check{m}-1}}{\mathcal{E}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}-1})} * \frac{1}{\frac{\theta^{\check{m}}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}})} + (1-\theta^{\check{m}})},\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) \leq &\theta^{\check{n}}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \theta^{\check{n}}\mathcal{F}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \cdots \circ \theta^{\check{m}-1}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \\ &\circ \theta^{\check{m}-1}\mathcal{F}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \theta^{\check{m}}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) \leq &\theta^{\check{n}}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \theta^{\check{n}}\mathcal{G}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \cdots \circ \theta^{\check{m}-1}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \\ &\circ \theta^{\check{m}-1}\mathcal{G}(\vartheta_1, \rho_0, \frac{\pi}{3^{\check{m}-1}}) \circ \theta^{\check{m}}\mathcal{G}(\vartheta_0, \rho_0, \frac{\pi}{3^{\check{m}-1}}).\end{aligned}$$

Using (3.12), for each case $\check{n}, \check{m} \to +\infty$, we deduce

$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{E}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) = 1 * 1 * \cdots * 1 = 1,$$
$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{F}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) = 0 \circ 0 \circ \cdots \circ 0 = 0$$

and

$$\lim_{\check{n},\check{m}\to+\infty} \mathcal{G}(\vartheta_{\check{n}}, \rho_{\check{m}}, \pi) = 0 \circ 0 \circ \cdots \circ 0 = 0.$$

Which implies that bisequence $(\vartheta_{\check{n}}, \rho_{\check{n}})$ is a Cauchy bisequence. From $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete **neutrosophic bipolar cone metric space**. Then, $\{\vartheta_{\check{n}}\} \to \vartheta$ and $\{\rho_{\check{n}}\} \to \vartheta$, where $\vartheta \in \mathcal{P} \cap \mathcal{Q}$. Using ((v)), ((xi)), ((xvii)) and (3.12), we get

$$\begin{aligned}\mathcal{E}(\vartheta, \check{p}\vartheta, \pi) &\geq \mathcal{E}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right) \\ &= \mathcal{E}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \mathcal{E}\left(\check{p}\vartheta_{\check{n}}, \check{p}\vartheta_{\check{n}}, \frac{\pi}{3}\right) * \mathcal{E}\left(\check{p}\vartheta_{\check{n}}, \check{p}\vartheta, \frac{\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}
&\geq \mathcal{E}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) * \frac{1}{\frac{\theta^{\check{n}+1}}{\mathcal{E}(\vartheta_0, \rho_0, \frac{\pi}{3})} + (1-\theta^{\check{n}+1})} * \mathcal{E}\left(\check{p}\vartheta_{\check{n}}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&\to 1 * 1 * 1 = 1 \quad \text{as} \quad \check{n} \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(\vartheta, \check{p}\vartheta, \pi) &\leq \mathcal{F}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&= \mathcal{F}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{F}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&\leq \mathcal{F}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \theta^{\check{n}+1}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \mathcal{F}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \check{n} \to +\infty
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta, \check{p}\vartheta, \pi) &\leq \mathcal{G}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}+1}, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&= \mathcal{G}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \mathcal{G}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&\leq \mathcal{G}\left(\vartheta, \vartheta_{\check{n}+1}, \frac{\pi}{3}\right) \circ \theta^{\check{n}+1}\mathcal{F}(\vartheta_0, \rho_0, \frac{\pi}{3}) \circ \mathcal{G}\left(\check{p}\vartheta_{\check{n}+1}, \check{p}\vartheta, \frac{\pi}{3}\right)\\
&\to 0 \circ 0 \circ 0 = 0 \quad \text{as} \quad \check{n} \to +\infty.
\end{aligned}$$

Hence, $\check{p}\vartheta = \vartheta$. Let $\check{p}\check{c} = \check{c}$ for some $\check{c} \in \mathcal{P}$, then

$$\begin{aligned}
\frac{1}{\mathcal{E}(\vartheta, \check{c}, \pi)} - 1 &= \frac{1}{\mathcal{E}(\check{p}\vartheta, \check{p}\check{c}, \pi)} - 1\\
&\leq \theta\left[\frac{1}{\mathcal{E}(\vartheta, \check{c}, \pi)} - 1\right] < \frac{1}{\mathcal{E}(\vartheta, \check{c}, \pi)} - 1,
\end{aligned}$$

which is a contradiction.

$$\mathcal{F}(\vartheta, \check{c}, \pi) = \mathcal{F}(\check{p}\vartheta, \check{p}\check{c}, \pi) \leq \theta\mathcal{F}(\vartheta, \check{c}, \pi) < \mathcal{F}(\vartheta, \check{c}, \pi),$$

which is a contradiction and

$$\mathcal{G}(\vartheta, \check{c}, \pi) = \mathcal{G}(\check{p}\vartheta, \check{p}\check{c}, \pi) \leq \theta\mathcal{G}(\vartheta, \check{c}, \pi) < \mathcal{G}(\vartheta, \check{c}, \pi),$$

which is a contradiction. Therefore, $\mathcal{E}(\vartheta, \check{c}, \pi) = 1, \mathcal{F}(\vartheta, \check{c}, \pi) = 0$ and $\mathcal{G}(\vartheta, \check{c}, \pi) = 0$, hence, $\vartheta = \check{c}$. □

Example 3.1. *Let* $\mathcal{A} = \mathbb{R}$, $\mathcal{R} = \{\vartheta \in \mathcal{A} | \vartheta \geq 0\}$, $\mathcal{P} = [0,1]$ *and* $\mathcal{Q} = \{0\} \cup \mathbb{N} - \{1\}$. *Define* $\mathcal{E}, \mathcal{F}, \mathcal{G} \colon \mathcal{P} \times \mathcal{Q} \times int(\mathcal{R}) \to [0,1]$ *as*

$$\begin{aligned}
\mathcal{E}(\vartheta, \rho, \pi) &= \frac{\pi}{\pi + |\vartheta - \rho|},\\
\mathcal{F}(\vartheta, \rho, \pi) &= \frac{|\vartheta - \rho|}{\pi + |\vartheta - \rho|},
\end{aligned}$$

$$\mathcal{F}(\vartheta,\rho,\pi)=\frac{|\vartheta-\rho|}{\pi}.$$

Then, $(\mathcal{P},\mathcal{Q},\mathcal{E},\mathcal{F},\mathcal{G},*,\circ,\mathcal{R})$ is a complete ***neutrosophic bipolar cone metric space*** *with con-t-nm $\breve{u}*\breve{v}=\breve{u}\breve{v}$ and con-t-co-nm $\breve{u}\circ\breve{v}=\max\{\breve{u},\breve{v}\}$.*
Define $\breve{p}\colon\mathcal{P}\cup\mathcal{Q}\to\mathcal{P}\cup\mathcal{Q}$ by

$$\breve{p}(\vartheta)=\begin{cases}\frac{1-3^{-\vartheta}}{4}, & \text{if } \vartheta\in[0,1],\\ 0, & \text{if } \vartheta\in\mathbb{N}-\{1\},\end{cases}$$

for all $\vartheta\in\mathcal{P}\cup\mathcal{Q}$ and take $\theta\in[\frac{1}{2},1)$, then

$$\begin{aligned}\mathcal{E}(\breve{p}\vartheta,\breve{p}\rho,\theta\pi)&=\mathcal{E}\left(\frac{1-3^{-\vartheta}}{4},\frac{1-3^{-\rho}}{4},\theta\pi\right)\\&=\frac{\theta\pi}{\theta\pi+\left|\frac{1-3^{-\vartheta}}{4}-\frac{1-3^{-\rho}}{4}\right|}=\frac{\theta\pi}{\theta\pi+\frac{|3^{-\vartheta}-3^{-\rho}|}{4}}\\&\geq\frac{\theta\pi}{\theta\pi+\frac{|\vartheta-\rho|}{4}}=\frac{4\theta\pi}{4\theta\pi+|\vartheta-\rho|}\geq\frac{\pi}{\pi+|\vartheta-\rho|}=\mathcal{E}(\vartheta,\rho,\pi),\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\breve{p}\vartheta,\breve{p}\rho,\theta\pi)&=\mathcal{F}\left(\frac{1-3^{-\vartheta}}{4},\frac{1-3^{-\rho}}{4},\theta\pi\right)\\&=\frac{\left|\frac{1-3^{-\vartheta}}{4}-\frac{1-3^{-\rho}}{4}\right|}{\theta\pi+\left|\frac{1-3^{-\vartheta}}{4}-\frac{1-3^{-\rho}}{4}\right|}=\frac{\frac{|3^{-\vartheta}-3^{-\rho}|}{4}}{\theta\pi+\frac{|3^{-\vartheta}-3^{-\rho}|}{4}}\\&=\frac{|3^{-\vartheta}-3^{-\rho}|}{4\theta\pi+|3^{-\vartheta}-3^{-\rho}|}\leq\frac{|\vartheta-\rho|}{4\theta\pi+|\vartheta-\rho|}\leq\frac{|\vartheta-\rho|}{\pi+|\vartheta-\rho|}=\mathcal{F}(\vartheta,\rho,\pi)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\breve{p}\vartheta,\breve{p}\rho,\theta\pi)&=\mathcal{G}\left(\frac{1-3^{-\vartheta}}{4},\frac{1-3^{-\rho}}{4},\theta\pi\right)\\&=\frac{\left|\frac{1-3^{-\vartheta}}{4}-\frac{1-3^{-\rho}}{4}\right|}{\theta\pi}=\frac{\frac{|3^{-\vartheta}-3^{-\rho}|}{4}}{\theta\pi}\\&=\frac{|3^{-\vartheta}-3^{-\rho}|}{4\theta\pi}\leq\frac{|\vartheta-\rho|}{4\theta\pi}\leq\frac{|\vartheta-\rho|}{\pi}=\mathcal{G}(\vartheta,\rho,\pi).\end{aligned}$$

Thus, all the axioms of Theorem 3.1 are fulfilled, and zero is the fixed point for $\breve{p}$.

4. Application

Let $\mathcal{A}=\mathbb{R}$, $\mathcal{R}=\{\vartheta\in\mathcal{A}|\vartheta\geq 0\}$, $\mathcal{P}=C([\mathfrak{c},\mathfrak{a}],[0,+\infty))$ is the set of all continuous functions defined on $[\mathfrak{c},\mathfrak{a}]$ with values in the interval $[0,+\infty)$ and $\mathcal{Q}=C([\mathfrak{c},\mathfrak{a}],(-\infty,0])$ is the set of all continuous functions defined on $[\mathfrak{c}, \mathfrak{a}]$ with values in the interval $(-\infty, 0]$. Suppose the integral equation:

$$\vartheta(\omega) = \wedge(\omega) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi \quad \text{for} \quad \omega, \varpi \in [\mathfrak{c}, \mathfrak{a}] \tag{4.1}$$

where $\delta > 0, \wedge(\varpi)$ is a fuzzy function of $\varpi\colon \varpi \in [\mathfrak{c}, \mathfrak{a}]$ and $\mho\colon \mathcal{C}([\mathfrak{c}, \mathfrak{a}] \times \mathbb{R}) \to \mathbb{R}^+$. Define $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ by means of

$$\mathcal{E}(\vartheta(\omega), \rho(\omega), \pi) = \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\pi}{\pi + |\vartheta(\omega) - \rho(\omega)|} \quad \text{for all} \quad \vartheta, \rho \in \mathcal{P} \text{ and } \pi \in int(\mathcal{R}),$$

$$\mathcal{F}(\vartheta(\omega), \rho(\omega), \pi) = 1 - \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\pi}{\pi + |\vartheta(\omega) - \rho(\omega)|} \quad \text{for all} \quad \vartheta, \rho \in \mathcal{P} \text{ and } \pi \in int(\mathcal{R}),$$

and

$$\mathcal{G}(\vartheta(\omega), \rho(\omega), \pi) = \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{|\vartheta(\omega) - \rho(\omega)|}{\pi} \quad \text{for all} \quad \vartheta, \rho \in \mathcal{P} \text{ and } \pi \in int(\mathcal{R}),$$

with con-t-nm and con-t-co-nm define by $\breve{u} * \breve{v} = \breve{u} \cdot \breve{v}$ and $\breve{u} \circ \breve{v} = \max\{\breve{u}, \breve{v}\}$. Then $(\mathcal{P}, \mathcal{E}, \mathcal{F}, \mathcal{G}, *, \circ)$ is a complete **neutrosophic bipolar cone metric space**. Assume $|\mho(\omega, \varpi)\vartheta(\omega) - \mho(\omega, \varpi)\rho(\omega)| \leq |\vartheta(\omega) - \rho(\omega)|$ for $\vartheta \in \mathcal{P}, \rho \in \mathcal{Q}, \theta \in (0, 1)$ and $\forall \omega, \varpi \in [\mathfrak{c}, \mathfrak{a}]$. Consider $\mho(\omega, \varpi)(\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \check{d}\varpi) \leq \theta < 1$. Then, the integral Equation (4.1) has a unique solution.

Proof. Define $\check{p}\colon \mathcal{P} \cup \mathcal{Q} \to \mathcal{P} \cup \mathcal{Q}$ by

$$\check{p}\vartheta(\omega) = \wedge(\omega) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi \quad \text{for all} \quad \omega, \varpi \in [\mathfrak{c}, \mathfrak{a}].$$

For all $\vartheta, \rho \in \mathcal{P} \cup \mathcal{Q}$, we obtain

$$\begin{aligned}
\mathcal{E}(\check{p}\vartheta(\omega), \check{p}\rho(\omega), \theta\pi) &= \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\check{p}\vartheta(\omega) - \check{p}\rho(\omega)|} \\
&= \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + | \wedge(\omega) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi - \wedge(\omega) - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi|} \\
&= \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi|} \\
&= \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\mho(\omega, \varpi)\vartheta(\omega) - \mho(\omega, \varpi)\rho(\omega)|(\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \check{d}\varpi)} \\
&\geq \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\pi}{\pi + |\vartheta(\omega) - \rho(\omega)|} \\
&\geq \mathcal{E}(\vartheta(\omega), \rho(\omega), \pi),
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(\check{p}\vartheta(\omega), \check{p}\rho(\omega), \theta\pi) &= 1 - \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\check{p}\vartheta(\omega) - \check{p}\rho(\omega)|} \\
&= 1 - \sup_{\omega \in [\mathfrak{c}, \mathfrak{a}]} \frac{\theta\pi}{\theta\pi + | \wedge(\omega) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi - \wedge(\omega) - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega, \varpi)\vartheta(\omega)\check{d}\varpi|}
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi|} \\
&= 1 - \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\mho(\omega,\varpi)\vartheta(\omega) - \mho(\omega,\varpi)\rho(\omega)|(\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \check{d}\varpi)} \\
&\leq 1 - \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\pi}{\pi + |\vartheta(\omega) - \rho(\omega)|} \\
&\leq \mathcal{F}(\vartheta(\omega),\rho(\omega),\pi),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{G}(\check{p}\vartheta(\omega),\check{p}\rho(\omega),\theta\pi) &= \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\check{p}\vartheta(\omega) - \check{p}\rho(\omega)|} \\
&= \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\wedge(\omega) + \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi - \wedge(\omega) - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi|} \\
&= \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi - \delta \int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\omega,\varpi)\vartheta(\omega)\check{d}\varpi|} \\
&= \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\theta\pi}{\theta\pi + |\mho(\omega,\varpi)\vartheta(\omega) - \mho(\omega,\varpi)\rho(\omega)|(\delta \int_{\mathfrak{c}}^{\mathfrak{a}} \check{d}\varpi)} \\
&\leq \sup_{\omega\in[\mathfrak{c},\mathfrak{a}]} \frac{\pi}{\pi + |\vartheta(\omega) - \rho(\omega)|} \\
&\leq \mathcal{G}(\vartheta(\omega),\rho(\omega),\pi),
\end{aligned}
$$

Hence, all the axioms of Theorem 3.1 are fulfilled and $\check{p}$ has a $\mathcal{UFP}$. Thus integral Equation (4.1) has a unique solution. □

5. Conclusion

In this paper, we introduced the concept of neutrosophic bipolar cone metric space and proved fixed point theorems. At last, we presented an application on integral equations.

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