International Journal of Analysis and Applications

Digital Transformation in Taiwan's Insurance Industries for MABAC Technology Based on Circular Modified Fuzzy Choquet Frank Network Data Envelopment Analysis

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Abstract: Fuzzy set theory has significant and dominant applications in Taiwan's insurance industry, especially in fields involving decision-making, uncertainty, and risk assessment. Providing the complexity and problems in assessing factors, for instance, natural disaster risks, customer creditworthiness, or health conditions, traditional binary logic often falls short. Taiwanese insurers have adopted fuzzy logic systems to enhance fraud detection, premium pricing, and privilege evaluations by catching the indistinctness characteristic in human ruling and imperfect data. The Taiwan insurance industry is a dynamic and spirited module of the commercial sector, contributing meaningfully to risk management and economic stability. For this, we study to propose an assessment of the proficiency of insurance enterprises using Network Data Envelopment Analysis. Toward this end, the frank operational laws for circular Pythagorean fuzzy (CPF) uncertainty are applied. Moreover, the CPF Choquet Frank averaging (CPFCFA) operator and CPF Choquet Frank geometric (CPFCFG) operator with three dominant properties for each operator have been studied. The study deliberates the multi-attributive border approximation area comparison (MABAC) model and verifies it with the help of numerical examples. This study enhances the industry's efficiency to offer adapted insurance products and handle risks precisely, aligning with Taiwan's push toward intelligent financial services and digital transformation. In the following, we establish the decision-making performance for assessing the proficiency of insurance enterprises using the network data envelopment analysis (NDEA) technique. Finally, we examine the ranking values of offered representations to compare them with the ranking values of prevailing models to show the capability and efficacy of the originated approaches.

Received Jul. 15, 2025

2020 Mathematics Subject Classification. 91B06.

Key words and phrases. Choquet Frank aggregation operators; circular Pythagorean fuzzy network information; decision-making problems; MABAC model; Taiwan insurance industry.

https://doi.org/10.28924/2291-8639-23-2025-245

ISSN: 2291-8639

1. Introduction

Taiwan's insurance industry, which represents a global market or global platform for businessmen, is experiencing a thoughtful digital transformation theme to enhance capability, regulation responsiveness, customer service, and product innovation. This technique contains the adoption of technologies, for instance, blockchain for secure information sharing, mobile applications for customer engagement, big data analytics for risk modeling, and artificial intelligence-driven underwriting, and cloud computing for operations scalability. NDEA gives a meaningful and well-structured technique to design the multi-stage efficiency of insurance firms by selecting their internal structure. NDEA is a modified and general form of traditional Data Envelopment Analysis (DEA) [1] that integrates a network structure of shape to address the validity and proficiency of decision-making units with interdependent procedures [2]. It is very famous, especially in sectors where operations involve various stages, for instance, healthcare, insurance, and manufacturing [3]. The decision-making model is also very famous for discovering your required decision or results from the collection of information [4, 5]. Further, we know that vagueness and convolution are part of life, and because of this reason, various decision-makers have failed to cope with it. For instance, before 1965, each expert has just two types of information like zero and one, nether more; for example, when we talk about the intelligent peoples, so we have different types of range or scale, if the people have done their master, we will include in intelligent (assigned 1), but the people have not educated, then we will include in the not intelligent (assigned 0), but what about those peoples, they have just failed completed their Bachler or they will be completing their master after studying their one or two more course, to cope with this types of problems, the classical information has been failed. For this, Zadeh [6] designed the fuzzy sets (FSs) model. FSs are a mathematical model that modifies crisp set theory to cope with the concept of partial membership. In the occurrence of traditional sets, an element either contains or does not, but in the case of FSs, we have a partial function or partial degree with zero or one. Every information in FSs has a truth function ranging from zero to one. FS theory is used in numerous fields, including control systems, image processing, neural language processing, and decision-making procedures in uncertain environments.

Intuitionistic fuzzy sets (IFSs) are a reform of FSs that incorporate the model of uncertainty in a massive, nuanced way. Developed by Atanassov [7, 8] in the 1980s, IFSs describe a third dimension to the truth function, falsity function, and refusal function. Each element in IFSs is denoted and defined in the form, for instance, the representation of the positive or truth function is $\mu_Z: \mathcal{X} \rightarrow [0,1]$ and the representation of the negative or falsity function is $\eta_Z: \mathcal{X} \rightarrow [0,1]$ with $0 \leq$

$\mu_Z + \eta_Z \leq 1$. Further, the refusal function is defined and defined by: $\pi_Z = 1 - (\mu_Z + \eta_Z)$, where $\mu_Z, \eta_Z \in [0, 1]$. IFSs were particularly useful in various areas, such as decision-making models where data is incomplete or uncertain. Further, Pythagorean fuzzy sets (PFSs) [9] are a reformed version of FSs and IFSs that incorporate the model of uncertainty in a massive, nuanced way. The structure of PFSs and IFSs is the same; they also describe a third dimension to the truth function, falsity function, and refusal function, and each element in PFSs is denoted and defined in the form, for instance, the representation of the positive or truth function is $\mu_Z: \mathcal{X} \to [0,1]$ and the representation of the negative or falsity function is $\eta_Z: \mathcal{X} \to [0,1]$ with $0 \leq (\mu_Z)^2 + (\eta_Z)^2 \leq 1$. Further, the refusal function is defined and defined by: $\pi_Z = \left(1 - ((\mu_Z)^2 + (\eta_Z)^2)\right)^{\frac{1}{2}}$, where $\mu_Z, \eta_Z \in [0, 1]$. PFSs are particularly useful in various areas, such as decision-making models where data is incomplete or uncertain, data mining, machine learning, and game theory.

To broaden the notion of IFSs, Atanassov [10] expanded the notion and originated the circular IFSs (CIFSs) in 2020. A CIFS represents a circle with a radius r that centers the truth and falsity function, where the representation of the truth function is $\mu_Z: \mathcal{X} \to [0,1]$ and the representation of the negative or falsity function is $\eta_Z: \mathcal{X} \to [0,1]$ with $0 \leq \mu_Z + \eta_Z \leq 1$. Further, the refusal function is defined and defined by: $\pi_Z = 1 - (\mu_Z + \eta_Z)$, where $\mu_Z, \eta_Z \in [0, 1]$ with radius functions such as $\xi_Z: \mathcal{X} \to [0,1]$. Additionally, in 2022, Bozyigit et al. [11] identified the perfect technique of circular PFSs (CPFSs), because of ambiguity and complications that are part of genuine life problems, because the structure of CPFSs and CIFSs is the same, they also describe a fourth dimension to the truth function, falsity function, refusal function, and radius function, and each element in CPFSs is denoted and defined in the form, for instance, the representation of the positive or truth function is $\mu_Z: \mathcal{X} \to [0,1]$ and the representation of the negative or falsity function is $\eta_Z: \mathcal{X} \to [0,1]$ with $0 \leq (\mu_Z)^2 + (\eta_Z)^2 \leq 1$. Further, the refusal function is defined and defined by: $\pi_Z = \left(1 - ((\mu_Z)^2 + (\eta_Z)^2)\right)^{\frac{1}{2}}$, where $\mu_Z, \eta_Z \in [0, 1]$ with radius functions such as $\xi_Z: \mathcal{X} \to [0,1]$. CPFSs are particularly useful in various areas, such as decision-making models where data is incomplete or uncertain, data mining, machine learning, and game theory.

In 2015, Pamucar and Cirovic [12] diagnosed the MABAC technique for classical information as a perfect model for coping with vague and complex data. Furthermore, Klement et al. [13] offered a new version of the book based on triangular norms for the unit interval. These norms can help us in the construction of the aggregation, but they contain many limitations. For this, Frank [14, 15] initiated new norms, called Frank t-norm (FTN) and Frank t-conorm (FTCN) for the unit interval, where the simple norms are a part of Frank norms. In 1953, Choquet [16] familiarized the model of the Choquet integral and fuzzy measures for unit intervals. Moreover, Xu [17] diagnosed the Choquet integral for weighted IFSs. Tan and Chen [18] derived the intuitionistic fuzzy Choquet integral for decision support systems. Tan and Chen [19] exposed the Choquet integral for induced IFSs. Xu and Yager [20] designed the geometric operators for IFSs. Yang et al. [21] derived the Frank operators for IFSs. Zhang et al. [22] presented the Frank power operators for IFSs. Xing et al. [23] invented the Choquet Frank operators for PFSs. Ali and Yang [24] described the Hamacher operators for CPFSs. EDAS model and improved Dombi operators for CPFSs were invented by Garg et al. [25]. Ali et al. [26] evaluated the Aczel-Alsina power operators for circular Pythagorean fuzzy linguistic sets. Jiang et al. [27] designed the Bonferroni mean operators and EDAS model for CPFSs. Verma [28] presented the MABAC model based on order-alpha divergence measures for IFSs. Jia et al. [29] designed the extended MABAC model for intuitionistic fuzzy rough sets. Zhao et al. [30] explored the intuitionistic fuzzy MABAC information and its applications. Petchimuthu et al. [31] discussed the modified power operators for Yager norms based on generalized FSs. Zhang and Gao [32] described the TODIM technique with interpretable decision-making fuzzy models. Shahin et al. [33] exposed the renewable energy source analysis based on the fuzzy MARCOS technique. Jameel et al. [34] invented the sustainable development model for renewable energy and energy prioritized techniques. Sarfarz and Gul [35] designed the Hamacher operators for the evaluation of the medical college projects. Saqlain [36] presented the bibliometric analysis with fuzzy decision-making models. Bhowmik et al. [37] discussed the modified fuzzy TOPDIM approach with Schweizer-Sklar power green energy source. Kang et al. [38] developed the hydraulic converters in tidal stream turbines with comprehensive distance-based ranking techniques. Mishra and Rani [39] initiated the blockchain network with prioritized decision-making models for fuzzy information. Sandra et al. [40] derived the smart decision technique and sustainable management model for a modified fuzzy model. Li and Rong [41] designed the emergency logistics outsourcing suppliers with hybrid modified fuzzy models. Li et al. [42] invented the divergence measures and post-flood assessment for complex modified fuzzy information. Konur Bilgen et al. [43] studied the SWARA and Q-ROF-EDAS technique for the shipyard industry. Finally, we concluded that the technique of CPFSs is very reliable, but to date, no one can derive any kind of information based on it. Therefore, we object to estimating the Choquet-frank operators and the MABAC model based on CPFSs will be proposed. The major influence of this article is listed below:

1) To establish the operational laws based on Frank norms for CPF uncertainty with various dominant results.

2) To study the CPFCFA operator and the CPFCFG operator with three dominant properties for each operator.
3) To assume the proposed models, we goal to discuss the MABAC model and also verify it with the help of numerical examples.
4) To establish the decision-making performance for assessing the proficiency of insurance enterprises using the NDEA approach.
5) To procedure the ranking values of proposed models for comparing them with the ranking values of existing models to show the capability and efficacy of the originated approaches.

This article is arranged in the following ways: in Section 2, we discussed the information on CPFSs, secondly, we briefly discussed the idea of Frank norms and their related cases, and last, we reviewed the model of Choquet integral. In Section 3, we present the operational laws based on Frank norms for CPF uncertainty with various dominant results. In Section 4, we studied the CPFCFA operator and the CPFCFG operator with three dominant properties for each operator. In Section 5, to assume the proposed models, we goal to discuss the MABAC model and also verify it with the help of numerical examples. In Section 6, we established the decision-making performance for assessing the proficiency of insurance enterprises using the NDEA approach. In Section 7, we evaluated the ranking values of proposed models by comparing them with the ranking values of existing models to show the capability and efficacy of the proposed approaches. Some final data are discussed in Section 8.

2. Preliminaries

This section is divided into three major sub-sections. First, we discussed the information on CPFSs; secondly, we briefly discussed the idea of Frank norms and their related cases, and last, we reviewed the model of Choquet integral.

2.1. CPFSs: Circular Pythagorean Fuzzy Sets [11]

In this subsection, we reviewed the old model of CPFSs and their fundamental laws.

Definition 1: Consider $\mathcal{X}$ to be a fixed ordinary set. The model of CPFS Z based on $\mathcal{X}$ is illustrated and defined in the following form such as

$$Z = \{\langle o, \mu_Z, \eta_Z, \xi_Z \rangle | o \in \mathcal{X}\}$$

The representation of the positive function and negative function with radius is as follows such as $\mu_Z, \eta_Z: \mathcal{X} \to [0,1]$ and $\xi_Z: \mathcal{X} \to [0,1]$, such as $\mu_Z, \eta_Z, \xi_Z \in [0,1]$. The prominent technique of CPFS is as follows such as $0 \leq (\mu_Z)^2 + (\eta_Z)^2 \leq 1$, where the mathematical form of the neutral function is defined as: $\pi_Z = \sqrt{1-(\mu_Z)^2-(\eta_Z)^2}$. For simplicity, the triplet $\langle \mu_Z, \eta_Z, \xi_Z \rangle$ represents the CPF numbers (CPFNs), such as $Z = \langle \mu_Z, \eta_Z, \xi_Z \rangle$.

Definition 2: Let $Z = \langle \mu_Z, \eta_Z, \xi_Z \rangle$ be any CPFN. Then

$$\dot{\mathrm{S}}(Z) = \begin{cases} \sqrt{((\mu_Z)^2-(\eta_Z)^2)*(\xi_Z)^2} & \mu_Z \geq \eta_Z \\ -\sqrt{((\eta_Z)^2-(\mu_Z)^2)*(\xi_Z)^2} & \mu_Z \leq \eta_Z \end{cases}$$

$$\hbar(Z) = \sqrt{((\mu_Z)^2+(\eta_Z)^2)*(\xi_Z)^2}$$

Called the score value and accuracy value, such as $\dot{\mathrm{S}}(Z) \in [-1,1]$, $\hbar(Z) \in [0,1]$. Further, we consider any CPFNs, $Z_1 = \langle \mu_{Z1}, \eta_{Z1}, \xi_{Z1} \rangle$ and $Z_2 = \langle \mu_{Z2}, \eta_{Z2}, \xi_{Z2} \rangle$, thus when $\dot{\mathrm{S}}(Z_1) > \dot{\mathrm{S}}(Z_2)$, thus $Z_1 > Z_1$; when $\dot{\mathrm{S}}(Z_1) = \dot{\mathrm{S}}(Z_2)$, thus when $\hbar(Z_1) > \hbar(Z_2)$, then $Z_1 > Z_2$; and when $\hbar(Z_1) = \hbar(Z_2)$, then $Z_1 = Z_1$.

Definition 3: Consider any three CPFNs $Z = \langle \mu_Z, \eta_Z, \xi_Z \rangle$, $Z_1 = \langle \mu_{Z1}, \eta_{Z1}, \xi_{Z1} \rangle$, and $Z_2 = \langle \mu_{Z2}, \eta_{Z2}, \xi_{Z2} \rangle$ with parameter $\psi \geq 1$, thus

$$(Z_1 \oplus_{\mathcal{F}} Z_2)_{TN} = \left(\sqrt{\mu_{\boldsymbol{\mathcal{P}1}}{}^2+\mu_{Z2}{}^2-\mu_{Z1}{}^2*\mu_{Z2}{}^2}, \eta_{Z1}*\eta_{Z2}, \sqrt{\xi_{Z1}{}^2+\xi_{Z2}{}^2-\xi_{Z1}{}^2*\xi_{Z2}{}^2}\right)$$

$$(Z_1 \oplus_{\mathcal{F}} Z_2)_{TCN} = \left(\sqrt{\mu_{Z1}{}^2+\mu_{Z2}{}^2-\mu_{Z1}{}^2*\mu_{Z2}{}^2}, \eta_{Z1}*\eta_{Z2}, \xi_{Z1}*\xi_{Z2}\right)$$

$$(Z_1 \otimes_{\mathcal{F}} Z_2)_{TN} = \left(\mu_{Z1}*\mu_{Z2}, \sqrt{\eta_{Z1}{}^2+\eta_{Z2}{}^2-\eta_{Z1}{}^2*\eta_{Z2}{}^2}, \xi_{Z1}*\xi_{Z2}\right)$$

$$(Z_1 \otimes_{\mathcal{F}} Z_2)_{TCN} = \left(\mu_{Z1}*\mu_{Z2}, \sqrt{\eta_{Z1}{}^2+\eta_{Z2}{}^2-\eta_{Z1}{}^2*\eta_{Z2}{}^2}, \sqrt{\xi_{Z1}{}^2+\xi^2-\xi_{Z1}{}^2*\xi_{Z2}{}^2}\right)$$

$$(\psi Z)_{TN} = \left(\sqrt{1-(1-\mu_Z{}^2)^{\psi}}, \eta_{\mathfrak{p}}{}^{\psi}, \sqrt{1-\left(1-\xi_Z{}^2\right)^{\psi}}\right)$$

$$(\psi Z)_{TCN} = \left(\sqrt{1-(1-\mu_Z{}^2)^{\psi}}, \eta_Z{}^{\psi}, \xi_Z{}^{\psi}\right)$$

$$\left(Z^{\psi}\right)_{TN} = \left(\mu_Z{}^{\psi}, \sqrt{1-(1-\eta_Z{}^2)^{\psi}}, \xi_Z{}^{\psi}\right)$$

$$\left(Z^{\psi}\right)_{TCN} = \left(\mu_Z{}^{\psi}, \sqrt{1-(1-\eta_Z{}^2)^{\psi}}, \sqrt{1-\left(1-\xi_Z{}^2\right)^{\psi}}\right)$$

2.2. Frank t-norm and t-conorm [14, 15]

In this subsection, we briefly reviewed the model of FTN and FTCN based on any two $o, \Psi \in [0,1]$ with parameters $\mathrm{F}ɟ \in (1, +\infty)$, such as

$$\overline{\mathrm{T}}_{\mathfrak{G}}(o, \Psi) = \mathrm{Log}_{\mathrm{F}ɟ}\left(1+\frac{(\mathrm{F}ɟ^{o}-1)(\mathrm{F}ɟ^{\Psi}-1)}{\mathrm{F}ɟ-1}\right), \forall o, \Psi \in [0,1], \mathrm{F}ɟ \in (1,+\infty)$$

$$\dot{\mathrm{S}}_{\mathfrak{G}}(o, \Psi) = 1-\mathrm{Log}_{\mathrm{F}ɟ}\left(1+\frac{(\mathrm{F}ɟ^{1-o}-1)(\mathrm{F}ɟ^{1-\Psi}-1)}{\mathrm{F}ɟ-10}\right), \forall o, \Psi \in [0,1], \mathrm{F}ɟ \in (1,+\infty)$$

With some necessary properties, for both norms, such as where $\overline{\mathrm{T}}_{\mathfrak{G}}(o, \Psi)$ used a t-conorm, if they satisfy the following properties, such as: (i) $\overline{\mathrm{T}}_{\mathfrak{G}}(1,1) = 1$, $\overline{\mathrm{T}}_{\mathfrak{G}}(o,0) = \overline{\mathrm{T}}_{\mathfrak{G}}(0,o) = o$; (ii) when $o_1 \leq o_2$

and $\Psi_1 \leq \Psi_2$, thus $\overline{\mathrm{T}}_{\mathfrak{G}}(o_1, o_2) \leq \overline{\mathrm{T}}_{\mathfrak{G}}(\Psi_1, \Psi_2)$; (iii) $\overline{\mathrm{T}}_{\mathfrak{G}}(o_1, o_2) = \overline{\mathrm{T}}_{\mathfrak{G}}(o_2, o_1)$; and (iv) $\overline{\mathrm{T}}_{\mathfrak{G}}(o_1, \overline{\mathrm{T}}_{\mathfrak{G}}(o_2, o_3)) = \overline{\mathrm{T}}_{\mathfrak{G}}(\overline{\mathrm{T}}_{\mathfrak{G}}(o_1, o_2), o_3)$. Similarly, where $\dot{\mathrm{S}}_{\mathfrak{G}}(o, \Psi)$ used is a t-norm, if they satisfy the following properties, such as: (i) $\dot{\mathrm{S}}_{\mathfrak{G}}(0,0) = 0$, $\dot{\mathrm{S}}_{\mathfrak{G}}(o, 1) = o$; (ii) when $o_1 \leq o_2$ and $\Psi_1 \leq \Psi_2$, thus $\dot{\mathrm{S}}_{\mathfrak{G}}(o_1, o_2) \leq \dot{\mathrm{S}}_{\mathfrak{G}}(\Psi_1, \Psi_2)$; (iii) $\dot{\mathrm{S}}_{\mathfrak{G}}(o_1, o_2) = \dot{\mathrm{S}}_{\mathfrak{G}}(o_2, o_1)$; and (iv) $\dot{\mathrm{S}}_{\mathfrak{G}}(o_1, \dot{\mathrm{S}}_{\mathfrak{G}}(o_2, o_3)) = \dot{\mathrm{S}}_{\mathfrak{G}}(\dot{\mathrm{S}}_{\mathfrak{G}}(o_1, o_2), o_3)$. Further, we stated some special parts of the Frank norms based on different values of parameter, such as: when Fꟷ $\rightarrow 1$, thus the basic idea of Frank norms will be reduced into algebraic norms, such as $\overline{\mathrm{T}}_{\mathfrak{G}}(o, \Psi) \rightarrow o\,\Psi$ and $\dot{\mathrm{S}}_{\mathfrak{G}}(o, \Psi) \rightarrow o + \Psi - o\Psi$, when Fꟷ $\rightarrow \infty$, thus the basic idea of Frank norms will be reduced into Lukasiewicz norms, such as $\overline{\mathrm{T}}_{\mathfrak{G}}(o, \Psi) \rightarrow max(0, o + \Psi - 1)$ and $S\mathfrak{G}\,(x, y) \rightarrow min(o + \Psi, 1)$.

2.3. Choquet Integral Operator [16, 17]

In this subsection, we revised the technique of fuzzy measure, the Choquet integral operator, and their related ideas.

Definition 4: The model of fuzzy measure based on fixed set $\mathcal{X}$ is a collection of mappings, such as $\Delta\nabla : \Gamma(o) \rightarrow [0,1]$, with two important properties, such as: (i) $\Delta\nabla(\phi) = 0$, $\Delta\nabla(\mathcal{X}) = 1$, called boundary condition, (ii) $\partial, \mathcal{B} \in \mathcal{X}$, and $\partial \subseteq \mathcal{B}$, then $\Delta\nabla(\partial) \leq \Delta\nabla(\mathcal{B})$, called monotonicity. Further, we discussed the model of $\rho -$ fuzzy measure, such as:

$$\Delta\nabla(\partial \cup \mathcal{B}) = \Delta\nabla(\partial) + \Delta\nabla(\mathcal{B}) + \rho\Delta\nabla(\partial)\Delta\nabla(\mathcal{B})$$

Further, when $\rho = 0$, thus we have

$$\Delta\nabla(\partial \cup \mathcal{B}) = \Delta\nabla(\partial) + \Delta\nabla(\mathcal{B})$$

Additionally, when all the values in $\mathcal{X}$ are independent, thus

$$\Delta\nabla(\partial) = \sum_{o_\tau \in \partial} \Delta\nabla(o_\tau)$$

When all values in $\mathcal{X}$ are finite, thus

$$\Delta\nabla(\partial) = \Delta\nabla\left(\bigcup_{\tau=1}^{\text{Å}} o_\tau\right) = \begin{cases} \dfrac{1}{\rho}\left[\prod_{\tau=1}^{\text{Å}}\left(1 + \psi\Delta\nabla(o_\tau)\right) - 1\right] & \rho \neq 0 \\ \sum_{o_\tau \in \partial} \Delta\nabla(o_\tau) & \rho = 0 \end{cases}$$

Moreover, $o_\tau \cap o_{\jmath} = \phi$, for $\tau, \jmath = 1,2 \ldots \text{Å}$, and $\tau \neq \jmath$, when $\rho \succ 0$, thus $\Delta\nabla(\partial \cup \mathcal{B}) \succ \Delta\nabla(\partial) + \Delta\nabla(\mathcal{B})$, when $-1 \leq \rho \prec 0$, thus $\Delta\nabla(\partial \cup \mathcal{B}) \prec \Delta\nabla(\partial) + \Delta\nabla(\mathcal{B})$.

Definition 5: A real-valued mapping g based on a fixed set $\mathcal{X}$ with fuzzy measure $\Delta\nabla$, thus

$$\int g\, d\Delta\nabla = \sum_{\tau=1}^{\text{Å}} \left[\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right] g_{\emptyset(\tau)}$$

Called the discrete Choquet integral of g concerning $\Delta\nabla$, where the representation of the permutation is as follows, such as $\emptyset(\tau)$ of $(1,2\cdots n)$ with $g_{\emptyset(1)} \geq g_{\emptyset(2)} \geq \cdots \geq g_{\emptyset(\text{Å})}$, and $\partial_{\emptyset(0)} = \phi$, $\partial_{\emptyset(\tau)} = \{\mathcal{G}_{\emptyset(1)}, \dots \mathcal{G}_{\emptyset(\tau)}\}$.

3. Frank Operational Laws for CPFNs

In this section, we develop a novel model of frank operational laws based on the collection of CPFNs. Further, we also simplify some major properties for the proposed operational laws.

Definition 6: Consider any two CPFNs $Z_1 = \langle \mu_{Z1}, \eta_{Z1}, \xi_{Z1} \rangle$, and $Z_2 = \langle \mu_{Z2}, \eta_{Z2}, \xi_{Z2} \rangle$. The frank operational laws are described in the following form such as

$$(Z_1 \oplus_{\mathcal{F}} Z_2)_{TN} = \left(\begin{array}{c} \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\mu_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\mu_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\eta_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\eta_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \\ \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\xi_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\xi_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)} \end{array} \right)$$

$$(Z_1 \oplus_{\mathcal{F}} Z_2)_{TCN} = \left(\begin{array}{c} \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\mu_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\mu_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\eta_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\eta_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \\ \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\xi_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\xi_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)} \end{array} \right)$$

$$(Z_1 \otimes_{\mathcal{F}} Z_2)_{TN} = \left(\begin{array}{c} \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\mu_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\mu_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\eta_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\eta_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \\ \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\xi_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\xi_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)} \end{array} \right)$$

$$(Z_1 \otimes_{\mathcal{F}} Z_2)_{TCN} = \left(\begin{array}{c} \sqrt{\text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{\mu_{Z1}^2} - 1\right)\left(\text{F}\daleth^{\mu_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\eta_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\eta_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)}, \\ \sqrt{1 - \text{Log}_{\text{F}\daleth}\left(1 + \frac{\left(\text{F}\daleth^{1-\xi_{Z1}^2} - 1\right)\left(\text{F}\daleth^{1-\xi_{Z2}^2} - 1\right)}{\text{F}\daleth - 1}\right)} \end{array} \right)$$

Theorem 1: Consider a CPFN $\mathrm{Z} = \langle \mu_{\mathrm{Z}}, \eta_{\mathrm{Z}}, \xi_{\mathrm{Z}} \rangle$, then the multiplication operation $\text{Å}.\mathcal{F}\mathrm{Z}$ is described in the following form, such as

$$(\text{Å}.Z)_{TN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\eta_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\xi_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)} \end{array} \right)$$

$$(\text{Å}.Z)_{TCN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\eta_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\xi_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)} \end{array} \right)$$

Further, the term Å represents a positive integer, such as $\text{Å}.\mathrm{Z} = \overbrace{Z \oplus Z \oplus \ldots \oplus Z}^{\text{Å}}$.

The Proof of Theorem 1 is discussed in Appendix A.

Theorem 2: Consider any CPFN $\mathrm{Z} = \langle \mu_{\mathrm{Z}}, \eta_{\mathrm{Z}}, \xi_{\mathrm{Z}} \rangle$, then the power operation $\mathrm{Z}^{\wedge\text{Å}}$ is described in the following form, such as

$$(\mathrm{Z}^{\wedge\text{Å}})_{\mathrm{TN}} = \left(\sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\eta_{Z_1}{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)} \right)$$

$$(\mathrm{Z}^{\wedge\text{Å}})_{\mathrm{TCN}} = \left(\sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\eta_{Z_1}{}^2} - 1\right)^{\text{Å}}}{(\text{FℲ} - 1)^{\text{Å}-1}}\right)}, \right)$$

where $\mathrm{Z}^{\wedge\text{Å}} = \overbrace{Z \otimes Z \otimes \ldots \otimes Z}^{\text{Å}}$. Hence, based on the above information in Theorem 1 and Theorem 2, we have

$$(\sigma.Z)_{TN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\mu_Z{}^2} - 1\right)^{\sigma}}{(\text{FℲ} - 1)^{\sigma-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{\eta_Z{}^2} - 1\right)^{\sigma}}{(\text{FℲ} - 1)^{\sigma-1}}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{FℲ}}\left(1 + \frac{\left(\text{FℲ}^{1-\xi_Z{}^2} - 1\right)^{\sigma}}{(\text{FℲ} - 1)^{\sigma-1}}\right)} \end{array} \right)$$

$$(\sigma.Z)_{TCN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{1-\mu_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{\xi_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)$$

$$\left(\mathrm{Z}^{\wedge\text{Å}}\right)_{TN}=\left(\begin{array}{c}\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{\mu_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{1-\eta_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{\xi_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)$$

$$\left(\mathrm{Z}^{\wedge\text{Å}}\right)_{TCN}=\left(\begin{array}{c}\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{\mu_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{1-\eta_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{\left(\mathrm{Fꟸ}^{1-\xi_Z{}^2}-1\right)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)$$

Further, we described some special cases of the above-proposed information, called $\sigma.Z$ and $\mathrm{Z}^{\wedge\sigma}$, such as

1) When $\mathrm{Z}=\langle\mu_{\mathrm{Z}},\eta_{\mathrm{Z}},\xi_{\mathrm{Z}}\rangle=(1,0,1)$, thus

$$\sigma.Z=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1-1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{0}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1-1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)=(1,0,1)$$

$$\mathrm{Z}^{\wedge\sigma}=\left(\begin{array}{c}\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)=(1,0,1)$$

That is $\sigma.(1,0,1)=(1,0,1)$, and $(1,0,1)^{\wedge\sigma}=(1,0,1)$.

2) When $\mathrm{Z}=\langle\mu_{\mathrm{Z}},\eta_{\mathrm{Z}},\xi_{\mathrm{Z}}\rangle=(1,0,1)$, thus

$$\sigma.Z=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1-0}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\sqrt{\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\mathrm{Fꟸ}}\left(1+\frac{(\mathrm{Fꟸ}^{1-1}-1)^{\sigma}}{(\mathrm{Fꟸ}-1)^{\sigma-1}}\right)}\end{array}\right)=(0,1,0)$$

$$Z^{\wedge\sigma} = \left(\begin{array}{c}\sqrt{\text{Log}_{\text{FℲ}}\left(1+\frac{(\text{FℲ}^{0}-1)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}, \sqrt{1-\text{Log}_{\text{FℲ}}\left(1+\frac{(\text{FℲ}^{0}-1)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}, \\ \sqrt{\text{Log}_{\text{FℲ}}\left(1+\frac{(\text{FℲ}^{0}-1)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}\end{array}\right) = (0,1,0)$$

That is $\sigma.(0,1,0) = (0,1,0)$, and $(0,1,0)^{\wedge\sigma} = (0,1,0)$.

3) When $\text{FℲ} \to 1$, thus

$$\lim_{\text{FℲ}\to 1} \sigma.Z = \lim_{\text{FℲ}\to 1}\left(\begin{array}{c}\sqrt{1-\text{Log}_{\text{FℲ}}\left(1+\frac{\left(\text{FℲ}^{1-\mu_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}, \sqrt{\text{Log}_{\text{FℲ}}\left(1+\frac{\left(\text{FℲ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}, \\ \sqrt{1-\text{Log}_{\text{FℲ}}\left(1+\frac{\left(\text{FℲ}^{1-\xi_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}\end{array}\right)$$

By using the technique of equivalent infinitesimal replacement $\ln(1+o) \sim o(o \succ 0)$, Logarithmic transform, such as

$$\lim_{\text{FℲ}\to 1}\left(\sqrt{\text{Log}_{\text{FℲ}}\left(1+\frac{\left(\text{FℲ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}\right) = \lim_{\text{FℲ}\to 1}\left(\sqrt{\frac{\ln\left(1+\frac{\left(\text{FℲ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}{\ln \text{FℲ}}}\right)$$

$$= \lim_{\text{FℲ}\to 1}\left(\sqrt{\frac{\left(\text{FℲ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}\ln \text{FℲ}}}\right)$$

To consider the Taylor expansion and $\ln \text{FℲ} \succ 0$, we have

$$\text{FℲ}^{\eta_Z{}^2} = 1 + \eta_Z{}^2 \ln \text{FℲ} + \frac{\eta_Z{}^2}{2}(\ln \text{FℲ})^2 + \cdots = 1 + \eta_Z{}^2 \ln \text{FℲ} + \mathcal{O}(\ln \text{FℲ})$$
$$\Rightarrow \vartheta^{\eta_Z{}^2} - 1 = \eta_Z{}^2 \ln \vartheta + \mathcal{O}(\ln \vartheta)$$

Then

$$\lim_{\text{FℲ}\to 1}\left(\sqrt{\frac{\left(\text{FℲ}^{\eta_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}\ln \text{FℲ}}}\right) = \lim_{\text{FℲ}\to 1}\left(\sqrt{\frac{(\eta_Z{}^2 \ln \text{FℲ})^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}\ln \text{FℲ}}}\right) = \lim_{\text{FℲ}\to 1}\left(\sqrt{\frac{(\eta_Z{}^2)^{\sigma}(\ln \text{FℲ})^{\sigma-1}}{(\text{FℲ}-1)^{\sigma-1}}}\right) = (\eta_Z)^{\sigma}$$

Using the same procedure, we have

$$\lim_{\text{FℲ}\to 1}\left(\sqrt{1-\text{Log}_{\text{FℲ}}\left(1+\frac{\left(\text{FℲ}^{1-\mu_Z{}^2}-1\right)^{\sigma}}{(\text{FℲ}-1)^{\sigma-1}}\right)}\right) = \sqrt{1-(1-\mu_Z{}^2)^{\sigma}}$$

Then,

$$\lim_{\text{FℲ}\to 1} \sigma.Z = \left(\sqrt{1-(1-\mu_Z{}^2)^{\sigma}}, (\eta_Z)^{\sigma}, \sqrt{1-\left(1-\xi_Z{}^2\right)^{\sigma}}\right)$$

Similarly,

$$\lim_{\text{Fꟾ}\to 1} Z^{\wedge\sigma} = \left((\mu_Z)^{\sigma}, \sqrt{1-(1-\eta_Z{}^2)^{\sigma}}, (\xi_Z)^{\sigma}\right)$$

That is

$$\sigma.(0,1,0) \longrightarrow \left(\sqrt{1-(1-\mu_Z{}^2)^{\sigma}}, (\eta_Z)^{\sigma}, \sqrt{1-\left(1-\xi_Z{}^2\right)^{\sigma}}\right)$$

and

$$(0,1,0)^{\wedge\sigma} \longrightarrow \left(\sqrt{1-(1-\mu_Z{}^2)^{\sigma}}, (\eta_Z)^{\sigma}, \sqrt{1-\left(1-\xi_Z{}^2\right)^{\sigma}}\right), \text{Fꟾ} \longrightarrow 1.$$

Theorem 3: Consider any three CPFNs $Z = \langle \mu_Z, \eta_Z, \xi_Z \rangle$, $Z_1 = \langle \mu_{Z1}, \eta_{Z1}, \xi_{Z1} \rangle$, and $Z_2 = \langle \mu_{Z2}, \eta_{Z2}, \xi_{Z2} \rangle$, then

1. $Z_1 \oplus_{\mathcal{F}} Z_2 = Z_2 \oplus_{\mathcal{F}} Z_1$.
2. $Z_1 \otimes_{\mathcal{F}} Z_2 = Z_2 \otimes_{\mathcal{F}} Z_1$.
3. $\sigma(Z_1 \oplus_{\mathcal{F}} Z_2) = \sigma Z_2 \oplus_{\mathcal{F}} \sigma Z_1, \sigma \geq 0$.
4. $\sigma_1 Z \oplus_{\mathcal{F}} \sigma_2 Z = (\sigma_1 + \sigma_2)Z, \sigma_1, \sigma_2 \geq 0$.
5. $Z^{\sigma_1} \otimes_{\mathcal{F}} Z^{\sigma_2} = Z^{(\sigma_1+\sigma_2)}, \sigma_1, \sigma_2 \geq 0$.
6. $(Z_1{}^{\sigma} \otimes_{\mathcal{F}} Z_2{}^{\sigma}) = (Z_2 \otimes_{\mathcal{F}} Z_1)^{\sigma}, \sigma \geq 0$.

Proof: Omitted.

The information in Theorem 3 is the same for both t-norm and t-conorm. Further, we derive the model of Choquet Frank operators based on the above information for CPFNs.

4. Choquet Frank Aggregation Operators for CPFNs

In this section, we justify the model of the CPFCFA operator and the CPFCFG operator based on the collection of CPFNs. Further, we also discuss some major properties and their related results to enhance the worth of the proposed theory. Consider the family of CPFNs $Z_\tau = \langle \mu_Z, \eta_Z, \xi_Z \rangle (\tau = 1, 2, \dots, \text{Å})$ with permutation $\emptyset(\tau)$ of $(1, 2, \dots, \text{Å})$ and $Z_{\emptyset(1)} \geq Z_{\emptyset(1)} \geq \cdots \geq Z_{\emptyset(\text{Å})}$, where $\mathcal{G}_{\emptyset(\tau)}$ is the attribute corresponding to $Z_{\emptyset(\tau)}$, $\partial_{\emptyset(\tau)} = \phi$, $\partial_{\emptyset(\tau)} = \{\mathcal{G}_{\emptyset(1)}, \dots \mathcal{G}_{\emptyset(\tau)}\}$.

Definition 7: The model of the CPFCFA operator is illustrated and defined in the following form such as

$$\mathcal{F}(\mathcal{C}_1)\int Zd\Delta\nabla = CPFCFA(Z_1, Z_2, \dots, Z_{\text{Å}})_{TN} = \oplus_{\mathcal{F}}{}_{\mathcal{F}}{}_{\tau=1}^{\text{Å}} \left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right) Z_{\emptyset(\tau)}$$

$$\mathcal{F}(\mathcal{C}_1)\int Zd\Delta\nabla = CPFCFA(Z_1, Z_2, \dots, Z_{\text{Å}})_{TCN} = \oplus_{\mathcal{F}}{}_{\mathcal{F}}{}_{\tau=1}^{\text{Å}} \left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right) Z_{\emptyset(\tau)}$$

Theorem 4: Prove that the aggregated values of the proposed operators are again a CPFN, such as

$$CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TN} = \left(\begin{array}{c} \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{array} \right)$$

$$CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TCN} = \left(\begin{array}{c} \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{array} \right)$$

The Proof of Theorem 4 is discussed in Appendix B.

Theorem 5: Prove that the following inequality holds for both FTN and FTCN, such as

$$CPFCFA(\psi.\mathcal{F}Z_1, \psi.\mathcal{F}Z_2, \ldots, \psi.\mathcal{F}Z_{\text{Å}})_{TN} = \psi.\mathcal{F}\; CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TN}$$
$$CPFCFA(\psi.\mathcal{F}Z_1, \psi.\mathcal{F}Z_2, \ldots, \psi.\mathcal{F}Z_{\text{Å}})_{TCN} = \psi.\mathcal{F}\; CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TCN}$$

The Proof of Theorem 5 is discussed in Appendix C.

Theorem 6: Prove that the following inequality holds such as

$$CPFCFA(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{\text{Å}} \oplus_{\mathcal{F}} Z)_{TN} = CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TN} \oplus_{\mathcal{F}} Z$$
$$CPFCFA(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{\text{Å}} \oplus_{\mathcal{F}} Z)_{TCN} = CPFCFA(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{\text{Å}} \oplus_{\mathcal{F}} Z)_{TCN} \oplus_{\mathcal{F}} Z$$

The Proof of Theorem 6 is discussed in Appendix D.

Theorem 7: Prove that the following inequality holds such as

$$CPFCFA(\psi.Z_1 \oplus_{\mathcal{F}} \mathrm{Z}\psi, \ldots, \psi.Z_{\text{Å}} \oplus_{\mathcal{F}} \mathrm{Z})_{TN} = \psi.CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TN} \oplus_{\mathcal{F}} \mathrm{Z}$$
$$CPFCFA(\psi.Z_1 \oplus_{\mathcal{F}} \mathrm{Z}\psi, \ldots, \psi.Z_{\text{Å}} \oplus_{\mathcal{F}} \mathrm{Z})_{TCN} = \psi.CPFCFA(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TCN} \oplus_{\mathcal{F}} \mathrm{Z}$$

Theorem 8: Prove that the following inequality holds such as

$$CPFCFA\left(Z_{\mathfrak{G}_1} \oplus_{\mathcal{F}} Z_{\eta_1}, \ldots, Z_{\mathfrak{G}_{\text{Å}}} \oplus_{\mathcal{F}} Z_{\eta_{\text{Å}}}\right)_{TN} = CPFCFA\left(Z_{\mathfrak{G}_1}, \ldots, Z_{\mathfrak{G}_{\text{Å}}}\right)_{TN} \oplus_{\mathcal{F}} CPFCFA\left(Z_{\eta_1}, \ldots, Z_{\eta_{\text{Å}}}\right)_{TN}$$
$$CPFCFA\left(Z_{\mathfrak{G}_1} \oplus_{\mathcal{F}} Z_{\eta_1}, \ldots, Z_{\mathfrak{G}_{\text{Å}}} \oplus_{\mathcal{F}} Z_{\eta_{\text{Å}}}\right)_{TCN} = CPFCFA\left(Z_{\mathfrak{G}_1}, \ldots, Z_{\mathfrak{G}_{\text{Å}}}\right)_{TCN} \oplus_{\mathcal{F}} CPFCFA\left(Z_{\eta_1}, \ldots, Z_{\eta_{\text{Å}}}\right)_{TCN}$$

The Proof of Theorem 8 is discussed in Appendix E.

Further, we described the fundamental properties of the proposed theory, called idempotency, monotonicity, and boundedness.

Theorem 9: When $Z_{\tau} = Z = \langle \mu_{Z}, \eta_{Z}, \xi_{Z} \rangle (\tau = 1, 2, \dots, Å)$, thus

$$CPFCFA(Z_1, Z_2, \dots, Z_{Å})_{TN} = Z$$
$$CPFCFA(Z_1, Z_2, \dots, Z_{Å})_{TCN} = Z$$

Theorem 10: When $\mu_{Z\mathbb{G}_{\tau}} \leq \mu_{Z\eta_{\tau}}$ and $\eta_{Z\mathbb{G}_{\tau}} \geq \eta_{Z\eta_{\tau}} \forall \tau$, thus

$$CPFCFA\left(Z_{\mathbb{G}_1}, \dots, Z_{\mathbb{G}_{Å}}\right)_{TN} \leq CPFCFA\left(Z_{\eta_1}, \dots, Z_{\eta_{Å}}\right)_{TN}$$
$$CPFCFA\left(Z_{\mathbb{G}_1}, \dots, Z_{\mathbb{G}_{Å}}\right)_{TCN} \leq CPFCFA\left(Z_{\eta_1}, \dots, Z_{\eta_{Å}}\right)_{TCN}$$

Theorem 11: When $Z^{+} = \left(\max_{\tau}(\mu_{Z\tau}), \min_{\tau}(\eta_{Z\tau}), \max_{\tau}(\xi_{Z\tau})\right)$ and $Z^{-} = \left(\min_{\tau}(\mu_{Z\tau}), \max_{\tau}(\eta_{Z\tau}), \min_{\tau}(\xi_{Z\tau})\right)$, thus

$$Z^{-} \leq CPFCFA(Z_1, Z_2, \dots, Z_{Å})_{TN} \leq Z^{+}$$
$$Z^{-} \leq CPFCFA(Z_1, Z_2, \dots, Z_{Å})_{TCN} \leq Z^{+}$$

Further, we describe some dominant and particular cases of the initiated operators for different values of parameters.

Theorem 12: Prove that the following information is held for different values of parameters, such as

1) When $\vartheta \to 1$, thus

$$\left(\lim_{\vartheta \to 1} CPFCFA\right)_{TN} = \left(\begin{array}{c} \sqrt{1 - \prod_{\tau=1}^{Å}\left(1 - \mu_{Z\emptyset(\tau)}{}^{2}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}, \prod_{\tau=1}^{Å}\left(\eta_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}, \\ \sqrt{1 - \prod_{\tau=1}^{Å}\left(1 - \xi_{Z\emptyset(\tau)}{}^{2}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}} \end{array}\right)$$

$$\left(\lim_{\vartheta \to 1} CPFCFA\right)_{TCN} = \left(\begin{array}{c} \sqrt{1 - \prod_{\tau=1}^{Å}\left(1 - \mu_{Z\emptyset(\tau)}{}^{2}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}, \prod_{\tau=1}^{Å}\left(\eta_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}, \\ \prod_{\tau=1}^{Å}\left(\xi_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})} \end{array}\right)$$

Called the CPF Choquet averaging operator (CPFCA).

2) When $\vartheta \to 1$, $\Delta\nabla_{\tau} = \Delta\nabla\left(\partial_{\sigma(\tau)}\right) - \Delta\nabla\left(\partial_{\sigma(\tau-1)}\right)$, then the proposed theory is converted for the CPF weighted averaging (CPFWA) operator.

3) When $\vartheta \to 1$, $\omega_\tau = \Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)})$, and $\Delta\nabla(\partial) = \sum_{\tau=1}^{|\partial|} \omega_\tau$, where $\omega = (\omega_1, \omega_2, \dots, \omega_{\Delta\nabla})^{\mathbb{T}}$, $\omega_{j} \in [0,1]$, $\sum_{j=1}^{Å} \omega_{j} = 1$, then the proposed theory is converted for the CPF-ordered weighted averaging (CPFOWA) operator.

4) When $\vartheta \to +\infty$, thus

$$\left(\lim_{\vartheta\to+\infty} CPFCFA\right)_{TN} = \begin{pmatrix} \sqrt{1-\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\mu_{Z\emptyset(\tau)}{}^2}, \sqrt{\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\eta_{Z\emptyset(\tau)}{}^2}, \\ \sqrt{1-\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\xi_{Z\emptyset(\tau)}{}^2} \end{pmatrix}$$

$$\left(\lim_{\vartheta\to+\infty} CPFCFA\right)_{TCN} = \begin{pmatrix} \sqrt{1-\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\mu_{Z\emptyset(\tau)}{}^2}, \sqrt{\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\eta_{Z\emptyset(\tau)}{}^2}, \\ \sqrt{1-\sum_{\tau=1}^{Å}\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)\xi_{Z\emptyset(\tau)}{}^2} \end{pmatrix}$$

Called a traditional arithmetic weighted average operator.

The Proof of Theorem 12 is discussed in Appendix F.

Definition 8: The model of the CPFCFG operator is illustrated and defined in the following form such as

$$\mathcal{F}(\mathcal{C}_1)\int Z d\Delta\nabla = CPFCFG(Z_1, Z_2, \dots, Z_{Å})_{TN} = \otimes_{\mathcal{F}\mathcal{F}}{}_{\tau=1}^{Å}\left(Z_{\emptyset(\tau)}\right)^{\left(\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}$$

$$\mathcal{F}(\mathcal{C}_1)\int Z d\Delta\nabla = CPFCFG(Z_1, Z_2, \dots, Z_{Å})_{TCN} = \otimes_{\mathcal{F}\mathcal{F}}{}_{\tau=1}^{Å}\left(Z_{\emptyset(\tau)}\right)^{\left(\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}$$

Theorem 13: Prove that the aggregated values of the proposed operators are again a CPFN, such as

$$CPFCFG(Z_1, Z_2, \dots, Z_{Å})_{TN} = \begin{pmatrix} \sqrt{\mathrm{Log}_{\mathrm{F}Ɉ}\left(1+\prod_{\tau=1}^{Å}\left(\mathrm{F}Ɉ^{\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1-\mathrm{Log}_{\mathrm{F}Ɉ}\left(1+\prod_{\tau=1}^{Å}\left(\mathrm{F}Ɉ^{1-\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\mathrm{F}Ɉ}\left(1+\prod_{\tau=1}^{Å}\left(\mathrm{F}Ɉ^{\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{pmatrix}$$

$$CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TCN} = \left(\begin{array}{c} \sqrt{\mathrm{Log}_{\mathrm{Fꟻ}}\left(1 + \prod_{\tau=1}^{Å}\left(\mathrm{Fꟻ}^{\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{Fꟻ}}\left(1 + \prod_{\tau=1}^{Å}\left(\mathrm{Fꟻ}^{1-\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{Fꟻ}}\left(1 + \prod_{\tau=1}^{Å}\left(\mathrm{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)} \end{array}\right)$$

Theorem 14: Prove that the following inequality holds for both FTN and FTCN, such as

$$CPFCFG(\psi.Z_1, \psi.Z_2, \ldots, \psi.Z_{Å})_{TN} = \psi.CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TN}$$

$$CPFCFG(\psi.Z_1, \psi.Z_2, \ldots, \psi.Z_{Å})_{TCN} = \psi.CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TCN}$$

Theorem 15: Prove that the following inequality holds, such as

$$CPFCFG(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{Å} \oplus_{\mathcal{F}} Z)_{TN} = CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TN} \oplus_{\mathcal{F}} Z$$

$$CPFCFG(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{Å} \oplus_{\mathcal{F}} Z)_{TCN} = CPFCFG(Z_1 \oplus_{\mathcal{F}} Z, \ldots, Z_{Å} \oplus_{\mathcal{F}} Z)_{TCN} \oplus_{\mathcal{F}} Z$$

Theorem 16: Prove that the following inequality holds, such as

$$CPFCFG(\psi.Z_1 \oplus_{\mathcal{F}} \mathrm{Z}\psi, \ldots, \psi.Z_{Å} \oplus_{\mathcal{F}} \mathrm{Z})_{TN} = \psi.CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TN} \oplus_{\mathcal{F}} \mathrm{Z}$$

$$CPFCFG(\psi.Z_1 \oplus_{\mathcal{F}} \mathrm{Z}\psi, \ldots, \psi.Z_{Å} \oplus_{\mathcal{F}} \mathrm{Z})_{TCN} = \psi.CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TCN} \oplus_{\mathcal{F}} \mathrm{Z}$$

Theorem 17: Prove that the following inequality holds, such as

$$CPFCFG\left(Z_{\mathbb{G}_1} \oplus_{\mathcal{F}} Z_{\eta_1}, \ldots, Z_{\mathbb{G}_{Å}} \oplus_{\mathcal{F}} Z_{\eta_{Å}}\right)_{TN} = CPFCFG\left(Z_{\mathbb{G}_1}, \ldots, Z_{\mathbb{G}_{Å}}\right)_{TN} \oplus_{\mathcal{F}} CPFCFG\left(Z_{\eta_1}, \ldots, Z_{\eta_{Å}}\right)_{TN}$$

$$CPFCFG\left(Z_{\mathbb{G}_1} \oplus_{\mathcal{F}} Z_{\eta_1}, \ldots, Z_{\mathbb{G}_{Å}} \oplus_{\mathcal{F}} Z_{\eta_{Å}}\right)_{TCN} = CPFCFG\left(Z_{\mathbb{G}_1}, \ldots, Z_{\mathbb{G}_{Å}}\right)_{TCN} \oplus_{\mathcal{F}} CPFCFG\left(Z_{\eta_1}, \ldots, Z_{\eta_{Å}}\right)_{TCN}$$

Theorem 18: When $\mathrm{Z}_\tau = \mathrm{Z} = \langle \mu_{\mathrm{Z}}, \eta_{\mathrm{Z}}, \xi_{\mathrm{Z}} \rangle (\tau = 1, 2, \ldots, Å)$, thus

$$CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TN} = Z$$

$$CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TCN} = Z$$

Theorem 19: When $\mu_{Z\mathbb{G}_\tau} \le \mu_{Z\eta_\tau}$ and $\eta_{Z\mathbb{G}_\tau} \ge \eta_{Z\eta_\tau} \forall \tau$, thus

$$CPFCFG\left(Z_{\mathbb{G}_1}, \ldots, Z_{\mathbb{G}_{Å}}\right)_{TN} \le CPFCFG\left(Z_{\eta_1}, \ldots, Z_{\eta_{Å}}\right)_{TN}$$

$$CPFCFG\left(Z_{\mathbb{G}_1}, \ldots, Z_{\mathbb{G}_{Å}}\right)_{TCN} \le CPFCFG\left(Z_{\eta_1}, \ldots, Z_{\eta_{Å}}\right)_{TCN}$$

Theorem 20: When $\mathrm{Z}^+ = \left(\max_\tau(\mu_{\mathrm{Z}\tau}), \min_\tau(\eta_{\mathrm{Z}\tau}), \max_\tau(\xi_{\mathrm{Z}\tau})\right)$ and $\mathrm{Z}^- = \left(\min_\tau(\mu_{\mathrm{Z}\tau}), \max_\tau(\eta_{\mathrm{Z}\tau}), \min_\tau(\xi_{\mathrm{Z}\tau})\right)$, thus

$$\mathrm{Z}^- \le CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TN} \le \mathrm{Z}^+$$

$$\mathrm{Z}^- \le CPFCFG(Z_1, Z_2, \ldots, Z_{Å})_{TCN} \le \mathrm{Z}^+$$

Further, we describe some dominant and particular cases of the initiated operators for different values of parameters.

Theorem 21: Prove that the following information holds for different values of parameters, such as

1) When $\vartheta \to 1$, thus

$$\left(\underset{\vartheta\to 1}{\mathrm{Lim}}\, CPFCFG\right)_{TN} = \begin{pmatrix} \prod_{\tau=1}^{\mathring{A}} \left(\mu_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}, \left(1-\prod_{\tau=1}^{\mathring{A}}\left(1-\left(\eta_{Z\emptyset(\tau)}\right)^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)^{\frac{1}{2}}, \\ \prod_{\tau=1}^{\mathring{A}} \left(\xi_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})} \end{pmatrix}$$

$$\left(\underset{\vartheta\to 1}{\mathrm{Lim}}\, CPFCFG\right)_{TCN} = \begin{pmatrix} \prod_{\tau=1}^{\mathring{A}} \left(\mu_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}, \left(1-\prod_{\tau=1}^{\mathring{A}}\left(1-\left(\eta_{Z\emptyset(\tau)}\right)^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)^{\frac{1}{2}}, \\ \left(1-\prod_{\tau=1}^{\mathring{A}}\left(1-\left(\xi_{Z\emptyset(\tau)}\right)^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)^{\frac{1}{2}} \end{pmatrix}$$

Called the CPF Choquet geometric operator (CPFCG).

2) When $\vartheta \to 1$, $\Delta\nabla_{\tau} = \Delta\nabla\left(\partial_{\sigma(\tau)}\right) - \Delta\nabla\left(\partial_{\sigma(\tau-1)}\right)$, then the proposed theory is converted to a CPF-weighted geometric (CPFWG) operator.

3) When $\vartheta \to 1$, $\omega_{\tau} = \Delta\nabla\left(\partial_{\sigma(\tau)}\right) - \Delta\nabla\left(\partial_{\sigma(\tau-1)}\right)$, and $\Delta\nabla(\partial) = \sum_{\tau=1}^{|\partial|}\omega_{\tau}$, where $\omega = (\omega_1, \omega_2, \dots, \omega_{\Delta\nabla})^{\mathrm{T}}$, $\omega_{j} \in [0,1]$, $\sum_{j=1}^{\mathring{A}}\omega_{j} = 1$, then the proposed theory is converted for the CPF-ordered weighted geometric (CPFOWG) operator.

5. MABAC Model Based on Proposed Theory

This section investigates the major information of MABAC data by considering the started operators for CPFSs. For this, we intend to construct the decision matrix by using m alternatives. $Z_1, Z_2, \dots, Z_m$ and n attributes $A_1, A_2, \dots, A_{\mathring{A}}$ with weight vectors $\omega = (\omega_1, \omega_2, \dots, \omega_{\Delta\nabla})^{\mathrm{T}}$, $\omega_{j} \in [0,1]$, $\sum_{j=1}^{\mathring{A}}\omega_{j} = 1$. Thus, the major information about the steps of the MABAC model is illustrated below:

Step 1: First, we calculate the matrix of data with the help of CPFNs, such as

$$D = \begin{bmatrix} Z_{11} & Z_{12} & \cdot\ \cdot & Z_{1n} \\ Z_{21} & Z_{22} & :\ : & Z_{2n} \\ \cdot & \cdot & & \cdot \\ . & . & \cdot\ \cdot & . \\ Z_{m1} & Z_{m2} & \cdot\ \cdot & Z_{mn} \end{bmatrix}$$

Further, we have two options: either we normalize or do not normalize the data in the decision matrix. If we have a cost type, then we normalize the data, such as

$$N=\begin{cases}\langle \mu_{Z1},\eta_{Z1},\xi_{Z1}\rangle & benefit\\ \langle \xi_{Z1},\eta_{Z1},\mu_{Z1}\rangle & cost\end{cases}$$

We have no option or requirement to normalize the data.

Step 2: Thus, we measure or interrogate the weighted data matrix in the occurrence of planned laws, such as

$$(\text{Å}.Z)_{TN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{1-\mu_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)},\sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{\eta_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{1-\xi_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)}\end{array}\right)$$

$$(\text{Å}.Z)_{TCN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{1-\mu_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)},\sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{\eta_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\frac{\left(\text{Fꟗ}^{\xi_Z{}^2}-1\right)^{\text{Å}}}{(\text{Fꟗ}-1)^{\text{Å}-1}}\right)}\end{array}\right)$$

Step 3: Then, we interrogated the aggregated data matrix in the occurrence of planned CPFCFA operators and CPFCFG operators for both norms, such as

$$CPFCFA(Z_1,Z_2,\dots,Z_{\text{Å}})_{TN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{1-\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}\end{array}\right)$$

$$CPFCFA(Z_1,Z_2,\dots,Z_{\text{Å}})_{TCN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fꟗ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟗ}^{\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}\end{array}\right)$$

$$CPFCFG(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TN} = \left(\begin{array}{c} \sqrt{\text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{\mu_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{1-\eta_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{\text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{\xi_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)} \end{array} \right)$$

$$CPFCFG(Z_1, Z_2, \ldots, Z_{\text{Å}})_{TCN} = \left(\begin{array}{c} \sqrt{\text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{\mu_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{1-\eta_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \text{Log}_{\text{Fɟ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fɟ}^{1-\xi_{Z\emptyset(\tau)}{}^{2}} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)} \end{array} \right)$$

Step 4: Estimate the distance data measures to use the data in the aggregated matrix and weighted data matrix, such as

$$DM_{\tau\text{ɉ}} = \begin{cases} D\left(Z_i, Z_j\right), & if\ Z_i > Z_j \\ 0, & if\ Z_i = Z_j \\ -D\left(Z_i, Z_j\right), & if\ Z_i < Z_j \end{cases}$$

Where the function of the distance measure is defined below:

$$D\left(Z_i, Z_j\right) = \frac{\text{I}}{3}\left(\left|\mu_{Zi} - \mu_{Zj}\right| + \left|\eta_{Zi} - \eta_{Zj}\right| + \left|\xi_{Zi} - \xi_{Zj}\right|\right)$$

Step 5: Demonstrate the function of appraisal value for distance techniques, such as

$$\text{Ṣ}_i = \frac{\text{I}}{\text{Å}} \sum_{j=\text{I}}^{\text{Å}} D\left(Z_i, Z_j\right)$$

Step 6: Estimate the model of ranking data for picking the best solution from the collection of data. To exploit the proposed model in the circumstance of various genuine-life problems, we assess the problems of the network data envelopment analysis approach: a case study on the Taiwan insurance industry for the proposed theory.

6. Network DEA: A Case Study on the Taiwan Insurance Industry

The NDEA model is a modified version of the traditional technique of Data Envelopment Analysis (DEA) that analyzes financial records for the internal body and interrelated procedures of decision-making units (DMU). The model of NDEA is used to find the flow among multiple stages or components within a DMU, making it perfectly useful in companies. NDEA is divided into many portions for how it techniques and derives the internal body or shape and procedure of DMU. This application model talks about the investigation of the selection of the primary kind of NDEA for use in Taiwan. For this, we deliberate on the ensuing five alternatives, such as

1) Dynamic NDEA.
2) Directional NDEA.
3) Network SBM (Slack-Based Model).
4) Stochastic NDEA.
5) Static NDEA.

Additionally, for each alternative, we have the following attributes such as

i) Growth Analysis.
ii) Social Impact.
iii) Political Impact.
iv) Environmental Impact.
v) Banking and Finance.

Thus, we will deliberate on the best one among the above five alternatives. For this, we have also considered different values of parameters, such as Fɟ $= 2, \rho = -0.7$, Å $= (0.3,0.2,0.2,0.2,0.1)$ for the weighted decision matrix and the value of Choquet is described below, such as $\Delta\nabla(Z_1) = 0.1, \Delta\nabla(Z_2) = 0.15, \Delta\nabla(Z_3) = 0.2, \Delta\nabla(Z_4) = 0.25, \Delta\nabla(Z_5) = 0.3$. Thus, the major information about the steps of the MABAC model is illustrated below:

Step 1: First, we calculate the matrix of data with the help of CPFNs, see Table 1.

Table 1. CPF information decision matrix.

	Z_1^A	Z_2^A	Z_3^A	Z_4^A	Z_5^A
Z_1	$\langle 0.9,0.3,0.1\rangle$	$\langle 0.91,0.31,0.11\rangle$	$\langle 0.92,0.32,0.12\rangle$	$\langle 0.93,0.33,0.13\rangle$	$\langle 0.94,0.34,0.14\rangle$
Z_2	$\langle 0.8,0.4,0.4\rangle$	$\langle 0.81,0.41,0.41\rangle$	$\langle 0.82,0.42,0.42\rangle$	$\langle 0.83,0.43,0.43\rangle$	$\langle 0.84,0.44,0.44\rangle$
Z_3	$\langle 0.6,0.5,0.2\rangle$	$\langle 0.61,0.51,0.21\rangle$	$\langle 0.62,0.52,0.22\rangle$	$\langle 0.63,0.53,0.23\rangle$	$\langle 0.64,0.54,0.24\rangle$
Z_4	$\langle 0.9,0.3,0.8\rangle$	$\langle 0.91,0.31,0.81\rangle$	$\langle 0.92,0.32,0.82\rangle$	$\langle 0.93,0.33,0.83\rangle$	$\langle 0.94,0.34,0.84\rangle$
Z_5	$\langle 0.8,0.4,0.1\rangle$	$\langle 0.81,0.41,0.11\rangle$	$\langle 0.82,0.42,0.12\rangle$	$\langle 0.83,0.43,0.13\rangle$	$\langle 0.84,0.44,0.14\rangle$

Further, we have two options: either we normalize or do not normalize the data in the decision matrix. If we have a cost type, then we normalize the data, such as

$$N=\begin{cases}\langle \mu_{Z1},\eta_{Z1},\xi_{Z1}\rangle & benefit\\ \langle \xi_{Z1},\eta_{Z1},\mu_{Z1}\rangle & cost\end{cases}$$

We have no option or requirement to normalize the data. So, we will go to the next step with the same data in Table 1.

Step 2: Thus, we measure or interrogate the weighted data matrix in the occurrence of planned laws, see Table 2.

Table 2. CPF weighted information matrix.

	Z_1^A	Z_2^A	Z_3^A	Z_4^A	Z_5^A
Z_1	$\langle 0.6023,0.7247,0.0548\rangle$	$\langle 0.5173,0.8155,0.0492\rangle$	$\langle 0.5302,0.8198,0.0537\rangle$	$\langle 0.5443,0.8239,0.0582\rangle$	$\langle 0.4094,0.9121,0.0444\rangle$
Z_2	$\langle 0.4971,0.7807,0.2238\rangle$	$\langle 0.4206,0.8542,0.1881\rangle$	$\langle 0.4286,0.8576,0.193\rangle$	$\langle 0.4369,0.861,0.1978\rangle$	$\langle 0.3212,0.9310,0.1439\rangle$
Z_3	$\langle 0.3473,0.8271,0.1100\rangle$	$\langle 0.2912,0.8859,0.0945\rangle$	$\langle 0.2968,0.8888,0.0990\rangle$	$\langle 0.3025,0.8917,0.1036\rangle$	$\langle 0.2198,0.9466,0.0766\rangle$
Z_4	$\langle 0.6023,0.7247,0.4971\rangle$	$\langle 0.5173,0.8155,0.4206\rangle$	$\langle 0.5302,0.8198,0.4286\rangle$	$\langle 0.5443,0.8239,0.4369\rangle$	$\langle 0.4094,0.9121,0.3212\rangle$
Z_5	$\langle 0.4971,0.7807,0.0548\rangle$	$\langle 0.4206,0.8542,0.0492\rangle$	$\langle 0.4286,0.8576,0.0537\rangle$	$\langle 0.4369,0.861,0.0582\rangle$	$\langle 0.3212,0.9310,0.0444\rangle$

Step 3: Then, we interrogated the aggregated data matrix in the occurrence of planned CPFCFA operators and CPFCFG operators for both norms, see Table 3.

Table 3. CPF aggregated data.

	CPFCFA operator	CPFCFG operator
Z_1	$\langle 0.2896,0.9485,0.028\rangle$	$\langle 0.8262,0.5427,0.4483\rangle$
Z_2	$\langle 0.2301,0.9596,0.1013\rangle$	$\langle 0.7805,0.5775,0.6334\rangle$
Z_3	$\langle 0.1570,0.9687,0.0517\rangle$	$\langle 0.7091,0.6106,0.5300\rangle$
Z_4	$\langle 0.2896,0.9485,0.2301\rangle$	$\langle 0.8262,0.5427,0.7805\rangle$
Z_5	$\langle 0.2301,0.9596,0.028\rangle$	$\langle 0.7805,0.5775,0.4483\rangle$

Step 4: Estimate the distance data measures to use the data in the aggregated matrix and weighted data matrix, see Table 4.

Table 4. CPF distance measures.

	CPFCFA operator	CPFCFG operator
Z_1	0.1877,0.1273,0.1316,0.1365,0.0575	0.2664,0.3269,0.3225,0.3177,0.3967
Z_2	0.1895,0.1276,0.1307,0.1340,0.0541	0.2987,0.3606,0.3574,0.3542,0.4341
Z_3	0.1300,0.0865,0.089,0.09146,0.0366	0.3327,0.3762,0.3738,0.3713,0.4262
Z_4	0.2678,0.1837,0.1892,0.1953,0.0824	0.2297,0.3138,0.3083,0.3022,0.4151
Z_5	0.1576,0.1057,0.1087,0.1119,0.0453	0.2933,0.3451,0.3421,0.3390,0.4055

Step 5: Demonstrate the function of appraisal value for distance techniques, see Table 5.

Table 5. Representation of appraisal values.

	CPFCFA operator	CPFCFG operator
Z_1	0.12815	0.32607
Z_2	0.1272	0.36102
Z_3	0.08674	0.37607
Z_4	0.18372	0.31387
Z_5	0.1059	0.34506

Step 6: Estimate the model of ranking data for picking the best solution between the collection of data in Table 6, such as

Table 6. Representation of ranking data.

Methods	Ranking values	Most valuable decision
CPFCFA operator	$Z_4 > Z_1 > Z_2 > Z_5 > Z_3$	Z_4
CPFCFG operator	$Z_3 > Z_2 > Z_5 > Z_1 > Z_4$	Z_3

Finally, we concluded that the best one is Z_4 with the MABAC-CPFCFA operator and Z_3 with MABAC-CPFCFG operator. Further, we simplify and describe the ranking values of the proposed theory by using the data in Table 1. Then, we interrogated the aggregated data matrix in the occurrence of planned CPFCFA operators and CPFCFG operators for both norms, see Table 7.

Table 7. CPF aggregated values.

	CPFCFA operator	CPFCFG operator
Z_1	$\langle 0.6372, 0.7387, 0.0682 \rangle$	$\langle 0.9767, 0.1797, 0.5719 \rangle$
Z_2	$\langle 0.5192, 0.7921, 0.2378 \rangle$	$\langle 0.9453, 0.2378, 0.7921 \rangle$
Z_3	$\langle 0.3631, 0.837, 0.1234 \rangle$	$\langle 0.8764, 0.2984, 0.6707 \rangle$
Z_4	$\langle 0.6372, 0.7387, 0.5192 \rangle$	$\langle 0.9767, 0.1797, 0.9453 \rangle$
Z_5	$\langle 0.5192, 0.7921, 0.0682 \rangle$	$\langle 0.9453, 0.2378, 0.5719 \rangle$

Demonstrate the function of appraisal value for distance techniques, see Table 8.

Table 8. Representation of appraisal values.

	CPFCFA operator	CPFCFG operator
Z_1	0.0255	0.549
Z_2	0.1422	0.7247
Z_3	0.0931	0.5527
Z_4	0.1941	0.9075
Z_5	0.0408	0.5232

Estimate the model of ranking data for picking the best solution between the collection of data in Table 9, such as

Table 9. Representation of ranking data.

Methods	Ranking values	Most valuable decision
CPFCFA operator	$Z_4 > Z_2 > Z_3 > Z_5 > Z_1$	Z_4
CPFCFG operator	$Z_4 > Z_2 > Z_3 > Z_1 > Z_5$	Z_4

Finally, we concluded that the best one is Z_4 with the CPFCFA operator and the CPFCFG operator. Moreover, we establish a comparative analysis to show the interpretation and validity of the initiated techniques.

7. Comparative Analysis

The performance of comparative analysis is a valuable and important part of every manuscript, especially when you need to prove the supremacy and validity of the derived theory. The key objective of this article is to design a comparison between invented information and various previous data. Therefore, we arranged the following old models, for instance, Xu [17] diagnosed the Choquet integral for weighted IFSs. Tan and Chen [18] derived the intuitionistic fuzzy Choquet integral for decision support systems. Tan and Chen [19} exposed the Choquet integral for induced IFSs. Xu and Yager [20] designed the geometric operators for IFSs. Yang et al. [21] derived the Frank operators for IFSs. Zhang et al. [22] presented the Frank power operators for IFSs. Xing et al. [23] invented the Choquet Frank operators for PFSs. Ali and Yang [24] described the Hamacher operators for CPFSs. EDAS model and improved Dombi operators for CPFSs were invented by Garg et al. [25]. Ali et al. [26] evaluated the Aczel-Alsina power operators for circular Pythagorean fuzzy linguistic sets. Jiang et al. [27] designed the Bonferroni mean operators and EDAS model for CPFSs. Verma [28] presented the MABAC model based on order-alpha divergence measures for IFSs. Jia et al. [29] designed the extended MABAC model for intuitionistic fuzzy rough sets. Zhao et al. [30] explored the intuitionistic fuzzy MABAC information and its applications. The comparative analysis is given in Table 10 for data in Table 1.

Table 10. For data in Table 1, the comparative analysis is given in Table 10.

Methods	Score values	Ranking model
Xu [17]	No	No
Tan and Chen [18]	No	No
Tan and Chen [19]	No	No
Xu and Yager [20]	No	No

Yang et al. [21]	No	No
Zhang et al. [22]	No	No
Xing et al. [23]	No	No
Ali and Yang [24]	0.0432,0.1027,0.0123,0.3026,0.0288	$Z_4 > Z_2 > Z_1 > Z_5 > Z_3$
Garg et al. [25]	0.0425,0.1024,0.0121,0.3015,0.0283	$Z_4 > Z_2 > Z_1 > Z_5 > Z_3$
Ali et al. [26]	No	No
Jiang et al. [27]	No	No
Verma [28]	No	No
Jia et al. [29]	No	No
Zhao et al. [30]	No	No
MABAC (averaging)	0.1281,0.1272,0.0867,0.1837,0.1059	$Z_4 > Z_1 > Z_2 > Z_5 > Z_3$
MABAC (geometric)	0.3260,0.3610,0.3760,0.3138,0.3450	$Z_3 > Z_2 > Z_5 > Z_1 > Z_4$
CPFCFA operator	0.0255,0.1422,0.0931,0.1941,0.0408	$Z_4 > Z_2 > Z_3 > Z_5 > Z_1$
CPFCFG operator	0.549,0.7247,0.5527,0.9075,0.5232	$Z_4 > Z_2 > Z_3 > Z_1 > Z_5$

Finally, we concluded that the best one is Z_4 with the MABAC-CPFCFA operator and Z_3 with the MABAC-CPFCFG operator. But we concluded that the best one is Z_4 with the CPFCFA operator and the CPFCFG operator. Further, the existing model of Ali and Yang [24] and the proposed theory of Garg et al. [25] also provided the same ranking values Z_4. The invented model is different from the existing models because the previous techniques were developed based on norms, but the proposed techniques are computed based on the Choquet integral, which is reliable for coping with vague data.

8. Concluding remarks

Network data envelopment analysis is used to assess the proficiency of insurance enterprises. To do this, we intended the frank operational laws for CPF uncertainty with various dominant results. Moreover, the CPFCFA operator and CPFCFG operator with three dominant properties for each operator have been studied. The MABAC model has been deliberated and verified by using numerical examples. In the following, a decision-making performance for assessing the proficiency of insurance enterprises using the Network DEA is proposed.

The major influence of this article is listed below:

1) We discussed the frank operational laws for CPF uncertainty with various dominant results.
2) We studied the CPFCFA operator and the CPFCFG operator with three dominant properties for each operator.
3) We deliberated on the MABAC model and verified it with the help of numerical examples.

4) We establish the decision-making performance for assessing the proficiency of insurance enterprises using the NDEA approach.
5) We proceed with the ranking values of proposed models for comparing them with the ranking values of existing models to show the capability and efficacy of the proposed approaches.

There are potential avenues for future studies: one can explore the circular Pythagorean hesitant fuzzy sets and their extensions. Moreover, the development of the various operators, measures, and methods for invented models and their extensions could be interesting. Finally, one can evaluate the decision-making technique, neural networks, game theory, and artificial data mining to enhance the worth of the proposed theory.

Appendix Section

Appendix A: Proof: Let $0 \leq \mu_Z, \eta_Z \leq 1$, $\text{Fꟻ} \succ 1$, and $\mu_Z{}^2 + \eta_Z{}^2 \leq 1$, thus $\eta_Z{}^2 \leq 1 - \mu_Z{}^2$, $0 \leq 1 - \mu_Z{}^2 \leq 1$. For $\text{Å}.\text{Z}$, thus

$$0 \leq \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \frac{(\text{Fꟻ}^{1-0} - 1)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)} \leq \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)}$$

$$\leq \sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \frac{(\text{Fꟻ}^{1-1} - 1)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)} = 1$$

$$0 = \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \frac{(\text{Fꟻ}^{0} - 1)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)} \leq \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\eta_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)} \leq \sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \frac{(\text{Fꟻ}^{1} - 1)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)} = 1$$

where

$$0 \leq \left(\sqrt{1 - \text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)}\right)^2 + \left(\sqrt{\text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\eta_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right)}\right)^2$$

$$\leq 1 - \text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\mu_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right) + \text{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\eta_Z{}^2} - 1\right)^{\text{Å}}}{(\text{Fꟻ} - 1)^{\text{Å}-1}}\right) = 1$$

The information of $\text{Å}.\text{Z}$ holds the condition of CPFSs. Further, by using mathematical induction, we prove the above information. For this, if $\text{Å} = 1$, thus

$$(1.Z)_{TN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)^{1}}{(\text{Fⅎ}-1)^{1-1}}\right)},\sqrt{\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)^{1}}{(\text{Fⅎ}-1)^{1-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)^{1}}{(\text{Fⅎ}-1)^{1-1}}\right)}\end{array}\right)=\langle\mu_Z,\eta_Z,\xi_Z\rangle=Z$$

For Å $=1$, the proposed theory holds. Further, we assume that the proposed theory also holds for Å $=\sigma$, such as

$$(\sigma.Z)_{TN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)},\sqrt{\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)}\end{array}\right)$$

Then, we prove it for Å $=\sigma+1$, such as

$$(\sigma+1).Z=(\sigma.Z)\oplus.Z=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)},\sqrt{\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)}\end{array}\right)\oplus\langle\mu_Z,\eta_Z,\xi_Z\rangle$$

$$=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\left(\text{Fⅎ}^{1-\left(1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)\right)}-1\right).\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)}{\text{Fⅎ}-1}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fⅎ}}\left(1+\left(\text{Fⅎ}^{\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)}\right).\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)}{\text{Fⅎ}-1}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\left(\text{Fⅎ}^{1-\left(1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)^{\sigma}}{(\text{Fⅎ}-1)^{\sigma-1}}\right)\right)}-1\right).\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)}{\text{Fⅎ}-1}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\mu_Z^2}-1\right)^{\sigma+1}}{(\text{Fⅎ}-1)^{\sigma}}\right)},\sqrt{\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{\eta_Z^2}-1\right)^{\sigma+1}}{(\text{Fⅎ}-1)^{\sigma}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fⅎ}}\left(1+\frac{\left(\text{Fⅎ}^{1-\xi_Z^2}-1\right)^{\sigma+1}}{(\text{Fⅎ}-1)^{\sigma}}\right)}\end{array}\right)$$

For $\text{Å} = \sigma + 1$, the proposed theory is held successfully. Similarly, we will be evaluating the remaining part of the proposed theory.

$$(\text{Å}.Z)_{TCN} = \left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\frac{\left(\text{FꟻF}^{1-\mu_{Z}^{2}}-1\right)^{\text{Å}}}{(\text{FꟻF}-1)^{\text{Å}-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FꟻF}}\left(1+\frac{\left(\text{FꟻF}^{\eta_{Z}^{2}}-1\right)^{\text{Å}}}{(\text{FꟻF}-1)^{\text{Å}-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{FꟻF}}\left(1+\frac{\left(\text{FꟻF}^{\xi_{Z}^{2}}-1\right)^{\text{Å}}}{(\text{FꟻF}-1)^{\text{Å}-1}}\right)}\end{array}\right).$$

Appendix B: Proof: For the verification of the above two proposed techniques, we use the model of mathematical induction. If $\text{Å} = 2$, thus

$$\left(\Delta\nabla(\partial_{\emptyset(1)}) - \Delta\nabla(\partial_{\emptyset(0)})\right).Z_{\emptyset 1} = \left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\mu_{Z\sigma(1)}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{\eta_{Z\sigma(1)}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)}, \\ \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\xi_{Z\sigma(1)}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)}\end{array}\right)$$

$$\left(\Delta\nabla(\partial_{\emptyset(2)}) - \Delta\nabla(\partial_{\emptyset(1)})\right).Z_{\emptyset 2} = \left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\mu_{Z\sigma(2)}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{\eta_{Z\sigma(2)}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)}, \\ \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\xi_{Z\sigma(2)}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)}\end{array}\right)$$

Thus,

$$CPFCFA(Z_1, Z_2) = \Delta\nabla(\partial_{\emptyset 2}) - \Delta\nabla(\partial_{\emptyset 1}).Z_{\emptyset 1} \oplus_{\mathcal{F}} \Delta\nabla(\partial_{\emptyset 3}) - \Delta\nabla(\partial_{\emptyset 2}).Z_{\emptyset 2}$$

$$= \begin{pmatrix} \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\mu_{Z\sigma(1)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{\eta_{Z\sigma_{(1)}}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)}, \\ \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\xi_{Z\sigma(1)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})}}{(\vartheta-1)^{\Delta\nabla(\partial_{\sigma(1)})-\Delta\nabla(\partial_{\sigma(0)})-1}}\right)} \end{pmatrix} \oplus \begin{pmatrix} \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\mu_{Z\sigma(2)}{}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{\eta_{Z\sigma_{(2)}}{}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)}, \\ \sqrt{1-\mathrm{Log}_{\vartheta}\left(1+\frac{\left(\vartheta^{1-\xi_{Z\sigma(2)}{}^{2}}-1\right)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)}}{(\vartheta-1)^{\left(\Delta\nabla(\partial_{\emptyset(2)})-\Delta\nabla(\partial_{\emptyset(1)})\right)-1}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\frac{\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta-1)^{\sum_{\tau=1}^{2}\Delta\nabla(\partial_{\sigma(\tau)})-\Delta\nabla(\partial_{\sigma(\tau-1)})-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1+\frac{\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta-1)^{\sum_{\tau=1}^{2}\Delta\nabla(\partial_{\sigma(\tau)})-\Delta\nabla(\partial_{\sigma(\tau-1)})-1}}\right)}, \\ \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\frac{\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta-1)^{\sum_{\tau=1}^{2}\Delta\nabla(\partial_{\sigma(\tau)})-\Delta\nabla(\partial_{\sigma(\tau-1)})-1}}\right)} \end{pmatrix}$$

Where, $\sum_{\tau=1}^{2}\Delta\nabla\left(\partial_{\sigma(\tau)}\right)-\Delta\nabla\left(\partial_{\sigma(\tau-1)}\right)=1$, thus

$$CPFCFA(Z_{\sigma1},Z_{\sigma2}) = \begin{pmatrix} \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{2}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{pmatrix}$$

For $\text{Å}=2$, the proposed theory holds. Further, we assume that the proposed theory holds for $\text{Å}=\sigma$, thus

$$CPFCFA(Z_{1},Z_{2},\dots,Z_{\sigma}) = \begin{pmatrix} \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{pmatrix}$$

Thus, we prove it for Å $= \sigma + 1$, such as

$$CPFCFA(Z_1, Z_2, \dots, Z_\sigma, Z_{\sigma+1}) = CPFCFA(Z_1, Z_2, \dots, Z_\sigma) \oplus_{\mathcal{F}} \Delta\nabla(\partial_{\emptyset(\sigma)}) - \Delta\nabla(\partial_{\emptyset(\sigma+1)}) Z_{\emptyset\sigma+1}$$

$$= \begin{pmatrix} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\mu_{Z\emptyset\sigma+1}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\sigma)}) - \Delta\nabla(\partial_{\emptyset(\sigma-1)})}}{(\vartheta - 1)^{\Delta\nabla(\partial_{\sigma(\sigma)}) - \Delta\nabla(\partial_{\sigma(\sigma-1)}) - 1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\eta_{Z\emptyset\sigma+1}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\sigma)}) - \Delta\nabla(\partial_{\emptyset(\sigma-1)})}}{(\vartheta - 1)^{\Delta\nabla(\partial_{\sigma(\sigma)}) - \Delta\nabla(\partial_{\sigma(\sigma-1)}) - 1}}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\xi_{Z\emptyset\sigma+1}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\sigma)}) - \Delta\nabla(\partial_{\emptyset(\sigma-1)})}}{(\vartheta - 1)^{\sum_{\tau=1}^{2}\Delta\nabla(\partial_{\sigma(\sigma)}) - \Delta\nabla(\partial_{\sigma(\sigma-1)}) - 1}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta - 1)^{\sum_{\tau=1}^{\sigma+1}\Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)}) - 1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta - 1)^{\sum_{\tau=1}^{\sigma+1}\Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)}) - 1}}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{(\vartheta - 1)^{\sum_{\tau=1}^{\sigma+1}\Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)}) - 1}}\right)} \end{pmatrix}$$

where $\sum_{\tau=1}^{\sigma+1}\Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)}) = 1$, thus

$$CPFCFA(Z_1, Z_2, \dots, Z_\sigma, Z_{\sigma+1}) = \begin{pmatrix} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\sigma+1}\left(\text{Fꟻ}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{pmatrix}$$

For $\text{Å} = \sigma + 1$, we successfully hold the required results. Further, we will also prove the remaining part, such as

$$CPFCFA(Z_1, Z_2, \dots, Z_{\text{Å}})_{TCN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{array} \right).$$

Appendix C: Proof: Using the information in Theorem 1 and Theorem 4, we have

$$(\psi . Z_\tau)_{TN} = \left(\sqrt{1 - \mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi-1}}\right)}, \sqrt{1 - \mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi}}\right)} \right)$$

$$(\psi . Z_\tau)_{TCN} = \left(\sqrt{1 - \mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi-1}}\right)}, \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\left(\text{FꟻI}^{\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi}}{(\vartheta - 1)^{\psi-1}}\right)} \right)$$

then

$$CPFCFA(\psi . Z_1, \psi . Z_2, \dots, \psi . Z_{\text{Å}}) = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta - 1)^{\psi-1}}\right)}, \\ \sqrt{\mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta - 1)^{\psi-1}}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{FꟻI}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\text{FꟻI}^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta - 1)^{\psi-1}}\right)} \end{array} \right)$$

$$\psi.CPFCFA(Z_1,Z_2,\dots,Z_{\mathring{A}})=\psi.\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fɟ}}\left(1+\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{1-\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta-1)^{\psi-1}}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta-1)^{\psi-1}}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\prod_{\tau=1}^{\mathring{A}}\left(\text{Fɟ}^{1-\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\psi\left(\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})\right)}}{(\vartheta-1)^{\psi-1}}\right)}\end{array}\right)$$

$$=CPFCFA(\psi.Z_1,\psi.Z_2,\dots,\psi.Z_{\mathring{A}})$$

Similarly, we will prove it for FTCN.

Appendix D: Proof: Let

$$(Z_\tau\oplus_{\mathcal{F}}Z)_{TN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{1-\mu_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{1-\mu_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)},\sqrt{\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{\eta_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{\eta_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{1-\xi_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{1-\xi_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)}\end{array}\right)$$

$$(Z_\tau\oplus_{\mathcal{F}}Z)_{TCN}=\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{1-\mu_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{1-\mu_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)},\sqrt{\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{\eta_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{\eta_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)},\\ \sqrt{\mathrm{Log}_{\text{Fɟ}}\left(1+\frac{\left(\text{Fɟ}^{\xi_{Z_\tau}{}^2}-1\right)\left(\text{Fɟ}^{\xi_Z{}^2}-1\right)}{\text{Fɟ}-1}\right)}\end{array}\right)$$

Thus

and

$$CPFCFA(Z_1 \oplus_{\mathcal{F}} Z, \dots, Z_{\text{Å}} \oplus_{\mathcal{F}} Z) = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\mu_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)}, \\ \sqrt{\mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{\eta_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\xi_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)} \end{array} \right)$$

and

$$CPFCFA(Z_1, Z_2, \dots, Z_{\text{Å}}) \oplus_{\mathcal{F}} Z = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{\mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)} \end{array} \right) \oplus_{\mathcal{F}} Z\langle \mu_Z, \eta_Z, \xi_Z \rangle$$

$$= \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\mu_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)}, \\ \sqrt{\mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{\eta_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\xi_Z{}^2} - 1\right)\prod_{\tau=1}^{\text{Å}}\left(\mathrm{F}Ⅎ^{1-\xi_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\vartheta - 1}\right)} \end{array} \right)$$

$$= \mathrm{CPFCFA}(\mathrm{Z}_1 \oplus_{\mathcal{F}} \mathrm{Z}, \dots, \mathrm{Z}_{\text{Å}} \oplus_{\mathcal{F}} \mathrm{Z})$$

Similarly, we will evaluate it for FTCN.

Appendix E: Proof: Let

$$\left(Z_{\mathbb{G}_\tau} \oplus_{\mathcal{F}} Z_{\eta_\tau}\right)_{TN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\mu_{Z\mathbb{G}_\tau}{}^2} - 1\right)\left(\mathrm{F}Ⅎ^{1-\mu_{Z\eta_\tau}{}^2} - 1\right)}{(\vartheta - 1)}\right)}, \sqrt{\mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{\eta_{Z\mathbb{G}_\tau}{}^2} - 1\right)\left(\mathrm{F}Ⅎ^{\eta_{Z\eta_\tau}{}^2} - 1\right)}{(\vartheta - 1)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\mathrm{F}Ⅎ}\left(1 + \frac{\left(\mathrm{F}Ⅎ^{1-\xi_{Z\mathbb{G}_\tau}{}^2} - 1\right)\left(\mathrm{F}Ⅎ^{1-\xi_{Z\eta_\tau}{}^2} - 1\right)}{(\vartheta - 1)}\right)} \end{array} \right)$$

$$\left(Z_{\mathfrak{E}_\tau} \oplus_{\mathcal{F}} Z_{\eta_\tau}\right)_{TCN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{1-\mu_{Z_{\mathfrak{E}_\tau}}{}^2} - 1\right)\left(\text{Fꟻ}^{1-\mu_{Z_{\eta_\tau}}{}^2} - 1\right)}{(\vartheta - 1)}\right)}, \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\eta_{Z_{\mathfrak{E}_\tau}}{}^2} - 1\right)\left(\text{Fꟻ}^{\eta_{Z_{\eta_\tau}}{}^2} - 1\right)}{(\vartheta - 1)}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\left(\text{Fꟻ}^{\xi_{Z_{\mathfrak{E}_\tau}}{}^2} - 1\right)\left(\text{Fꟻ}^{\xi_{Z_{\eta_\tau}}{}^2} - 1\right)}{(\vartheta - 1)}\right)} \end{array} \right)$$

$$CPFCFA\left(Z_{\mathfrak{E}_1} \oplus_{\mathcal{F}} Z_{\eta_1}, \dots, Z_{\mathfrak{E}_{\text{Å}}} \oplus_{\mathcal{F}} Z_{\eta_{\text{Å}}}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\frac{\left(\text{Fꟻ}^{1-\mu_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{1-\mu_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)}{(\vartheta - 1)}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\frac{\left(\text{Fꟻ}^{\eta_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{\eta_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)}{(\vartheta - 1)}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\frac{\left(\text{Fꟻ}^{1-\xi_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{1-\xi_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)}{(\vartheta - 1)}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)} \end{array} \right)$$

Since

$$CPFCFA\left(Z_{\mathfrak{E}_1}, \dots, Z_{\mathfrak{E}_{\text{Å}}}\right)_{TN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{1-\mu_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{\eta_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}} \left(\text{Fꟻ}^{1-\xi_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)} \end{array} \right)$$

$$CPFCFA\left(Z_{\eta_1}, \dots, Z_{\eta_{\text{Å}}}\right)_{TN} = \left(\begin{array}{c} \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\left(\text{Fꟻ}^{1-\mu_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{1-\mu_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}{(\vartheta - 1)}\right)}, \\ \sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\left(\text{Fꟻ}^{\eta_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{\eta_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}{(\vartheta - 1)}\right)}, \\ \sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \frac{\prod_{\tau=1}^{\text{Å}} \left(\left(\text{Fꟻ}^{1-\xi_{Z_{\mathfrak{E}(\emptyset\tau)}}{}^2} - 1\right)\left(\text{Fꟻ}^{1-\xi_{Z_{\eta(\emptyset\tau)}}{}^2} - 1\right)\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right) - \Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}{(\vartheta - 1)}\right)} \end{array} \right)$$

$$= CPFCFA\left(Z_{\mathfrak{E}_1}, \dots, Z_{\mathfrak{E}_{\text{Å}}}\right)_{TN} \oplus_{\mathcal{F}} CPFCFA\left(Z_{\eta_1}, \dots, Z_{\eta_{\text{Å}}}\right)_{TN}$$

Similarly, we will evaluate the remaining part.

Appendix F: Proof: Using the proposed information, we aim to evaluate part (1) and part (4), where the proof of part (2) and part (3) is similar to the proof of part (1) and part (4).

1) Let

$$\lim_{\vartheta\to+\infty} CPFCFA = \lim_{\vartheta\to+\infty}\left(\begin{array}{c}\sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)},\\ \sqrt{1-\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{1-\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}\end{array}\right)$$

Thus, we only derive that

$$\lim_{\vartheta\to 1}\sqrt{1-\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{1-\mu_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}$$

$$=\sqrt{1-\prod_{\tau=1}^{\text{Å}}\left(1-\mu_{Z\emptyset(\tau)}{}^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}}$$

$$\lim_{\vartheta\to 1}\sqrt{\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}=\prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}$$

$$\lim_{\vartheta\to 1}\sqrt{1-\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{1-\xi_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}$$

$$=\sqrt{1-\prod_{\tau=1}^{\text{Å}}\left(1-\xi_{Z\emptyset(\tau)}{}^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}}$$

First, we evaluate that

$$\lim_{\vartheta\to 1}\sqrt{\mathrm{Log}_{\text{FꟻF}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{\eta_{Z\emptyset(\tau)}{}^2}-1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}=\prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)})-\Delta\nabla(\partial_{\emptyset(\tau-1)})}$$

where $\vartheta \to 1$, then $\left(\vartheta^{\eta_{Z\sigma(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\sigma(\tau)}) - \Delta\nabla(\partial_{\sigma(\tau-1)})} \to 0$. Further, by using the technique of equivalent infinitesimal replacement $\ln(1 + o) \sim o(o \succ 0)$, and Logarithmic transform, such as

$$\sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\text{FꟻF}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}$$

$$= \sqrt{\frac{\ln\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}{\ln \text{Fꟻ}}}$$

$$\sim \sqrt{\frac{\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\ln \text{Fꟻ}}}$$

Using the following technique, we have

$$\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 = 1 + \eta_{Z\emptyset(\tau)}{}^2 \ln \text{Fꟻ} + \frac{\eta_{Z\emptyset(\tau)}{}^2}{2}(\ln \text{Fꟻ})^2 + \cdots = 1 + \eta_{Z\emptyset(\tau)}{}^2 \ln \text{Fꟻ} + O(\ln \text{Fꟻ})$$

$$\Rightarrow \text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 = \eta_{Z\emptyset(\tau)}{}^2 \ln \text{Fꟻ} + O(\ln \text{Fꟻ})$$

Then, $\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1 \longrightarrow \eta_{Z\emptyset(\tau)}{}^2 \ln \text{Fꟻ}$, thus

$$\lim_{\vartheta \to 1}\sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}$$

$$= \lim_{\vartheta \to 1}\sqrt{\frac{\ln\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}{\ln \text{Fꟻ}}}$$

$$= \lim_{\vartheta \to 1}\sqrt{\frac{\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\ln \text{Fꟻ}}} = \lim_{\vartheta \to 1}\sqrt{\frac{\prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}{}^2 \ln \text{Fꟻ}\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}{\ln \text{Fꟻ}}}$$

$$= \lim_{\vartheta \to 1}\sqrt{\frac{\prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}{}^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}(\ln \text{Fꟻ})}{\ln \text{Fꟻ}}} = \prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}{}^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}$$

thus

$$\lim_{\vartheta \to 1}\sqrt{1 - \mathrm{Log}_{\text{Fꟻ}}\left(1 + \prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^2} - 1\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}\right)}$$

$$= \sqrt{1 - \prod_{\tau=1}^{\text{Å}}\left(1 - \mu_{Z\emptyset(\tau)}{}^2\right)^{\Delta\nabla(\partial_{\emptyset(\tau)}) - \Delta\nabla(\partial_{\emptyset(\tau-1)})}}$$

thus,

$$\underset{\vartheta\to 1}{\mathrm{Lim}}\, CPFCFA = \left(\begin{array}{c}\sqrt{1-\prod_{\tau=1}^{\text{Å}}\left(1-\mu_{Z\emptyset(\tau)}{}^{2}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}, \prod_{\tau=1}^{\text{Å}}\left(\eta_{Z\emptyset(\tau)}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}, \\ \sqrt{1-\prod_{\tau=1}^{\text{Å}}\left(1-\xi_{Z\emptyset(\tau)}{}^{2}\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}\end{array}\right)$$

Further, we derive part (4), such as

$$\underset{\vartheta\to+\infty}{\mathrm{Lim}}\sqrt{1-\mathrm{Log}_{\text{FƋ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FƋ}^{1-\mu_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$

$$=\sqrt{1-\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\mu_{Z\emptyset(\tau)}{}^{2}}$$

$$\underset{\vartheta\to+\infty}{\mathrm{Lim}}\sqrt{\mathrm{Log}_{\text{FƋ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FƋ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$

$$=\sqrt{\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\eta_{Z\emptyset(\tau)}{}^{2}}$$

$$\underset{\vartheta\to+\infty}{\mathrm{Lim}}\sqrt{1-\mathrm{Log}_{\text{FƋ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FƋ}^{1-\xi_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$

$$=\sqrt{1-\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\xi_{Z\emptyset(\tau)}{}^{2}}$$

thus

$$\underset{\vartheta\to+\infty}{\mathrm{Lim}}\sqrt{\mathrm{Log}_{\text{FƋ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FƋ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$

$$=\sqrt{\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\eta_{Z\emptyset(\tau)}{}^{2}}$$

Since

$\mathrm{Log}_{\text{FƋ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{FƋ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)$ is continuous, thus

$$\operatorname*{Lim}_{\vartheta\to+\infty}\sqrt{\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$
$$=\sqrt{\operatorname*{Lim}_{\vartheta\to+\infty}\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$

Thus, by using L'Hospital's rule, we have

$$\sqrt{\operatorname*{Lim}_{\vartheta\to+\infty}\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}$$
$$=\sqrt{\operatorname*{Lim}_{\vartheta\to+\infty}\frac{\ln\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}{\ln\text{Fꟻ}}}$$
$$=\sqrt{\operatorname*{Lim}_{\vartheta\to+\infty}\frac{\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}}{\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\frac{\eta_{Z\emptyset(\tau)}{}^{2}\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1}{\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1}}\Big/\frac{1}{\text{Fꟻ}}}$$
$$=\sqrt{\operatorname*{Lim}_{\vartheta\to+\infty}\frac{1}{\left(1+1/\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\frac{\eta_{Z\emptyset(\tau)}{}^{2}}{1-1/\text{Fꟻ}^{\eta_{Z\emptyset(\tau)}{}^{2}}}}$$
$$=\sqrt{\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\eta_{Z\emptyset(\tau)}{}^{2}}$$

thus

$$\operatorname*{Lim}_{\vartheta\to+\infty}\sqrt{1-\mathrm{Log}_{\text{Fꟻ}}\left(1+\prod_{\tau=1}^{\text{Å}}\left(\text{Fꟻ}^{1-\mu_{Z\emptyset(\tau)}{}^{2}}-1\right)^{\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)}\right)}=\sqrt{1-\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\mu_{Z\emptyset(\tau)}{}^{2}}$$

Thus

$$\operatorname*{Lim}_{\vartheta\to+\infty}CPFCFA$$
$$=\left(\begin{array}{c}\sqrt{1-\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\mu_{Z\emptyset(\tau)}{}^{2}},\sqrt{\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\eta_{Z\emptyset(\tau)}{}^{2}},\\ \sqrt{1-\sum_{\tau=1}^{\text{Å}}\left(\Delta\nabla\left(\partial_{\emptyset(\tau)}\right)-\Delta\nabla\left(\partial_{\emptyset(\tau-1)}\right)\right)\xi_{Z\emptyset(\tau)}{}^{2}}\end{array}\right).$$

Hence, the proposed theory holds.

Acknowledgments: This work was supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 114-2410-H-224-001 and internal number 114-1011.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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