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Solving Split Mixed Equilibrium With Multiple Output Sets and Fixed Point of Certain Multi-Valued Mappings

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Abstract. In this paper, we study the split mixed equilibrium problem which includes the equilibrium problem, split convex minimization problem and split equilibrium problem, to mention a few. In addition, we propose a Halpern iterative method for solving split mixed equilibrium problem with multiple output sets and fixed point of a finite family of multi-valued strictly pseudo-contractive mappings in the framework of real Hilbert spaces. We prove a strong convergence theorem without imposing any compactness condition. Lastly, we present some consequences and give applications of our main result to split mixed variational inequality and split convex minimization problems. The result discussed in this article extends and complements many related results in literature.

1. Introduction

Let K be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let $F : K \times K \to \mathbb{R}$ be a bifunction, the Equilibrium Problem (EP) is to find $x \in K$ such that

$$F(x, y) \geq 0, \ \forall \ y \in K. \tag{1.1}$$

The solution set of EP is denoted by EP(F). The EP was first introduced by Fan [20] in 1972, then Blum and Oetlli [9], Noor and Oettli [30], and Abass [1] made significant contributions to this problem. The EP has a great impact in the development of several branches of pure and applied sciences and it provides a natural and unified framework for solving several problems arising in physics, economics, game theory, transportation network and elasticity. The EP can be applied to solve other different mathematical problems such as convex feasibility problem, variational

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inequality problem, minimization problem and fixed point problem to mention a few, the reader may consult ([4], [5], [39]). More so, the EP can be generalize into other optimization problems. For instance, the Mixed Equilibrium Problem (MEP), which is to find $x \in K$ such that

$$F(x,y) + \psi(y) - \psi(x) \geq 0, \ \forall \ y \in K, \tag{1.2}$$

where $\psi : K \to \mathbb{R}$ is a function. In particular, if $\psi = 0$, MEP (1.2) reduces to EP (1.1). We denote by $MEP(F,\psi)$ the solution set of MEP (1.2), for more generalizations of EP, the reader may consult [12]. Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point

$$x^* \in K_1 \text{ such that } Ax^* \in K_2, \tag{1.3}$$

where $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ are nonempty, closed and convex sets. The SFP in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [15] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [8]. Motivated by SFP (1.3) and EP (1.1), Kazmi and Rizvi [25] studied the following Split Equilibrium Problem (SEP): Let K_1 and K_2 be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, $F_1 : K_1 \times K_1 \to \mathbb{R}$ and $F_2 : K_2 \times K_2 \to \mathbb{R}$ be nonlinear bifunctions with $A : H_1 \to H_2$ being a bounded linear operator. The SEP is to find $x^* \in K_1$ such that

$$F_1(x^*, x) \geq 0, \ \forall \ x \in K_1 \tag{1.4}$$

and such that

$$y^* = Ax^* \in K_2 \text{ solves } F_2(y^*, y) \geq 0, \ \forall \ y \in K_2. \tag{1.5}$$

We denote by $SEP(F_1, F_2)$ the solution set of (1.4)-(1.5).

Recently, Suantai et al. [38] introduced the following iterative algorithm to approximate a common elements of the set of solution of SEP (1.4)-(1.5) and fixed point of a nonspreading multi-valued mappings: Given a sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 \in K_1 \text{ arbitrarily}, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n, \ \forall \ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, $r_n \subset (0,\infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and $S : K_1 \to CB(K_1)$ is a $\frac{1}{2}-$ nonspreading multi-valued mappings. The authors proved that under some mild conditions, the sequence $\{x_n\}$ converges weakly to an element of $Fix(S) \cap SEP(F_1, F_2)$, readers can consult [41]. In 2018, Abass et al. [6] considered a viscosity iterative method for approximating a common element of the set of solutions of $SEP(F_1, F_2)$ and fixed point of an infinite family of quasi-nonexpansive multi-valued mappings in the framework of real Hilbert spaces. The authors proved that under some mild conditions, the sequence $\{x_n\}$ converges strongly to an element of $Fix(S) \cap SEP(F_1, F_2)$, readers can as well consult ([2], [3], [7], [11], [25], [38], [44]) for convergence analysis of SEP.

The Split Mixed Equilibrium Problem (SMEP) is known to contain the EP, SEP and MEP. It is to find $x^* \in K_1$ such that

$$F_1(x^*, x) + \psi_1(x) - \psi_1(x^*) \geq 0, \ \forall \ x \in K_1 \tag{1.6}$$

and such that

$$y^* = Ax^* \in K_2 \text{ solves } F_2(y^*, y) + \psi_2(y) - \psi_2(y^*) \geq 0, \ \forall \ y \in K_2, \tag{1.7}$$

where K_1 and K_2 are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $F_1 : K_1 \times K_1 \to \mathbb{R}$ and $F_2 : K_2 \times K_2 \to \mathbb{R}$ are bifunctions satisfying $F_1(x, x) = 0, \ \forall \ x \in K_1$ and $F_2(y, y) = 0, \ \forall \ y \in K_2$, $\psi_1 : K_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : K_2 \to \mathbb{R} \cup \{+\infty\}$ being proper lower semicontinuous and convex functions such that $K_1 \cap dom\psi_1 \neq \emptyset$ and $K_2 \cap dom\psi_2 \neq \emptyset$. We denote by $SMEP(F_1, \psi_1, F_2, \psi_2) := \{x^* \in K_1 : x^* \in MEP(F_1, \psi_1) \text{ and } Ax^* \in MEP(F_2, \psi_2)\}$ the solution set of (1.6)-(1.7).

Recently, Onjai-uea and Phuengrattana [31] introduced the following iterative method for finding a solution of SMEP for λ-hybrid multi-valued mappings. They proved that the sequence generated by their iterative algorithm converges weakly to a common solution of fixed point problem of λ-hybrid multi-valued mappings and SMEP (1.6)-(1.7). Given a sequence $\{x_n\}$ generated by

$$\begin{cases} u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n u_n + (1 - \alpha_n)w_n, \ w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n)z_n, \ z_n \in Sy_n, \ n \in \mathbb{N}, \end{cases}$$

where $T_{r_n}^{F_1}$ and $T_{r_n}^{F_2}$ denotes the resolvents of bifunctions F_1 and F_2, respectively.

In this paper, we introduce the following SMEP with multiple output sets and fixed point of multi-valued strictly pseudo-contractive mappings as follows: find

$$x^* \in \bigcap_{i=1}^{m} Fix(S_i) \cap MEP(F, \psi) \tag{1.8}$$

and

$$y^* = A_j x^* \in \bigcap_{j=1}^{N} MEP(F_j, \psi_j). \tag{1.9}$$

Remark 1.1. (1) *If $S = I$, where I is an identity operator and $j = 1$, then we obtain SMEP* (1.6)-(1.7).
(2) *If (1) above holds and $\psi = 0$, then we obtain SEP* (1.4)-(1.5).

Questions:

(1) We observed that in ([38], [31]) and other problems related to SEP and SMEP, there are always two resolvents $T_{r_n}^{F_1}$ and $T_{r_n}^{F_2}$ respectively. Can we solve SMEP with just one of the aforementioned resolvents?

(2) Can we prove a strong convergence result of (1.8)-(1.9) as this is desirable to the weak convergence obtained in [31, 38].

(3) Is it possible to prove a strong convergence result without imposing the compactness conditions on our class of mappings ? (see [3, 17]).

Inspired by the results of ([6], [31], [38]) and other related results in literature, we propose a Halpern iterative method for approximating a common solution of (1.8)-(1.9). A strong convergence result was proved without imposing any compactness conditions. Consequences and applications of our main results were discussed. The result present in this paper extends and complements some related results in literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$" respectively.
Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : H \to 2^H$ be a multi-valued mapping. A vector $p \in K$ is called a fixed point of T, if $p \in Tp$. We denote the set of fixed point of T by $Fix(T)$. Let $CB(K)$ denote the family of nonempty closed bounded subset of K, the Hausdorff metric on $CB(K)$ is defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \text{ for } A,B \in CB(K),$$

where $d(x,K) = \inf\{||x-y|| : y \in K\}$.
A multi-valued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx,Ty) \le L||x-y||,\ x,y \in C. \tag{2.1}$$

In (2.1), if $L \in (0,1)$, then T is called a contraction while T is called nonexpansive if $L = 1$.
$T : K \to CB(K)$ is said to be

(i) *quasi-nonexpansive*, if $Fix(T) \neq \emptyset$ and

$$\mathcal{H}(Tx,Ty) \le ||x-y||,\ \forall\ x \in C, y \in Fix(T),$$

(ii) *k-strictly pseudocontractive* in the sense of [18] if there exists $k \in (0,1)$ such that $\forall x,y \in K$ and $u \in Tx$ there exists $v \in Ty$ such that

$$\mathcal{H}^2(Tx,Ty) \le ||x-y||^2 + k||x-y-(u-v)||^2.$$

Remark 2.1. *If $k = 0$ in (ii), we have (i).*

The metric projection P_K is a map defined on H onto K which assign to each $x \in H$, the unique point in K, denoted by P_Kx such that

$$||x-P_Kx|| = \inf\{||x-y|| : y \in K\}.$$

It is well known that P_Kx is characterized by the inequality $\langle x - P_Kx, z - P_Kx\rangle \leq 0, \ \forall\ z \in K$ and P_K is a firmly nonexpansive mapping. For more information on metric projections, (see [22]) and the references therein.

For solving mixed equilibrium problems, we assume that the bifunction $F : K \times K \to \mathbb{R}$ satisfies the following conditions:

$(A1)$ $F(x,x) = 0, \forall\ x \in K$;

$(A2)$ $F(x,y) + F(y,x) \leq 0, \forall\ x, y \in K$;

$(A3)$ For all $x, y, w \in K, \lim_{t\downarrow 0} F(tw + (1-t)x, y) \leq F(x,y)$;

(A4) For each $x \in K$, the function $y \mapsto F(x,y)$ is convex and lower semi-continuous;

(A5) For fixed $r > 0$ and $w \in K$, there exists a nonempty compact convex subset V of H and $x \in K \cap V$

such that

$$F(w,x) + \frac{1}{r}\langle y - x, x - w\rangle \geq 0, \forall\ y \in K \setminus V.$$

We now list some important results that we will need in the proof of our main result.

Lemma 2.1. *[35] Let K be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)-(A5), and let $\psi : K \to R$ be a proper lower semi-continuous and convex function such that $K \cap dom\psi \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to K$ as follows:*

$$T_r^F(x) = \left\{ w \in K : F(w,y) + \psi(y) - \psi(w) + \frac{1}{r}\langle y - w, w - x\rangle \geq 0, \forall y \in K \right\}. \tag{2.2}$$

Then

(1) *For each $x \in H, T_r^F(x) \neq \emptyset$;*

(2) *T_r^F is single valued;*

(3) *T_r^F is firmly nonexpansive, that is $\forall x, y \in H$,*

$$||T_r^F x - T_r^F y||^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(4) *$Fix(T_r^F) = MEP(F,\psi)$;*

(5) *$MEP(F,\psi)$ is closed and convex.*

Lemma 2.2. *[16] Let H be a real Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0,1)$, we have*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2$,

(ii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$,

(iii) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$

Lemma 2.3. *[17] Let H be a real Hilbert space. Let $\{x_i, i = 1, \cdots m\} \subset H$. For $\alpha_i \in (0,1), i = 1, \cdots m$ such that $\sum\limits_{i=1}^{m} \alpha_i = 1$, the following identity holds:*

$$||\sum_{i=1}^{m} \alpha_i x_i||^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{i,j=1, i\neq j}^{m} \alpha_i\alpha_j ||x_i - x_j||^2.$$

Lemma 2.4. *[36] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n d_n, n\geq 1.$$

If $\limsup_{k\to\infty} d_{n_k}\leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1}-a_{n_k}\}\geq 0,$$

then, $\lim_{n\to\infty}a_n=0$.

3. Main Result

In this section, we introduce a Halpern iterative algorithm for approximating a solution of split mixed equilibrium problem and fixed point of a finite family for multi-valued pseudo-contractive mappings with multiple output sets and prove its strong convergence theorem in the framework of real Hilbert spaces. We state the following assumptions that are needed in our result.

Assumption 3.1

(1) Let $H, H_j, j=0,1,2,\cdots,N$, be real Hilbert spaces and K, K_j be nonempty, closed and convex subsets of H and H_j respectively. Let $S_i: H\to CB(H)$ be a finite family of multi-valued κ_i-strictly pseudo-contractive mapping with $\kappa_i\in(0,1)$ such that $S_ip=\{p\}$, with $H_0=H$.

(2) Let $F: K\times K\to\mathbb{R}, F_j: K_j\times K_j\to\mathbb{R}$ be bifunctions satisfying Assumption (A1)-(A5) and $\psi: K\to\mathbb{R}\cup\{+\infty\}, \psi_j: K_j\to\mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous and convex functions such that $K\cap dom\psi\neq\emptyset$ and $K_j\cap dom\psi_j\neq\emptyset$, respectively. Suppose that F_j is upper semicontinuous in the first argument and $A_j: H\to H_j$, $j=1,2,\cdots N$ is a bounded linear operator then $\Omega:=\{x^*\in MEP(F,\psi)\cap\bigcap_{i=1}^{m}Fix(S_i): A_jx^*\in\bigcap_{j=1}^{N}MEP(F_j,\psi_j)\}\neq\emptyset$.

Algorithm 3.1. Split Mixed Equilibrium Problem and Multi-valued strictly Pseudo-contractive Mappings with Multiple Output Set.

For $u, x_1\in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n=\sum_{j=1}^{N}\theta_{j,n}\Big[(I^H-\gamma_{j,n}A_j^*(I^{H_j}-T_{r_n}^{F_j})A_jx_n)\Big],\\ y_n=\beta_0u_n+\sum_{i=1}^{m}\beta_iz_n^i,\ z_n^i\in S_iu_n,\\ x_{n+1}=\alpha_nu+(1-\alpha_n)y_n,\ \forall\ n\in\mathbb{N}, \end{cases}\tag{3.1}$$

where $\{\alpha_n\}\subset(0,1), \beta_i\in(\kappa_i,1), i=1,2,\cdots m$, such that $\sum_{i=0}^{m}\beta_i=1$ and $\kappa:=\max\{\kappa_i, i=1,2,\cdots,m\}, \{r_n\}\subset(0,\infty)$ and $\gamma_{j,n}\in(0,\frac{1}{L})$ such that L is the spectral radius of $A_j^*A_j, j=1,2,\cdots m$ and A_j^* is the adjoint of A_j. Assume that the following conditions hold:

(1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(2) $\{\theta_{j,n}\} \subset (0,1)$ *and* $\sum_{j=1}^{N} \theta_{j,n} = 1,$

(3) $0 < \liminf_{n\to\infty} r_n.$
Then, the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* = P_\Omega u$*, where* P_Ω *is the metric projection of* H *onto* Ω.

Remark 3.1. *For clarity, we assume in* (3.1) *that* $H_0 = H, F_0 = F$ *and* $\psi_0 = \psi$.

Proof. We need to show $A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j$ is $\frac{1}{L}-$ ism for all $j = 0, 1, 2, \cdots N$. Using the fact that $T_{r_n}^{F_j}$ is firmly nonexpansive and $I - T_{r_n}^{F_j}$ is 1-ism, we obtain that

$$\begin{aligned}
||A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x - A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j y||^2 &= \langle A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j(x-y), A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j(x-y)\rangle \\
&= \langle (I^{H_j} - T_{r_n}^{F_j})A_j(x-y), A_jA_j^*(I^{H_j} - T_{r_n}^{F_j})A_j(x-y)\rangle \\
&\le L\langle (I^{H_j} - T_{r_n}^{F_j})A_j(x-y), (I^{H_j} - T_{r_n}^{F_j})A_j(x-y)\rangle \\
&= L||(I^{H_j} - T_{r_n}^{F_j})A_j(x-y)||^2 \\
&\le L\langle A_j(x-y), (I^{H_j} - T_{r_n}^{F_j})A_j(x-y)\rangle \\
&= L\langle x-y, A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j - A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j y\rangle,
\end{aligned}$$

for all $x, y \in H$. So, we conclude that $A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j$ is a $\frac{1}{L}-$ ism for all $j = 0, 1, 2, \cdots, N$. More so, since $0 < \gamma_{j,n} < \frac{1}{L}$, we obtain that $I^H - \gamma_{j,n}A_j(I^{H_j} - T_{r_n}^{F_j})A_j$, $j = 0, 1, 2, \cdots N$ is nonexpansive.
Step 1: We show that $\{x_n\}, \{u_n\}$ and $\{y_n\}$ are bounded. Let $v \in \Omega$, then we have that $T_{r_n}^{F_j}(A_j v) = A_j vandv = (I^H - \gamma_{j,n})A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j v$. By applying the nonexpansive property of v defined above, we have that

$$\begin{aligned}
||u_n - v|| &= ||\sum_{j=0}^{N} \theta_{j,n}\Big[(I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n - (I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j v))\Big]|| \\
&\le \sum_{j=0}^{N} \theta_{j,n}||(I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n - (I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j v))|| \\
&\le ||x_n - v||. \qquad (3.2)
\end{aligned}$$

By applying Lemma 2.3 and inequality (3.2), we obtain that

$$\begin{aligned}
||y_n - v||^2 &= \beta_0||u_n - v||^2 + \sum_{i=1}^{m} \beta_i||z_n^i - v||^2 - \sum_{i=1}^{m} \beta_0\beta_i||u_n - z_n^i||^2 \\
&\quad - \sum_{i,j=1, i\ne k}^{m} \beta_i\beta_k||z_n^i - z_n^k||^2 \\
&\le \beta_0||u_n - v||^2 + \sum_{i=1}^{m} \beta_i||z_n^i - v||^2 - \sum_{i=1}^{m} \beta_0\beta_i||u_n - z_n^i||^2
\end{aligned}$$

$$\leq \beta_0\|u_n - v\|^2 + \sum_{i=1}^{m} \beta_i(\mathcal{H}(S_i u_n, S_i v))^2 - \sum_{i=1}^{m} \beta_0\beta_i\|u_n - z_n^i\|^2$$

$$= \|u_n - v\|^2 - \sum_{i=1}^{m} \beta_i(\beta_0 - \kappa)\|u_n - z_n^i\|^2 \tag{3.3}$$

$$\leq \|x_n - v\|^2 - \sum_{i=1}^{m} \beta_i(\beta_0 - \kappa)\|u_n - z_n^i\|^2. \tag{3.4}$$

We obtain from (3.1) and (3.4) that

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n u + (1-\alpha_n)y_n - v\| \\
&\leq \alpha_n\|u - v\| + (1-\alpha_n)\|y_n - v\| \\
&\leq \alpha_n\|u - v\| + (1-\alpha_n)\|x_n - v\| \\
&\leq \max\{\|u - v\|, \|x_n - v\|\} \\
&\vdots \\
&\leq \max\{\|u - v\|, \|x_1 - v\|\}.
\end{aligned}$$

Hence the sequence $\{x_n\}$ is indeed bounded as claimed. Consequently, it follows from (3.2)-(3.4) that the sequence $\{u_n\}$ and $\{y_n\}$ are also bounded.

Step 2: We show that $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Let $v \in \Omega$, we see from (3.1) that

$$\begin{aligned}
\|u_n - v\|^2 &= \|\sum_{j=0}^{N} \theta_{j,n}\Big[I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n\Big] - v\|^2 \\
&\leq \sum_{j=0}^{N} \theta_{j,n}\|(I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n) - v\|^2 \\
&\leq \|x_n - v\|^2 + \gamma_{j,n}^2\|A_j^*(I - T_{r_n}^{F_j})A_j x_n\|^2 + 2\gamma_{j,n}\langle v - x_n, A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n\rangle \\
&\leq \|x_n - v\|^2 + \gamma_{j,n}\langle A_j x_n - T_{r_n}^{F_j}A_j x_n, A_j A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n\rangle \\
&+ 2\gamma_{j,n}\langle A_j(v - x_n), A_j x_n - T_{r_n}^{F_j}A_j x_n\rangle \\
&\leq \|x_n - v\|^2 + L\gamma_{j,n}^2\langle A_j x_n - T_{r_n}^{F_j}A_j x_n, A_j x_n - T_{r_n}^{F_j}A_j x_n\rangle \\
&+ 2\gamma_{j,n}\langle A_j(v - x_n) + (A_j x_n - T_{r_n}^{F_j}A_j x_n) - (A_j x_n - T_{r_n}^{F_j}A_j x_n), A_j x_n - T_{r_n}^{F_j}A_j x_n\rangle \\
&\leq \|x_n - v\|^2 + L\gamma_{j,n}^2\|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2 + 2\gamma_{j,n}\Big[\langle A_j v - T_{r_n}^{F_j}A_j x_n, A_j x_n - T_{r_n}^{F_j}A_j x_n\rangle \\
&- \|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2\Big] \\
&= \|x_n - v\|^2 + L\gamma_{j,n}^2\|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2 + 2\gamma_{j,n}\left[\frac{1}{2}\|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2 - \|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2\right] \\
&\leq \|x_n - v\|^2 + \gamma_{j,n}(L\gamma_{j,n} - 1)\|A_j x_n - T_{r_n}^{F_j}A_j x_n\|^2.
\end{aligned} \tag{3.5}$$

Hence, we have from (3.1) and (3.5) that

$$\|u_n - v\|^2 = \|x_n - v\|^2 + \sum_{j=0}^{N} \theta_{j,n}\gamma_{j,n}(L\gamma_{j,n} - 1)\|A_j x_n - T_{r_n}^{F_j} A_j x_n\|^2. \tag{3.6}$$

From (3.1), we obtain

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \langle \alpha_n u + (1-\alpha_n) y_n - v, x_{n+1} - v\rangle \\ &= (1-\alpha_n)\langle y_n - v, x_{n+1} - v\rangle + \alpha_n \langle u - v, x_{n+1} - v\rangle \\ &\le \frac{(1-\alpha_n)}{2}\Big(\|x_{n+1} - v\|^2 + \|y_n - v\|^2\Big) + \alpha_n\langle u - v, x_{n+1} - v\rangle. \end{aligned}\tag{3.7}$$

This implies from (3.3) and (3.6) that

$$\begin{aligned}\|x_{n+1} - v\|^2 &\le (1-\alpha_n)\|u_n - v\|^2 - (1-\alpha_n)\sum_{i=1}^{m}\beta_i(\beta_0 - \kappa)\|u_n - z_n^i\|^2 + 2\alpha_n\langle u - v, x_{n+1} - v\rangle \\ &= (1-\alpha_n)\|x_n - v\|^2 + (1-\alpha_n)\sum_{j=0}^{N}\theta_{j,n}\gamma_{j,n}(L\gamma_{j,n} - 1)\|A_j x_n - T_{r_n}^{F_j} A_j x_n\|^2 \\ &- (1-\alpha_n)\sum_{i=1}^{m}\beta_i(\beta_0 - \kappa)\|u_n - z_n^i\|^2 + 2\alpha_n\langle u - v, x_{n+1} - v\rangle. \end{aligned}\tag{3.8}$$

Following the approach in (3.7), (3.8) and applying (3.2) and (3.4), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n(2\langle u - x^*, x_{n+1} - x^*\rangle) \\ &= (1-\alpha_n)\|u_n - x^*\|^2 + \alpha_n(2\langle u - x^*, x_{n+1} - x^*\rangle) \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n d_n, \end{aligned}\tag{3.9}$$

where $d_n = 2\langle u - x^*, x_{n+1} - x^*\rangle$. According to Lemma 2.4, to conclude our proof, it sufficies to show that $\limsup_{k\to\infty} d_{n_k} \le 0$ for every subsequence $\{\|x_{n_k} - x^*\|\}$ satisfying the condition

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|\Big) \ge 0. \tag{3.10}$$

To show this, suppose that $\{\|x_{n_k} - x^*\|\}$ is a subsequence of $\{\|x_n - x^*\|\}$ such that (3.10) holds. Then

$$\begin{aligned}\liminf_{k\to\infty}\Big(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2\Big) &= \liminf_{k\to\infty}\Big((\|x_{n_k+1} - x^*\| - \|x_{n_k} - x^*\|)(\|x_{n_k+1} - x^*\| + \|x_{n_k} - x^*\|)\Big) \\ &\ge 0. \end{aligned}\tag{3.11}$$

From (3.8), we obtain that

$$\begin{aligned}&\limsup_{k\to\infty}\Big(-(1-\alpha_{n_k})\sum_{j=0}^{N}\theta_{j,n_k}\gamma_{j,n_k}(L\gamma_{j,n_k} - 1)\|A_j x_{n_k} - T_{r_{n_k}}^{F_j} A_j x_{n_k}\|^2\Big) \\ &\le \limsup_{k\to\infty}\Big((1-\alpha_{n_k})\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2\Big) + \limsup_{k\to\infty}\Big(2\alpha_{n_k}\langle u - x^*, x_{n_k+1} - x^*\rangle\Big)\end{aligned}$$

$$
\begin{aligned}
&\le \limsup_{k\to\infty}\left(\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\right)+\limsup_{k\to\infty}\left(2\alpha_{n_k}\langle u-x^*,x_{n_k+1}-x^*\rangle\right)\\
&=-\liminf_{k\to\infty}\left(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\right)\\
&\le 0. \qquad (3.12)
\end{aligned}
$$

Since $\gamma_{j,n_k}(L\gamma_{j,n_k}-1)<0$, it follows from (3.12) that

$$
\lim_{k\to\infty}\|A_jx_{n_k}-T^{F_j}_{r_{n_k}}A_jx_{n_k}\|=0. \qquad (3.13)
$$

Also, from (3.8), we have that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left((1-\alpha_{n_k})\sum_{i=1}^m\beta_i(\beta_0-\kappa)\|u_{n_k}-z^i_{n_k}\|^2\right)\\
&\le \limsup_{k\to\infty}\left((1-\alpha_{n_k})\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\right)+\limsup_{k\to\infty}\left(2\alpha_{n_k}\langle u-x^*,x_{n_k+1}-x^*\rangle\right)\\
&\le \limsup_{k\to\infty}\left(\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\right)+\limsup_{k\to\infty}\left(2\alpha_{n_k}\langle u-x^*,x_{n_k+1}-x^*\rangle\right)\\
&=-\liminf_{k\to\infty}\left(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\right)\\
&\le 0. \qquad (3.14)
\end{aligned}
$$

Thus, since $\beta_i\in(\kappa,1)$, we obtain that

$$
\lim_{k\to\infty}\|u_{n_k}-z^i_{n_k}\|=0,\ i=1,2,\cdots m. \qquad (3.15)
$$

Since $z^i_{n_k}\in S_iu_{n_k}, i=1,2,\cdots m$, we have $0\le d(u_{n_k},S_iu_{n_k})\le\|u_{n_k}-z^i_{n_k}\|$ and so

$$
\lim_{k\to\infty}d(u_{n_k},S_iu_{n_k})=0. \qquad (3.16)
$$

From (3.1) and (3.15), we obtain the following

$$
\begin{cases}
\lim\limits_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0,\\
\lim\limits_{k\to\infty}\|y_{n_k}-u_{n_k}\|=0,\\
\lim\limits_{k\to\infty}\|x_{n_k+1}-y_{n_k}\|=0,\\
\lim\limits_{k\to\infty}\|y_{n_k}-x_{n_k}\|=0,\\
\lim\limits_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0.
\end{cases} \qquad (3.17)
$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}}\rightharpoonup x^*$. Also, from (3.17) that there exist subsequences $\{u_{n_{k_l}}\}$ of $\{u_{n_k}\}$ and $\{y_{n_{k_l}}\}$ of $\{y_{n_k}\}$ which converge weakly to x^*, respectively. Moreover, by continuity of S_i, $i=1,2,\cdots m$ and (3.16), we obtain that $x^*\in$

$\bigcap_{i=1}^m Fix(S_i)$. Also, since A_j is a bounded linear operator, we have that $A_j x_{n_{k_l}} \rightharpoonup A_j x^*$. Then, it follows from (3.13) that

$$T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}} \rightharpoonup A_j x^*, \text{ as } l \to \infty. \tag{3.18}$$

By the definition of $T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}}$, we have

$$F_j(T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}}, y) + \psi_j(y) - \psi_j(T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}}) + \frac{1}{r_{n_{k_l}}}\langle y - T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}}, T_{r_{n_{k_l}}}^{F_j} A_j x_{n_{k_l}} - A_j x_{n_{k_l}}\rangle \geq 0, \ \forall\, y \in K.$$

Since F_j is upper semicontinuous in the first argument, it implies from (3.18) that

$$F_j(A_j x^*, y) + \psi_j(y) - \psi_j(A_j x^*) \geq 0, \ \forall\, y \in K.$$

This shows that $A_j x^* \in MEP(F_j, \psi_j)$ for $j = 0, 1, 2, \cdots N$. Hence $A_j x^* \in \cap_{j=0}^N MEP(F_j, \psi_j)$, that is $A_j x^* \in \Omega$. Let $x^* = P_\Omega u$, suppose that $\{x_{n_{k_l}}\}$ is a subsequence of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\} \rightharpoonup v \in \Omega$, then we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle u - x^*, x_{n_k} - x^*\rangle &= \lim_{l\to\infty}\langle u - x^*, x_{n_{k_l}} - x^*\rangle \\
&= \langle u - x^*, v - x^*\rangle \\
&\leq 0.
\end{aligned} \tag{3.19}$$

On substituting (3.19) into (3.9), we obtain that $\limsup_{k\to\infty} d_{n_k} \leq 0$. Thus by applying Lemma 2.4 to (3.9), we conclude that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $x^* = P_\Omega u$. □

Corollary 3.1.

Algorithm 3.2. Split Mixed Equilibrium Problem and Multi-valued Nonexpansive Mappings with Multiple Output Set.

For $u, x_1 \in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases}
u_n = \sum\limits_{j=1}^N \theta_{j,n}\Big[(I^H - \gamma_{j,n} A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n)\Big] \\
y_n = \beta_0 u_n + \sum\limits_{i=1}^m \beta_i z_n^i, \ z_n^i \in S_i u_n \\
x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \ \forall\, n \in \mathbb{N},
\end{cases} \tag{3.20}$$

where $\{\alpha_n\} \subset (0,1), \beta_i \in (0,1), i = 1, 2, \cdots m$, such that $\sum\limits_{i=0}^m \beta_i = 1$ and $\{r_n\} \subset (0, \infty)$ and $\gamma_{j,n} \in (0, \frac{1}{L})$ such that L is the spectral radius of $A_j^* A_j$, $j = 1, 2, \cdots m$ and A_j^* is the adjoint of A_j. Assume that the following conditions hold:

(1) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^\infty \alpha_n = \infty$,

(2) $\{\theta_{j,n}\} \subset (0,1)$ *and* $\sum\limits_{j=1}^N \theta_{j,n} = 1$,

(3) $0 < \liminf_{n\to\infty} r_n$.
Then, the sequence $\{x_n\}$ *generated by* (3.20) *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the metric projection of* H *onto* Ω.

Corollary 3.2.

Algorithm 3.3. Weak Convergence of Split Mixed Equilibrium Problem and Multi-valued Nonexpansive Mappings with Multiple Output Set.
For $x_1 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n = \sum_{j=1}^{N} \theta_{j,n}\Big[(I^H - \gamma_{j,n}A_j^*(I^{H_j} - T_{r_n}^{F_j})A_j x_n)\Big] \\ y_n = \beta_0 u_n + \sum_{i=1}^{m} \beta_i z_n^i,\ z_n^i \in S_i u_n \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)y_n,\ \forall\, n \in \mathbb{N}, \end{cases} \tag{3.21}$$

where $\{\alpha_n\} \subset (0,1), \beta_i \in (0,1), i = 1,2,\cdots m$, *such that* $\sum_{i=0}^{m} \beta_i = 1$ *and* $\{r_n\} \subset (0,\infty)$ *and* $\gamma_{j,n} \in (0,\frac{1}{L})$ *such that L is the spectral radius of* $A_j^*A_j$, $j = 1,2,\cdots m$ *and* A_j^* *is the adjoint of* A_j. *Assume that the following conditions hold:*

(1) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(2) $\{\theta_{j,n}\} \subset (0,1)$ *and* $\sum_{j=1}^{N} \theta_{j,n} = 1$,

(3) $0 < \liminf_{n\to\infty} r_n$.
Then, the sequence $\{x_n\}$ *generated by* (3.21) *converges weakly to* $x^* \in \Omega$.

4. Applications

4.1. Application to split mixed variational inequality problem

Let K be a nonempty, closed and convex subset of a real Hilbert space H. Given a nonlinear mapping $B : K \to H$, then the Variational Inequality Problem (VIP) is to find $x^* \in K$ such that

$$\langle Bx^*, z - x^*\rangle \ge 0,\ \forall z\ \in K. \tag{4.1}$$

We will denote the solution set of VIP by $VI(B,C)$.
A mapping $B : K \to H$ is said to be $\nu-$ inverse strongly monotone mapping if there exists a constant $\nu > 0$ such that $\langle Bx - By, x - y\rangle \ge \nu\|Bx - By\|^2$ for any $x, y \in K$.
Let H_1 and H_2 be real Hilbert spaces, K_1 and K_2 be nonempty, closed and convex subsets of H_1 and H_2 respectively. Then the Split Mixed Variational Inequality Problem $(SMVIP)$ is to find $x^* \in K_1$ such that

$$\langle B_1(x^*), x - x^*\rangle + \psi_1(x) - \psi_1(x^*) \ge 0\ \forall\ x \in K_1, \tag{4.2}$$

and

$$y^* = Ax^* \in K_2 \quad \text{solves} \qquad \langle B_2(y^*), y - y^* \rangle + \psi_2(y) - \psi_2(y^*) \geq 0 \ \forall \ y \in K_2, \tag{4.3}$$

where $B_1 : H_1 \to H_1,\ B_2 : H_2 \to H_2$ are nonlinear mappings and $A : H_1 \to H_2$ is a bounded linear operator. We denote by $SMVIP(B_1, \psi_1, B_2, \psi_2)$ the solution set of problem (4.2)-(4.3).

Setting $F_1(x, y) = \langle B_1 x, y - x \rangle$ and $F_2(x, y) = \langle B_2 x, y - x \rangle$, it is easy to see that F and G satisfy condition (A1)-(A5) since $B_1 : K_1 \to H_1$ and $B_2 : K_2 \to H_2$ are $\nu-$ inverse strongly monotone mappings. Then from (3.1), the following result holds.

Algorithm 4.1. Weak Convergence of Split Mixed Variational Inequality Problem and Multi-valued Nonexpansive Mappings with Multiple Output Set. *For $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = \sum\limits_{j=1}^{N} \theta_{j,n} \Big[(I^H - \gamma_{j,n} A_j^* (I^{H_j} - P_{K_j}(I - \lambda B_j)) A_j x_n) \Big] \\ y_n = \beta_0 u_n + \sum\limits_{i=1}^{m} \beta_i z_n^i,\ z_n^i \in S_i u_n \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) y_n,\ \forall\ n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{\alpha_n\} \subset (0,1), \beta_i \in (0,1), i = 1, 2, \cdots m$, such that $\sum\limits_{i=0}^{m} \beta_i = 1$ and $\gamma_{j,n} \in (0, \frac{1}{L})$ such that L is the spectral radius of $A_j^* A_j, j = 1, 2, \cdots m$ and A_j^* is the adjoint of A_j. Assume that the following conditions hold:

(1) $0 < \liminf\limits_{n \to \infty} \alpha_n \leq \limsup\limits_{n \to \infty} \alpha_n < 1,$

(2) $\{\theta_{j,n}\} \subset (0,1)$ *and* $\sum\limits_{j=1}^{N} \theta_{j,n} = 1.$

Then, the sequence $\{x_n\}$ generated by (4.4) *converges weakly to $x^* \in SMVIP(B, \psi, B_j, \psi_j) \cap \bigcap\limits_{i=1}^{m} Fix(S_i)$.*

Remark 4.1. *In a real Hilbert space, if B is μ-inverse strongly monotone with $\lambda \in (0, 2\mu)$, then $P_K(I - \lambda B)$ is firmly nonexpansive. Thus, $P_K(I - \lambda B)$ is nonexpansive. In addition, $x^* \in VI(K, B)$ if and only if $x^* = P_K(I - \lambda B)(x^*), \forall\ \lambda > 0$, see [13].*

4.2. Application to split convex minimization problem

One of the most important problems in optimization theory and non-linear analysis is the problem of approximating solutions of Minimization Problem (MP) which is defined as follows: Find $x \in H$ such that

$$\psi(x) = \min_{y \in H} \psi(y), \tag{4.5}$$

where $\psi : H \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function. Recall that a mapping ψ is convex if

$$\psi(\lambda x + (1 - \lambda) y) \leq \lambda \psi(x) + (1 - \lambda) \psi(y)\ \forall x, y \in H,\ \lambda \in (0,1).$$

ψ is proper, if $D(\psi) := \{x \in H : \psi(x) < +\infty\} \neq \emptyset$, where $D(\psi)$ denotes the domain of ψ. The mapping $\psi : D(\psi) \to (-\infty, \infty]$ is lower semi-continuous at a point $x \in D(\psi)$ if

$$\psi(x) \leq \liminf_{n\to\infty} \psi(x_n), \tag{4.6}$$

for each sequence $\{x_n\}$ in $D(\psi)$ such that $\lim\limits_{n\to\infty} x_n = x$; ψ is said to be lower semicontinuous on $D(\psi)$ if it is lower semi-continuous at any point in $D(\psi)$. An example of a convex and lower semicontinuous function is the indicator function $\delta_K : H \to \mathbb{R}$ of a nonempty closed and convex subset K of H defined by

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

For any $\lambda > 0$, the resolvent (or Moreau-Yosida approximation) of ψ in H is defined as (see [43]).

$$J_\lambda^\psi(x) = \arg\min_{y\in H} \left[\psi(y) + \frac{1}{2\lambda}\|y - x\|^2\right].$$

It is generally known that J_λ^ψ is well-defined and firmly nonexpansive for all $\lambda > 0$. Hence, J_λ^ψ is nonexpansive for all $\lambda > 0$. For simplicity, we shall write J_λ^ψ for the resolvent of a proper, convex and lower semi-continuous mapping ψ. Furthermore, we denote the solution set of problem (4.5) by $\operatorname{argmin}_{y\in H} \psi(y)$. It is also known that $Fix(J_\lambda^\psi)$ coincides with $\operatorname{argmin}_{y\in H} \psi(y)$.

If $F_1 = 0$ and $F_2 = 0$ in SMEP (1.6)-(1.7), then SMEP (1.6)-(1.7) reduces to the following Split Convex Minimization Problem (SCMP): find $x^* \in K_1$ such that

$$\psi_1(x) \geq \psi_1(x^*), \ \forall \, x \in K_1, \tag{4.7}$$

and

$$y^* = Ax^* \in K_2 \text{ solves } \psi_2(y) \geq \psi_2(y^*), \ \forall \, y \in K_2. \tag{4.8}$$

We denote by $SCMP(\psi_1, \psi_2)$ the solution set of (4.7)-(4.8). We therefore obtained from (3.1) a new iterative method to solve SCMP with multiple output set as follows:

Algorithm 4.2. Weak Convergence of Split Convex Minimization Problem and Multi-valued Nonexpansive Mappings with Multiple Output Set.

For $x_1 \in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = \sum\limits_{j=0}^{N} \theta_{j,n}\Big[(I^H - \gamma_{j,n}A_j^*(I^{H_j} - J_\lambda^{\psi_j})A_j x_n)\Big] \\ y_n = \beta_0 u_n + \sum\limits_{i=1}^{m} \beta_i z_n^i, \ z_n^i \in S_i u_n \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n) y_n, \ \forall \, n \in \mathbb{N}, \end{cases} \tag{4.9}$$

where $\{\alpha_n\} \subset (0,1), \beta_i \in (0,1), i = 1, 2, \cdots m,$ such that $\sum\limits_{i=0}^{m} \beta_i = 1$ and $\{r_n\} \subset (0, \infty)$ and $\gamma_{j,n} \in (0, \frac{1}{L})$ such that L is the spectral radius of $A_j^* A_j$, $j = 1, 2, \cdots m$ and A_j^* is the adjoint of A_j. Assume that the following conditions hold:

(1) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(2) $\{\theta_{j,n}\} \subset (0,1)$ *and* $\sum_{j=0}^{N} \theta_{j,n} = 1$.

Then, the sequence $\{x_n\}$ *generated by* (4.9) *converges weakly to* $x^* \in SCMP(\psi, \psi_j) \cap \bigcap_{i=1}^{m} Fix(S_i)$.

5. Conclusion

In this article, we give an affirmative answer to the questions raised in the introductory part of this article by introducing a Halpern iterative method for approximating a common solution of (1.8)-(1.9). More so, we proved a strong convergence result for approximating the solutions of the aforementioned problems. We also present some applications of our results to SCMP with multiple output sets and SMVIP with multiple output sets. Our results complements and extends the results of ([6], [31], [38]) and many related results in literature.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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