Int. J. Anal. Appl. (2025), 23:301

International Journal of Analysis and Applications

On Piecewise Fractional Differential Operator with Application to Biological Model of Prey Predator

Muhammad Nisarul Haq[1], **Muhammad Sarwar**[1,2,*], **Kamaleldin Abodayeh**[2], **Saowaluck Chasreechai**[3,5,*], **Thanin Sitthiwirattham**[4,5]

[1]*Department of Mathematics, University of Malakand, Chakdara Dir(L), 18000, Khyber Pakhtunkhwa, Pakistan*

[2]*Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia*

[3]*Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*

[4]*Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand*

[5]*Research Group for Fractional Calculus Theory and Applications, Science and Technology Research Institute, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*

**Corresponding authors:* sarwarswati@gmail.com; saowaluck.c@sci.kmutnb.ac.th

Abstract. In this work, using the concept of piecewise fractional order differential operators and fixed point theory, existence results for the general Cauchy type dynamical system are studied. Further, Ulam-Hyers (U.H) stability and generalized U.H stability results are also investigated for the considered systems under the Caputo-Fabrizio type piecewise derivative. As an application of the considered system, biological prey-predator model, under piecewise fractional order differential operators having non-singular kernel are also studied.

1. Introduction

Fractional Calculus (FC) is a special area of applied analysis, which explore integrals and derivatives to non-integer orders. It provides the generalization of classical concepts. The notion of (FC) has very remarkable uses in the concept of analytic functions. The generalization of the classical definitions of fractional, For instance, distortion inequalities [1], convolution structures for different subclasses of analytical functions and the works in the research monographs, coefficient

Received: Jul. 1, 2025.

2020 *Mathematics Subject Classification.* 34A08.

Key words and phrases. dynamical system; Cauchy type problem; prey-predator model; Caputo-Fabrizio derivative (CFD); piecewise fractional order derivative; non-singular kernel.

https://doi.org/10.28924/2291-8639-23-2025-301
ISSN: 2291-8639

estimates, and characterization properties [2]. It plays a vital role in mathematical modeling, being very applicable in engineering and some other areas of science [3–6]. During the last few decades, it is becomes an active research field and got the attention of many researchers. The time fractional derivative possesses a memory effect because the information it contains about the functions is from earlier points. Such types of derivatives are considered historical and non-local distributed properties, that are needed for more accurate and better explanation and understanding of dynamic and complex system behavior [7–9].

The furthermost, meaningful definitions are Riemann-Liouville and Caputo [10], fractional derivatives; however, despite the results calculated by Riemann-Liouville and Caputo fractional derivatives having limitations, such as the singularity of their kernel at the endpoint of the interval. To overcome this problem, the Caputo-Fabrizio derivative (CFD) in [11], is studied, which is a new derivative with fractional order having no singularity in its kernel. The main benefit of Caputo-Fabrizio derivative is that, it uses an exponential kernel instead of a power-law kernel, making it easier to use in real-world applications, numerical calculations, and theoretical analysis. Based on this new approach (CFD), some interesting results have been investigated in [12, 13].

The Cauchy problem has prevalent applications across numerous an engineering and scientific disciplines. The study of partial differential equations (PDEs) and their characteristics is greatly aided by the presence of Cauchy problems [14, 15]. Numerous mathematical methods, such the Fourier and Laplace transforms, which are used to solve PDEs in a variety of situations, have also been developed as a result of it. Fixed point theory plays an important role in non-linear functional analysis. In metric spaces, S. Banach developed the fixed point theorem, also known as the Banach contraction principle, which is used in the existence theory.This theorem ensures that fixed points exist uniquely for self-mappings. The investigation of real-world problems through mathematical models usually involves the use of differential operators. Since real world problems involve major or short term changes that often cannot be explained by using usual ordinary or fractional operators [16], because usual fractional order operators also involve long memory concepts rather than short memory. This difficulty mostly causes the production of insufficient information, and hence the phenomenon cannot be well explained. Therefore, recently researchers pointed out that using piecewise calculus instead of classical or fractional calculus can explain the said effect more brilliantly [17–19].

As these operators are built on kernels that exhibit some qualities in nature, some of these could be captured using the idea of piecewise differentiation and integration. Examples include processes that resemble the generalized Mittag-Leffler function, power law processes, and fading memory processes. A coupled system of Cauchy type problems under the piecewise derivative in the Caputo sense was studied by Shah et al. [20], keeping in mind the significance of the novel idea of piecewise approach of fractional order differential equations. In the same way, in [21], a Cauchy type non local dynamical coupled system under the Caputo piecewise derivative was studied.

Motivated by the above contribution, in this work we are going to study the existence results for the following general Cauchy problem of piecewise equation with Caputo-Fabrizio derivative under initial conditions as

$$\begin{cases} {}^{PCF}\mathrm{D}^{\sigma}\mathcal{R}(t) = F_1(t, \mathcal{R}(t), \mathcal{S}(t));\ t \in [0, T], \\ {}^{PCF}\mathrm{D}^{\sigma}\mathcal{S}(t) = F_2(t, \mathcal{R}(t), \mathcal{S}(t));\ t \in [0, T], \\ \mathcal{R}(0) = \mathcal{R}_0, \\ \mathcal{S}(0) = \mathcal{S}_0, \end{cases} \tag{1.1}$$

where ${}^{PCF}\mathrm{D}^{\sigma}$ represents the Piecewise Caputo-Fabrizio derivative of order $0 < \sigma \leq 1$, and $F_1,\ F_2 :\ [0, T] \times \mathcal{IR} \times \mathcal{IR} \longrightarrow \mathcal{IR}$, are given to continuous functions. Morovere, the U.H and generalized U.H stability results are also investigated for the considered systems under the Caputo-Fabrizio type piecewise derivative. As an application of the considered system, prey-predator model under piecewise fractional order differential operators having non-singular kernel are also studied.

2. Preliminaries

Next we recall some basic definitions and results of piecewise fractional calculus which can be found in [20].

Definition 2.1. *If* f *is a continuous function, then the fractional-order piecewise derivative of the classical and exponential decay kernel* $\sigma \in (0, 1]$ *is defined as*

$${}^{PCF}_{0}\mathrm{D}^{\sigma}_{x}\mathrm{f}(t) = \begin{cases} \mathrm{f}'(t); & if\ t \in \beta_1 = [0, t_1], \\ {}^{CF}_{t_1}\mathrm{D}^{\sigma}_{t}\mathrm{f}(t); & if\ t \in \beta_2 = (t_1, T]. \end{cases}$$

Here, ${}^{CF}_{t_1}D^{\sigma}_{t}$ *represents a Caputo-Fabrizio derivative (CFD), for* $t \in \beta_2$ *which is defined as*

$${}^{CF}_{t_1}\mathrm{D}^{\sigma}_{t}\mathrm{f}(t) = \frac{M(\sigma)}{1-\sigma}\int_{t_1}^{t} exp\left(\frac{-\sigma(t-x)}{1-\sigma}\right)\left[\frac{df(x)}{dx}\right]dx,$$

where $\mathrm{M}(\sigma)$ *is the normalization function and* $\mathrm{M}(0) = \mathrm{M}(1) = 1$.

Definition 2.2. *Consider a continuous function* f, *then the piecewise integral with fractional order* $\sigma \in (0, 1]$ *is defined by*

$${}^{PCF}_{0}\mathrm{I}^{\sigma}_{t}\mathrm{f}(t) = \begin{cases} \displaystyle\int_0^{t_1} f(x)dx;\ \ if\ t \in \beta_1 = [0, t_1], \\ \displaystyle\frac{1-\sigma}{M(\sigma)}f(t) + \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} f(x)dx;\ if\ t \in \beta_2 = (t_1, T]. \end{cases}$$

Definition 2.3. *[21] Suppose f be a continuous function, then piecewise Riemann Liouville integral, with order $\sigma \in (0,1]$, is defined as*

$$
{}_0^{PRL}\mathrm{I}_t^{\sigma}\mathrm{f}(t) = \begin{cases} \displaystyle\int_0^{t_1} f(x)dx; \ \ if\ t \in \beta_1 = [0,t_1], \\ \displaystyle\frac{1}{\Gamma(\sigma)}\int_{t_1}^{t}(t-x)^{1-\sigma}f(x)dx; \ if\ t \in \beta_2 = (t_1,T]. \end{cases}
$$

Clearly, ${}^{PRL}I^{\sigma}$ represents ordinary integral in $[0,t_1]$, and Riemann-Liouville integral in $(t_1,T]$.

Definition 2.4. *[21] If f is a continuous function, then piecewise Caputo fractional derivatives is defined as*

$$
{}_0^{PC}\mathrm{D}_t^{\sigma}\mathrm{f}(t) = \begin{cases} \mathrm{f}'(t); & if\ \ t \in \beta_1 = [0,t_1], \\ {}_{t_1}^{C}\mathrm{D}_t^{\sigma}\mathrm{f}(t); & if\ \ t \in \beta_2 = (t_1,T]. \end{cases}
$$

Here, ${}_{t_1}^{C}D_t^{\sigma}$ represents a Caputo derivative, for $t \in \beta_2$.

Lemma 2.1. *If f is a continuous function, then the solution of the following problem under piecewise equation with CFD,*

$$
{}^{PCF}\mathrm{D}_t^{\sigma}\mathrm{y}(t) = \mathrm{f}(t); \ \textit{is given by}
$$

$$
\mathrm{y}(t) = \begin{cases} y(0) + \displaystyle\int_0^{t_1} f(x)dx; \ \ if\ t \in \beta_1 = [0,t_1], \\ y(t_1) + \displaystyle\frac{1-\sigma}{M(\sigma)}f(t) + \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} f(x)dx; \ if\ t \in \beta_2 = (t_1,T]. \end{cases}
$$

Theorem 2.1. *[20] In $Z \times Z$, if $E \in Z \times Z$ is a closed, convex, and non-empty subset, then $A = (A_1, B_1)$ and $B = (A_2, B_2)$ operators exist such that:*
1). $A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S}) \in E; \forall(\mathcal{R},\mathcal{S}) \in \mathbb{Z} \times \mathbb{Z}$.
2).When B is a completely continuous operator and A is contraction. Then there exist at least one fixed point $(\mathcal{R},\mathcal{S})$, such that $A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S}) = (\mathcal{R},\mathcal{S})$.

3. Existence Results

In this section we will study the existence for the consider model (1.1), for this we need the following definitions. Let

$$
Z_1 = \{\mathcal{R} : [0,T] \to IR; \mathcal{R} \in C[0,T]\},
$$
$$
Z_2 = \{\mathcal{S} : [0,T] \to IR; \mathcal{S} \in C[0,T]\},
$$

be the Banach spaces and the norm defined by $\|\mathcal{R}(t)\| = \sup_{t\in[0,T]} |\mathcal{R}(t)|$, $\|\mathcal{S}(t)\| = \sup_{t\in[0,T]} |\mathcal{S}(t)|$ respectively. Another Banach space is the product of Z_1 and Z_2, with a norm defined as follows:

$$
\| (\mathcal{R},\mathcal{S}) \| = \|\mathcal{R}\| + \|\mathcal{S}\|.
$$

Lemma 3.1. *By using Lemma (2.1), the solution of the proposed dynamical system (1.1) is given by*

$$\begin{cases} \mathcal{R}(t) = \begin{cases} \mathcal{R}_0 + \int_0^{t_1} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx; & if\ t \in \beta_1, \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} F_1(t, \mathcal{R}(t), \mathcal{S}(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx; & if\ t \in \beta_2. \end{cases} \\ \mathcal{S}(t) = \begin{cases} \mathcal{S}_0 + \int_0^{t_1} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx; & if\ t \in \beta_1, \\ \mathcal{S}(x_1) + \frac{1-\sigma}{M(\sigma)} F_2(t, \mathcal{R}(t), \mathcal{S}(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx; & if\ t \in \beta_2. \end{cases} \end{cases}$$

Further, we will utilize the following assumptions:

(H_1). For fixed real numbers $L_{F_i} > 0$, with $F_i : [0, \eta] \times IR \times IR \to IR$, and at each $(\mathcal{R}, \mathcal{S})$, $(\bar{\mathcal{R}}, \bar{\mathcal{S}})$ in $Z_1 \times Z_2$, with $i = 1, 2$. We have

$$|F_i(t, \mathcal{R}, \mathcal{S}) - F_i(t, \bar{\mathcal{R}}, \bar{\mathcal{S}})| \leq L_{F_i}\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\}.$$

(H_2). Let for constants $a > 0$, $C_{F_i} > 0$, $0 < D_{F_i} < 1$, where i=1, 2. We have

$$|F_i(t, \mathcal{R}(t), \mathcal{S}(t))| \leq a_{F_i}(t) + C_{F_i}(t)|\mathcal{R}(t)| + D_{F_i}|\mathcal{S}(t)|.$$

Moreover, we assume that

$$a^* = \sup_{t\in\beta} |a_{F_i}(t)|,\ b^* = \sup_{t\in\beta} |C_{F_i}(t)|,\ c^* = \sup_{t\in\beta} |D_{F_i}(t)| < 1,\ \beta = [0, T].$$

To derive existence theory, we used the following operators.

$$O = (O_1, O_2) : Z_1 \times Z_2 \to Z_1 \times Z_2;\ by$$

$$O(\mathcal{R}, \mathcal{S}) = (O_1(\mathcal{R}), O_2(\mathcal{S}))$$

These operators further can be expressed as:

$$O_1(\mathcal{R}, \mathcal{S}) = \begin{cases} \mathcal{R}_0 + \displaystyle\int_0^{t_1} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx;\ \ if\ t \in \beta_1, \\ \mathcal{R}(t_1) + \dfrac{1-\sigma}{M(\sigma)} F_1(t, \mathcal{R}(t), \mathcal{S}(t)) + \dfrac{\sigma}{M(\sigma)} \displaystyle\int_{t_1}^{t} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx;\ if\ t \in \beta_2. \end{cases} \tag{3.1}$$

$$O_2(\mathcal{R}, \mathcal{S}) = \begin{cases} \mathcal{S}_0 + \displaystyle\int_0^{t_1} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx;\ \ if\ t \in \beta_1, \\ \mathcal{S}(t_1) + \dfrac{1-\sigma}{M(\sigma)} F_2(t, \mathcal{R}(t), \mathcal{S}(t)) + \dfrac{\sigma}{M(\sigma)} \displaystyle\int_{t_1}^{t} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx;\ if\ t \in \beta_2. \end{cases} \tag{3.2}$$

Theorem 3.1. *The proposed coupled system (1.1), has a unique solution under the assumption* (H_1), *if* $\max\{K_1, K_2\} < 1$, *where*

$$K_1 = t_1(L_{F_1} + L_{F_2}),$$
$$K_2 = (L_{F_1} + L_{F_2})\left(\frac{1 - \sigma + \sigma(T - t_1)}{M(\sigma)}\right).$$

Proof. Let $\mathcal{R}, \mathcal{S}, \bar{\mathcal{R}}, \bar{\mathcal{S}}$ in $Z_1 \times Z_2$, then proceeds the following cases.
Case 1. When $t \in [0, t_1]$, then from (3.1).

$$\begin{aligned} |O_1(\mathcal{R}, \mathcal{S}) - O_1(\bar{\mathcal{R}}, \bar{\mathcal{S}})| &\leq \int_0^{t_1} |F_1(x, \mathcal{R}, \mathcal{S}) - F_1(x, \bar{\mathcal{R}}, \bar{\mathcal{S}})| dx \\ &\leq \int_0^{t_1} L_{F_1}\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\} dx \\ &\leq t_1 L_{F_1}\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\} \end{aligned}$$

by taking supremum, we get

$$\|O_1(\mathcal{R}, \mathcal{S}) - O_1(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq t_1 L_{F_1}\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\} \tag{3.3}$$

Similarly

$$\|O_2(\mathcal{R}, \mathcal{S}) - O_2(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq t_1 L_{F_2}\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\} \tag{3.4}$$

Adding (3.3) and (3.4), after simplification,we get

$$\|O(\mathcal{R}, \mathcal{S}) - O(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq t_1(L_{F_1} + L_{F_2})\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\}$$

Since $K_1 = t_1(L_{F_1} + L_{F_2})$, therefore,

$$\|O(\mathcal{R}, \mathcal{S}) - O(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq K_1\{|\mathcal{R} - \bar{\mathcal{R}}| + |\mathcal{S} - \bar{\mathcal{S}}|\} \tag{3.5}$$

Case 2. When $t \in (t_1, T]$, then from (3.2).

$$|O_1(\mathcal{R}, \mathcal{S}) - O_1(\bar{\mathcal{R}}, \bar{\mathcal{S}})| \leq \frac{1-\sigma}{M(\sigma)} |F_1(t, \mathcal{R}, \mathcal{S}) - F_1(t, \bar{\mathcal{R}}, \bar{\mathcal{S}})| + \frac{\sigma}{M(\sigma)} \int_{t_1}^{T} |F_1(x, \mathcal{R}, \mathcal{S}) - F_1(x,, \bar{\mathcal{R}}, \bar{\mathcal{S}})| dx$$

Using (H_1)

$$|O_1(\mathcal{R}, \mathcal{S}) - O_1(\bar{\mathcal{R}}, \bar{\mathcal{S}})| \leq \frac{1-\sigma}{M(\sigma)} L_{F_1}\{\|\mathcal{R} - \bar{\mathcal{R}}\| + \|\mathcal{S} - \bar{\mathcal{S}}\|\} + \frac{\sigma}{M(\sigma)} \int_{t_1}^{T} L_{F_1}\{\|\mathcal{R} - \bar{\mathcal{R}}\| + \|\mathcal{S} - \bar{\mathcal{S}}\|\} dx$$

Taking supermum of both side and simplifying.

$$\|O_1(\mathcal{R}, \mathcal{S}) - O_1(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq L_{F_1}\left(\frac{1 - \sigma + \sigma(T - t_1)}{M(\sigma)}\right)\{\|\mathcal{R} - \bar{\mathcal{R}}\| + \|\mathcal{S} - \bar{\mathcal{S}}\|\} \tag{3.6}$$

On the same way

$$\|O_2(\mathcal{R}, \mathcal{S}) - O_2(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq L_{F_2}\left(\frac{1 - \sigma + \sigma(T - t_1)}{M(\sigma)}\right)\{\|\mathcal{R} - \bar{\mathcal{R}}\| + \|\mathcal{S} - \bar{\mathcal{S}}\|\} \tag{3.7}$$

Combining (3.6) and (3.7), and after simplication, we get

$$\|O(\mathcal{R}, \mathcal{S}) - O(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq (L_{F_1} + L_{F_2})\left(\frac{1 - \sigma + \sigma(T - t_1)}{M(\sigma)}\right)\{\|\mathcal{R} - \bar{\mathcal{R}}\| + \|\mathcal{S} - \bar{\mathcal{S}}\|\}$$

since

$$K_2 = (L_{F_1} + L_{F_2})\left(\frac{1 - \sigma + \sigma(T - t_1)}{M(\sigma)}\right).$$

$$\|O(\mathcal{R}, \mathcal{S}) - O(\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq K_2\{\|(\mathcal{R}, \mathcal{S}) + (\bar{\mathcal{R}}, \bar{\mathcal{S}})\|\} \tag{3.8}$$

From relations (3.5) and (3.8), we have

$$\|O(\mathcal{R},\mathcal{S}) - O(\bar{\mathcal{R}},\bar{\mathcal{S}})\| \leq \begin{cases} K_1\{\|(\mathcal{R},\mathcal{S}) - (\bar{\mathcal{R}},\bar{\mathcal{S}})\|\} \\ K_2\{\|(\mathcal{R},\mathcal{S}) - (\bar{\mathcal{R}},\bar{\mathcal{S}})\|\} \end{cases}$$

It is obvious that O is contraction if $\max\{K_1, K_2\} < 1$. According to the Banach contraction principle, O has a single fixed point, which is the solutions of dynamical system (1.1). □

Theorem 3.2. *Under the assumptions (H_1)-(H_2), the proposed coupled system (1.1), has at least one solution if the condition* $\max\{L_{F_1} + L_{F_2},\ \frac{1-\sigma}{M(\sigma}(L_{F_1} + L_{F_2})\} < 1$ *hold.*

Proof. Here, we first defined the operators $A = (A_1, B_1)$ and $B = (A_2, B_2)$.

$$A_1(\mathcal{R}) = \begin{cases} \mathcal{R}_0 + F_1(t,\mathcal{R},\mathcal{S});\ t \in \beta_1, \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma}F_1(t,\mathcal{R},\mathcal{S});\ t \in \beta_2, \end{cases}$$

$$B_1(\mathcal{S}) = \begin{cases} \mathcal{S}_0 + F_2(t,\mathcal{R},\mathcal{S});\ t \in \beta_1, \\ \mathcal{S}(t_1) + \frac{1-\sigma}{M(\sigma)}F_2(t,\mathcal{R},\mathcal{S});\ t \in \beta_2, \end{cases}$$

$$A_2(\mathcal{R}) = \begin{cases} \int_0^{t_1} F_1(x,\mathcal{R},\mathcal{S})dx;\ t \in \beta_1, \\ \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} F_1(x,\mathcal{R},\mathcal{S})dx;\ t \in \beta_2, \end{cases}$$

$$B_2(\mathcal{S}) = \begin{cases} \int_0^{t_1} F_2(x,\mathcal{R},\mathcal{S})dx;\ t \in \beta_1, \\ \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} F_2(x,\mathcal{R},\mathcal{S})dx;\ t \in \beta_2. \end{cases}$$

Case(1). Let E be a non-empty, closed, convex subset of $Z \times Z$, such that

$$E = \{(\mathcal{R},\mathcal{S}) \in Z \times Z : \|(\mathcal{R},\mathcal{S})\| \leq r\},$$

where

$$r \geq \frac{t^* a + \lambda}{1 - \rho x^*},\ \lambda = |\mathcal{R}_0| + |\mathcal{S}_0|,\ a = a^*_{F_1} + a^*_{F_2},\ \rho = \max\{a^*_{F_1} + a^*_{F_2},\ c^*_{F_1} + c^*_{F_2}\},\ x^* = t_1 + 1.$$

Step(1) When $t \in [0, t_1]$, then we have

$$\begin{aligned} \|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \leq & \sup_{t\in\beta_1}\Big\{|\mathcal{R}_0| + |F_1(t,\mathcal{R},\mathcal{S})| + |\mathcal{S}_0| + |F_2(t,\mathcal{R},\mathcal{S})| \\ & + \int_0^{t_1} |F_1(x,\mathcal{R},\mathcal{S})|dx + \int_0^{t_1} |F_2(x,\mathcal{R},\mathcal{S})|dx\Big\}, \\ \leq & \sup_{t\in\beta_1}\Big\{\big|\mathcal{R}_0| + |\mathcal{S}_0| + a_{F_1}(t) + C_{F_1}|\mathcal{R}| + D_{F_1}|\mathcal{S}| + a_{F_2}(t) + C_{F_2}|\mathcal{R}| + D_{F_2}|\mathcal{S}| \\ & + (a_{F_1}(t) + C_{F_1}|\mathcal{R}| + D_{F_1}|\mathcal{S}|)t_1 + (a_{F_2}(t) + C_{F_2}|\mathcal{R}| + D_{F_2}|\mathcal{S}|)t_1\Big\}, \\ = & \Big\{|\mathcal{R}_0| + |\mathcal{S}_0| + a^*_{F_1}(t) + b^*_{F_1}\|\mathcal{R}\| + c^*_{F_1}\|\mathcal{S}\| + a^*_{F_2}(t) + b^*_{F_2}\|\mathcal{R}\| + c^*_{F_2}\|\mathcal{S}\| + (a^*_{F_1}(t) \\ & + b^*_{F_1}\|\mathcal{R}\| + c^*_{F_1}\|\mathcal{S}\|)t_1 + (a^*_{F_2}(t) + b^*_{F_2}\|\mathcal{R}\| + c^*_{F_2}\|\mathcal{S}\|)t_1\Big\}, \end{aligned}$$

$$= \Big\{|\mathcal{R}_0| + |\mathcal{S}_0| + (a^*_{F_1}(t) + a^*_{F_2}(t))(t_1+1) + (b^*_{F_1} + b^*_{F_2})\|\mathcal{R}\|(t_1+1)$$
$$+ (c^*_{F_1}| + c^*_{F_2}|)|\mathcal{S}\|(t_1+1)\Big\},$$

Since

$$\lambda = |\mathcal{R}_0| + |\mathcal{S}_0|,\ a = a^*_{F_1}(t) + a^*_{F_2}(t),\ b = b^*_{F_1} + b^*_{F_2},\ c = c^*_{F_1}| + c^*_{F_2}|,\ t^* = t_1 + 1.$$

Therefore,

$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le \{|\lambda + at^* + bt^*\|\mathcal{R}\| + ct^*\|\mathcal{S}\|\},$$
$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le \{|\lambda + at^* + t^*\rho\|(\mathcal{R},\mathcal{S})\|\},$$

$$As,\ r \ge \frac{t^*a+\lambda}{1-\theta t^*}\ So;$$
$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le r$$
$$A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S}) \in E$$

Step(2). When $t \in (t_1, T]$; then we have

$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le \sup_{t\in\beta_2}\{|\mathcal{R}(t_1)| + \frac{1-\sigma}{M(\sigma)}|F_1(t,\mathcal{R},\mathcal{S})| + \mathcal{S}_(t_1) + \frac{1-\sigma}{M(\sigma)}|F_2(t,\mathcal{R},\mathcal{S})| +$$
$$\frac{\sigma}{M(\sigma)}\int_{t_1}^{t}|F_1(x,\mathcal{R},\mathcal{S})|dx + \frac{\sigma}{M(\sigma)}\int_{t_1}^{t}|F_2(x,\mathcal{R},\mathcal{S})|dx\},$$
$$\le \{|\mathcal{R}(t_1)| + |\mathcal{S}(t_1)| + \frac{1-\sigma}{M(\sigma)}(a^*_{F_1} + a^*_{F_2}) + (b^*_{F_1} + b^*_{F_2})\|\mathcal{R}\| + (c^*_{F_1} + c^*_{F_2})\|\mathcal{S}\|$$
$$+\frac{\sigma}{M(\sigma)}\left((a^*_{F_1} + a^*_{F_2}) + (b^*_{F_1} + b^*_{F_2})\|\mathcal{R}\| + (c^*_{F_1} + c^*_{F_2})\|\mathcal{S}\|\right)(T-t_1),$$
$$Let\ \beta = |\mathcal{R}(t_1)| + |\mathcal{S}(t_1)|$$
$$\le \left\{\beta + \frac{1-\sigma}{M(\sigma)}(a + b\|\mathcal{R}\| + c\|\mathcal{S}\|) + \frac{\sigma}{M(\sigma)}(a + b\|\mathcal{R}\| + c\|\mathcal{S}\|)(T-t_1)\right\},$$

Since

$$max\{b,c\} = \rho,$$

$$\le \left\{\beta + \frac{1-\sigma}{M(\sigma)}(a + \rho\|(\mathcal{R},\mathcal{S})\|) + \frac{\sigma}{M(\sigma)}(a + \rho\|(\mathcal{R},\mathcal{S})\|)(T-t_1)\right\} \le r,$$

$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le \{|\lambda + ax^* + x^*\rho\|(\mathcal{R},\mathcal{S})\|\},$$

$$As,\ r \ge \frac{t^*a+\lambda}{1-\rho t^*}\ So;$$
$$\|A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S})\| \le r,$$
$$A(\mathcal{R},\mathcal{S}) + B(\mathcal{R},\mathcal{S}) \in E.$$

Case(2). We show that A is contraction, for this

$$\|A_1\mathcal{R}-A_1\bar{\mathcal{R}}\| \leq \sup_{t\in\beta_1}\begin{cases}|F_1(t,\mathcal{R},\mathcal{S})-F_1(t,\bar{\mathcal{R}},\bar{\mathcal{S}})|;\ t\in\beta_1,\\ \frac{1-\sigma}{M(\sigma)}|F_1(t,\mathcal{R},\mathcal{S})-F_1(t,\bar{\mathcal{R}},\bar{\mathcal{S}})|;\ t\in\beta_2.\end{cases}$$

$$\|A_1\mathcal{R}-A_1\bar{\mathcal{R}}\| \leq \begin{cases}L_{F_1}\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_1,\\ \frac{1-\sigma}{M(\sigma)}L_{F_1}\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_2.\end{cases}$$

similarly

$$\|B_1\mathcal{R}-B_1\bar{\mathcal{R}}\| \leq \begin{cases}L_{F_2}\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_1,\\ \frac{1-\sigma}{M(\sigma)}L_{F_2}\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_2.\end{cases}$$

$$\|A_1\mathcal{R}-A_1\bar{\mathcal{R}}\|+\|B_1\mathcal{R}-B_1\bar{\mathcal{R}}\| \leq \begin{cases}(L_{F_1}+L_{F_2})\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_1,\\ \frac{1-\sigma}{M(\sigma)}(L_{F_1}+L_{F_2})\{\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|\};\ t\in\beta_2.\end{cases}$$

$$\max\{L_{F_1}+L_{F_2}, \tfrac{1-\sigma}{M(\sigma)}(L_{F_1}+L_{F_2})\}=K<1$$

$$\|A_(\mathcal{R},\mathcal{S})-A(\bar{\mathcal{R}},\bar{\mathcal{S}})\| \leq K\|(\mathcal{R},\mathcal{S})-(\bar{\mathcal{R}},\bar{\mathcal{S}})\|$$

A is contraction. Next we show that $B=(A_2,B_2)$ is bounded.

$$\|A_2\mathcal{R}\| \leq \sup_{t\in\beta}\begin{cases}\int_0^{t_1}|F_1(x,\mathcal{R},\mathcal{S})|dx;\ t\in\beta_1,\\ \dfrac{\sigma}{M(\sigma)}\int_{t_1}^{T}|F_1(x,\mathcal{R},\mathcal{S})|dx;\ t\in\beta_2,\end{cases}$$

$$\|A_2\mathcal{R}\| \leq \sup_{t\in\beta}\begin{cases}\int_0^{t_1}\{|a_{F_1}(t)|+C_{F_1}|\mathcal{R}|+D_{F_1}|\mathcal{S}|\}dx;\ t\in\beta_1,\\ \dfrac{\sigma}{M(\sigma)}\int_{t_1}^{T}\{|a_{F1}(t)|+C_{F_1}|\mathcal{R}|+D_{F_1}|\mathcal{S}|\}dx;\ t\in\beta_2,\end{cases}$$

$$\|A_2\mathcal{R}\| \leq \begin{cases}\int_0^{t_1}\{|a^*_{F1}(t)|+b^*_{F_1}\|\mathcal{R}\|+c^*_{F_1}\|\mathcal{S}\|\}dx;\ t\in\beta_1,\\ \dfrac{\sigma}{M(\sigma)}\int_{t_1}^{T}\{|a^*_{F1}(t)|+b^*_{F_1}\|\mathcal{R}\|+c^*_{F_1}\|\mathcal{S}\|\}dx;\ t\in\beta_2,\end{cases}$$

$$\|A_2\mathcal{R}\| \leq \begin{cases}\{|a^*_{F1}(t)|+b^*_{F_1}\|\mathcal{R}\|+c^*_{F_1}\|\mathcal{S}|\}t_1;\ t\in\beta_1,\\ \frac{\sigma}{M(\sigma)}\{|a^*_{F1}(t)|+b^*_{F_1}\|\mathcal{R}\|+c^*_{F_1}\|\mathcal{S}\|\}(T-t_1);\ t\in\beta_2.\end{cases} \tag{3.9}$$

Similarly

$$\|B_2\mathcal{R}\| \leq \begin{cases}\{|a^*_{F1}(t)|+b^*_{F_2}\|\mathcal{R}\|+c^*_{F_2}\|\mathcal{S}|\}t_1;\ t\in\beta_1,\\ \frac{\sigma}{M(\sigma)}\{|a^*_{F1}(t)|+b^*_{F_2}|\mathcal{R}\||+c^*_{F_2}\|\mathcal{R}\|\}(T-t_1);\ x\in\beta_2.\end{cases} \tag{3.10}$$

Adding (3.9) and (3.10),

$$\|A_2\mathcal{R}\| + \|B_2\mathcal{R}\| \leq \begin{cases} \{(a^*_{F_1} + a^*_{F_2}) + (b^*_{F_1} + b^*_{F_2})\|\mathcal{R}\| + (c^*_{F_1} + c^*_{F_2})\|\mathcal{S}\|)\}t_1;\ t \in \beta_1, \\ \{(a^*_{F_1} + a^*_{F_2}) + (b^*_{F_1} + b^*_{F_2})\|\mathcal{R}\| + (c^*_{F_1} + c^*_{F_2})\|\mathcal{S}\|)\}(T - t_1);\ t \in \beta_2, \end{cases}$$

let $a = a^*_{F_1} + a^*_{F_2},\ b = b^*_{F_1} + b^*_{F_2},\ c = c^*_{F_1} + c^*_{F_2}$

$$\|A_2\mathcal{R}\| + \|B_2\mathcal{R}\| \leq \begin{cases} \{a + b\|\mathcal{R}\| + c\|\mathcal{S}\|)\}t_1;\ t \in \beta_1, \\ \{a + b\|\mathcal{R}\| + c\|\mathcal{S}\|)\}(T - t_1);\ t \in \beta_2, \end{cases}$$

Let $max\{b, c\} = \rho\ and\ \|(\mathcal{R}, \mathcal{S}\| \leq r$,

$$\|B(\mathcal{R}, \mathcal{S})\| \leq \begin{cases} (a + r\rho)t_1;\ t \in \beta_1, \\ \frac{\sigma}{M(\sigma)}(a + r\rho)(T - t_1);\ t \in \beta_2. \end{cases}$$

Let

$$max\{(a + \rho r)x_1, \frac{\sigma}{M(\sigma)}(a + \rho r)(T - t_1)\} = \Omega$$

$$\|B(\mathcal{R}, \mathcal{S})\| \leq \Omega$$

B is bounded. All the conditions are satisfied, so by theorem 2.1, the proposed dynamical system has at least one solution. □

4. Stability Analysis

In this section we will investigate Ulam-Hyers (U-H) and generalized (U-H) stability results for our proposed coupled system. We recall basic definition of Ulam-Hyers stability and its generalization from [20].

Consider the operatpr ψ : Z $\longrightarrow$Z, such that ZZ=Z×Z satisfies

$$\psi(\nu) = \nu, for\ \nu \in Z. \tag{4.1}$$

Definition 4.1. *The solution of the operator (4.1), is U-H stable if for every $\varepsilon > 0$, and let $\nu \in Z$, be any solution of the inequality*

$$\|\nu - \psi(\nu)\| \leq \varepsilon, \tag{4.2}$$

there exist a unique solution $\bar{\nu}$ of (4.2), with a constant $K > 0$, satisfies the following inequality

$$\|\nu - \bar{\nu}\| \leq K\varepsilon \tag{4.3}$$

Definition 4.2. *Our proposed coupled system (1.1), is said to U-H stable if there exist a constant $K_{1,2} = (K_1, K_2)$, such that for some $\varepsilon_{1,2} = (\varepsilon_1 + \varepsilon_2)$ >0, and for every solution $(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{V}}) \in Z_1 \times Z_2$, of the inequalities*

$$\begin{cases} |^{PCF}D^{\sigma}\mathcal{R}(t) - F_1(t, \mathcal{R}(t), \mathcal{S}(t))| < \varepsilon_1, \\ |^{PCF}D^{\sigma}\mathcal{S}(t) - F_2(t, \mathcal{R}(t), \mathcal{S}(t))| < \varepsilon_2. \end{cases} \tag{4.4}$$

There exist a unique solution $(\bar{\mathcal{R}}, \bar{\mathcal{S}}) \in Z_1 \times Z_2$ with

$$|(\mathcal{R}, \mathcal{S}) - (\bar{\mathcal{R}}, \bar{\mathcal{S}})| < K_{1,2}\varepsilon_{1,2} \tag{4.5}$$

Definition 4.3. *Our proposed coupled system (1.1), is said to be generalized Ulam-Hyer stable, if there exist a nondecreasing function $\Psi(\varepsilon) \in C(R^+, R^+)$, with $\Psi(0)=0$, such that for any solution $(\mathcal{R}, \mathcal{S}) \in Z_1 \times Z_2$ of the inequality (4.5), there exist a unique solution $(\bar{\mathcal{R}}, \bar{\mathcal{S}}) \in Z_1 \times Z_2$ of (1.1), which satisfies*

$$\|(\mathcal{R}, \mathcal{S}) - (\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq K_\psi \Psi(\varepsilon) \tag{4.6}$$

Remark 4.1. *Consider a function $h:[0, T] \to R$ is independent of the solution $(\mathcal{U}, \mathcal{V}) \in Z_1 \times Z_2$, such that $h(0)=0$,then*

$$|h(t)| < \varepsilon;\ t \in [0, T].$$

Lemma 4.1. *The solution of the problem*

$$\begin{cases} {}^{PCF}\mathrm{D}^\sigma \mathcal{R}(t) = F_1(t, \mathcal{R}(t), \mathcal{R}(t)) + h(t), \\ {}^{PCF}\mathrm{D}^\sigma \mathcal{S}(t) = F_2(t, \mathcal{R}(t), \mathcal{S}(t)) + h(t). \end{cases} \tag{4.7}$$

Satisfies the following relations.

$$\begin{cases} |\mathcal{R}(t) - \left(\mathcal{R}_0 + \int_0^{t_1} F_1(x, \mathcal{R}, \mathcal{S})dx\right)| \leq t_1\varepsilon_1, \\ |\mathcal{R}(t) - \left(\mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} F_1(t, \mathcal{R}, \mathcal{S}) + \int_{t_1}^{t} F_1(x, \mathcal{R}, \mathcal{S})dx\right)| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_1. \end{cases}$$

and

$$\begin{cases} |\mathcal{S}(t) - \left(\mathcal{S}_0 + \int_0^{t_1} F_2(x, \mathcal{R}, \mathcal{S})dx\right)| \leq t_1\varepsilon_2, \\ |\mathcal{S}(t) - \left(\mathcal{S}(t_1) + \frac{1-\sigma}{M(\sigma)} F_2(t, \mathcal{R}, \mathcal{S}) + \int_{t_1}^{t} F_2(x, \mathcal{R}, \mathcal{S})dx\right)| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_2. \end{cases}$$

Proof. By using lemma (2.1), the solution of system (4.7) given by

$$\begin{cases} \mathcal{R}(t) = \begin{cases} \mathcal{R}_0 + \int_0^{t_1} F_1(x, \mathcal{R}, \mathcal{S})dx + \int_0^{t_1} h(x)dx; & \text{if } t \in \beta_1, \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)}(F_1(t, \mathcal{R}, \mathcal{S}) + h(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} (F_1(x, \mathcal{R}, \mathcal{S}) + h(x))dx; & \text{if } t \in \beta_2. \end{cases} \\ \mathcal{S}(t) = \begin{cases} \mathcal{S}_0 + \int_0^{t_1} F_2(x, \mathcal{R}, \mathcal{S})dx + \int_0^{t_1} h(x)dx; & \text{if } t \in \beta_1, \\ \mathcal{S}(t_1) + \frac{1-\sigma}{M(\sigma)}(F_2(t, \mathcal{R}, \mathcal{S}) + h(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} (F_2(x, \mathcal{R}, \mathcal{S}) + h(x))dx; & \text{if } t \in \beta_2. \end{cases} \end{cases}$$

Applying remark (4.1), we can get

$$\begin{cases} |\mathcal{R}(t) - \left(\mathcal{R}_0 + \int_0^{t_1} F_1(x, \mathcal{R}, \mathcal{S})dx\right)| \leq t_1\varepsilon_1, \\ |\mathcal{R}(t) - \left(\mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} F_1(t, \mathcal{R}, \mathcal{S}) + \int_{t_1}^{t} F_1(x, \mathcal{R}, \mathcal{S})dx\right)| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_1. \end{cases}$$

and

$$\begin{cases} |\mathcal{S}(t) - \left(\mathcal{S}_0 + \int_0^{t_1} F_2(x, \mathcal{R}, \mathcal{S})dx\right)| \leq t_1\varepsilon_2, \\ |\mathcal{S}(t) - \left(\mathcal{S}(t_1) + \frac{1-\sigma}{M(\sigma)} F_2(t, \mathcal{R}, \mathcal{S}) + \int_{t_1}^{t} F_2(x, \mathcal{R}, \mathcal{S})dx\right)| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_2. \end{cases}$$

□

Theorem 4.1. *The solution of the proposed system (1.1), is Ulam-Hyers stable if the following conditions are hold*

$$K_1 = t_1(L_{F_1} + L_{F_2}) < 1,$$
$$K_2 = (L_{F_1} + L_{F_2})(\frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}) < 1.$$

Proof. To prove the above theorem we used assumption (H_1) and lemma (4.7). Let $(\mathcal{R}, \mathcal{S}) \in Z_1 \times Z_2$ is a unique solution and $(\bar{\mathcal{R}}, \bar{\mathcal{S}}) \in Z_1 \times Z_2$ be any solution of the coupled system (1.1).

Case(1). When $t \in \beta_1 = [o, t_1]$,

$$\begin{aligned}
\|\mathcal{R}-\bar{\mathcal{R}}\| &= \sup_{t\in\beta_1} |\mathcal{R} - (\mathcal{R}_0 + \int_0^{t_1} [F_1(x, \bar{\mathcal{R}}, \bar{\mathcal{S}}) + g(x)]dx)|, \\
&\leq t_1\varepsilon_1 + \sup_{t\in\beta_1} |\mathcal{R} - (\mathcal{R}_0 + \int_0^{t_1} [F_1(x, \mathcal{R}, \mathcal{S})dx)|, \\
&\leq t_1\varepsilon_1 + \sup_{t\in\beta_1} | + \int_0^{t_1} [F_1(x, \mathcal{R}, \mathcal{S}) + F_1(x, \bar{\mathcal{R}}, \bar{\mathcal{S}})]dx|,
\end{aligned}$$

$$\|\mathcal{R}-\bar{\mathcal{R}}\| \leq t_1\varepsilon_1 + t_1 L_{F_1}(\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\|), \tag{4.8}$$

Similarly

$$\|\mathcal{S}-\bar{\mathcal{S}}\| \leq t_1\varepsilon_2 + x_1 L_{F_2}(\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\|). \tag{4.9}$$

Adding (4.8) and (4.9) we get,

$$\begin{aligned}
\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\| &\leq t_1(\varepsilon_1+\varepsilon_2) + t_1(L_{F_1} + L_{F_2})(\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\|), \\
&\leq \frac{t_1}{1 - t_1(L_{F_1} + L_{F_2})}\varepsilon_{1,2},
\end{aligned}$$

$$\|(\mathcal{R}, \mathcal{S}) - (\bar{\mathcal{R}}, \bar{\mathcal{S}})\| \leq \frac{t_1}{1-K_1}\varepsilon_{1,2} \tag{4.10}$$

Case(2). When $x \in \beta_2 = (t_1, T]$, we have

$$\begin{aligned}
\|\mathcal{R}-\bar{\mathcal{R}}\| &= \sup_{t\in\beta_2} |\mathcal{R} - (\mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)}[F_1(t, \bar{\mathcal{R}}, \bar{\mathcal{S}}) + h(t)] + \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} [F_1(x, \bar{\mathcal{R}}, \bar{\mathcal{S}}) + h(x)]dx)|, \\
&\leq \frac{1-\sigma}{M(\sigma)}\varepsilon_1 + \frac{\sigma(T-t_1)}{M(\sigma)}\varepsilon_1 + \sup_{t\in\beta_2} |\mathcal{R} - (\mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)}F_1(t, \mathcal{R}, \mathcal{S}) \\
&+ \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} F_1(x, \mathcal{R}, \mathcal{S})dx)| + \frac{1-\sigma}{M(\sigma)}\sup_{t\in\beta_2} |F_1(t, \mathcal{R}, \mathcal{S}) - F_1(t, \bar{\mathcal{R}}, \bar{\mathcal{S}})| \\
&+ \frac{\sigma}{M(\sigma)}\int_{t_1}^{t} |F_1(x, \mathcal{R}, \mathcal{S}) - F_1(x, \bar{\mathcal{R}}, \bar{\mathcal{S}})|dx, \\
&\leq \frac{1-\sigma}{M(\sigma)}\varepsilon_1 + \frac{\sigma(T-t_1)}{M(\sigma)}\varepsilon_1 + \frac{1-\sigma}{M(\sigma)}L_{F_1}(\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\|) \\
&+ \frac{\sigma(T-t_1)}{M(\sigma)}L_{F_1}(\|\mathcal{R}-\bar{\mathcal{R}}\| + \|\mathcal{S}-\bar{\mathcal{S}}\|),
\end{aligned}$$

$$\|\mathcal{R}-\bar{\mathcal{R}}\| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_1 + \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}L_{F_1}(\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|). \tag{4.11}$$

On the same way

$$\|\mathcal{S}-\bar{\mathcal{S}}\| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\varepsilon_2 + \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}L_{F_2}(\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|). \tag{4.12}$$

Adding (4.11) and (4.12),

$$\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\| \leq \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}(\varepsilon_1+\varepsilon_2) + \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}(L_{F_1}+L_{F_2})\|\mathcal{R}-\bar{\mathcal{R}}\|+\|\mathcal{S}-\bar{\mathcal{S}}\|.$$

$$Let\ \Delta = \frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)},\ then\ K_2 = \Delta(L_{F_1}+L_{F_2}),$$

$$\|(R,S)-(\bar{R},\bar{S})\| \leq \frac{\Delta}{1-K_2))}\varepsilon_{1,2} \tag{4.13}$$

$$Let\ K = max\{\frac{x_1}{1-K_1}, \frac{\Delta}{1-K_2))}\},$$

From (4.10) and (4.13), we have

$$\|(\mathcal{R},\mathcal{S})-(\bar{\mathcal{R}},\bar{\mathcal{S}})\| \leq K\varepsilon_{1,2}, \tag{4.14}$$

Hence the solution is Ulam-Hyer stable. Replacing $\varepsilon_{1,2}$ by $\Psi(\varepsilon)$ in Eq.(25), we have

$$\|(\mathcal{R},\mathcal{S})-(\bar{\mathcal{R}},\bar{\mathcal{S}}\| \leq K\Psi(\varepsilon) \tag{4.15}$$

with $\Psi(0)$=0. Which shows that the solution is generalized Ulam-Hyer stable. □

5. Application of Cauchy problem to Prey-predator model

Consider the following coupled system of Prey-predator model [22].

$$\begin{cases} {}^{PCF}\mathrm{D}^{w}\mathcal{R}(t) = a_1\mathcal{R}(t) - b_1\mathcal{R}(t)\mathcal{S}(t) = \phi_1(t,R(t),S(t)), \\ {}^{PCF}\mathrm{D}^{w}\mathcal{S}(t) = a_2\mathcal{R}(t)\mathcal{S}(t) - b_2\mathcal{S}(t) = \phi_1(t,R(t),S(t)), \\ \mathcal{R}(0) = \gamma_1, \\ \mathcal{S}(0) = \gamma_2, \end{cases} \tag{5.1}$$

where $w \in (0,1]$ and $\gamma_1, \gamma_2 \geq 0$. Additionally, it is stated that the functions $\phi_i(i=1,2) : \tau \times IR^2 \to IR$ are continuous. The populations of predators and prey are indicated by the variables $\mathcal{S}(t)$ and $\mathcal{R}(t)$, respectively. The maximum per capita growth rate and the impact of predators on the prey growth rate are described by the prey's parameters a_1 and b_1, respectively. The parameters a_2 and b_2 of the predator, respectively, represent the death rate per capita of the predator and the impact of prey on the predator's growth.

Lemma 5.1. *The solution of piecewise differential equation*

$$^{PCF}\mathrm{D}_t^w G(t) = H(t, G(t)),$$

is given by

$$G(t) = \begin{cases} G(0) + \int_0^{t_1} H(x, G(x))dx; \ \ if\ t \in \beta_1 = [0, t_1], \\ G(t_1) + \dfrac{1-w}{M(w)} H(t, G(t)) + \dfrac{w}{M(w)} \displaystyle\int_{t_1}^{t} H(x, G(x))dx; if\ t \in \beta_2 = (t_1, T]. \end{cases}$$

6. Uniqueness and Existence Analysis of the Model

In this section, both uniqueness and existence results for the considered model, under piecewise Caputo- Fabrizio derivatives are presented. For this we further, elaborate lemma (5.1) as:

$$^{PCF}\mathrm{D}_t^w L(t) = \mathcal{N}(t, L(t)), 0 < w \leq 1.$$

$$L(t) = \begin{cases} L_0 + \int_0^{t_1} \mathcal{N}(x, L(x))dx; \ if\ t \in \beta_1 = [0, t_1], \\ L(t_1) + \dfrac{1-w}{M(w)} \mathcal{N}(t, L(t)) + \dfrac{w}{M(w)} \displaystyle\int_{t_1}^{t} \mathcal{N}(x, L(x))dx; if\ t \in \beta_2 = (t_1, T]. \end{cases}$$

where

$$L(t) = \begin{cases} \mathcal{R}(t), \\ \mathcal{S}(t), \end{cases} \quad L(0) = \begin{cases} \gamma_1, \\ \gamma_2, \end{cases} \quad L(t_1) = \begin{cases} \mathcal{R}(t_1), \\ \mathcal{S}(t_1), \end{cases}$$

$$\mathcal{N}(t, L(t)) = \begin{cases} \mathcal{N}_1 = \begin{cases} D\mathcal{R}(t); \ t \in \beta_1, \\ ^{CF}D^w \mathcal{R}(t); \ t \in \beta_2, \end{cases} \\ \mathcal{N}_2 = \begin{cases} D\mathcal{S}(t); \ t \in \beta_1, \\ ^{CF}D^w \mathcal{S}(t); \ t \in \beta_2. \end{cases} \end{cases}$$

Consider the Banach space E_1=C[0,T], such that $0 < T < \infty$ and norm defined by E_1 is

$$||L|| = \max_{t \in [0,T]} \mid L(t) \mid .$$

The Lipschitz and growth condition [23] can be defined as

$$C_1 : \textit{There exist constant} K_{\mathcal{N}} \geq 0, \forall \mathcal{N} \textit{ and } \widetilde{L} \in E_1 \textit{ such that}$$
$$|\mathcal{N}(t, L) - \mathcal{N}(t, \widetilde{L})| \leq K_{\mathcal{N}} |L - \widetilde{L}|.$$

$$C_2 : \textit{There exist constants}, C_{\mathcal{N}} > 0, \textit{ and } \mathcal{N}_{\mathcal{N}} > 0,$$
$$|\mathcal{N}(t, L(t))| \leq C_{\mathcal{N}} |L| + \mathcal{N}_{\mathcal{N}}.$$

Theorem 6.1. *Under the assumption (C_2) the proposed model (5.1), has at least one solution.*

Proof. Let B be a non-empty, convex, and closed subset of E_1 such that

$$B = \{L \in E_1 : ||L|| \leq R_{1,2}\}$$

For any $L \in E_1$, consider the operator $Q : B \to B$

$$|Q(l)| \leq \begin{cases} |l(0)| + \int_0^{t_1} |\mathcal{N}(x, L(x))|dx;\ if\ t \in \beta_1 = [0, t_1], \\ |L(t_1)| + \frac{1-w}{M(w)}|\mathcal{N}(t, l(t))| + \frac{w}{M(w)} \int_{t_1}^{t} |\mathcal{N}(x, L(x)|dx; if\ t \in \beta_2 = (t_1, T]. \end{cases}$$

$$\leq \begin{cases} |L(0)| + \int_0^{t_1} [C_{\mathcal{N}}|L| + \mathcal{N}_{\mathcal{N}}]ds;\ if\ t \in \beta_1, \\ |L(t_1)| + \frac{1-w}{M(w)}[C_{\mathcal{N}}|K| + \mathcal{N}_{\mathcal{N}}] + \frac{w}{M(w)} \int_{t_1}^{t} [C_{\mathcal{N}}|L| + \mathcal{N}_{\mathcal{N}}]dx; if\ t \in \beta_2. \end{cases}$$

$$||Q(L)|| \leq \begin{cases} |L(0)| + [C_{\mathcal{N}}||L|| + \mathcal{N}_{\mathcal{N}}]t_1;\ if\ t \in \beta_1 \\ |L(t_1)| + \frac{1-w}{M(w)}[C_{\mathcal{N}}||L|| + \mathcal{N}_{\mathcal{N}}] + \frac{w}{M(w)}[C_{\mathcal{N}}||L|| + \mathcal{N}_{\mathcal{N}}](t - t_1); if\ t \in \beta_2. \end{cases}$$

$$||Q(L)|| \quad \leq \begin{cases} r_1;\ if\ t\ \in \beta_1 \\ r_2;\ if\ t\ \in \beta_2. \end{cases} \tag{6.1}$$

From (6.1), it is clear that Q(B)⊆B. Further, we show that Q is completely continuous.
Case(1): Let $t_i < t_j \in [0, t_1]$

$$|Q(L)(t_i) - Q(L)(t_j)| \leq \int_0^{t_i} |\mathcal{N}(x, L(x))|dx - \int_0^{t_j} |\mathcal{N}(x, K(x))|dx$$

$$\leq (C_{\mathcal{N}}||L|| + \mathcal{N}_{\mathcal{N}})(t_i - t_j)$$

$$|Q(L)(t_i) - Q(L)(t_j)| \to 0, as\ t_i \to t_j$$

Case(2): When $t_i, t_j \in (t_1, T]$, then we have

$$\begin{aligned} |Q(L)(t_i) - Q(L)(t_j)| &\leq \frac{w}{M(w)} \int_{t_1}^{t_i} |N(x, L(x))|dx - \frac{w}{M(w)} \int_{t_1}^{t_j} |N(x, L(x))|dx \\ &\leq \frac{w}{M(w)} \left(\int_{t_1}^{t_i} |N(x, L(x))|dx - \int_{t_1}^{t_j} |N(x, L(x))|dx \right) \\ &\leq \frac{w}{M(w)} (C_{\mathcal{N}}||L|| + \mathcal{N}_{\mathcal{N}})(t_i - t_j) \\ |Q(L)(t_i) - Q(L)(t_j)| &\to 0, as\ t_i \to t_j \end{aligned}$$

So Q is bounded and equi-continuous. Thus, Q has at least one fixed point according to the Schauder fixed point theorem. There is thus at least one solution for the suggested model (5.1) in this way. □

Theorem 6.2. *Under the hypothesis (C_1) the considered model (5.1), has unique solution, if the following condition is satisfied.*

$$max\left\{\theta_1 = \mathcal{K}_N t_1,\ \theta_2 = \mathcal{K}_{\mathcal{N}} \frac{1-w+w(t-t_1)}{M(w)}\right\} \leq 1$$

Proof. Let L, $\bar{L} \in B$ on $[0, t_1]$.

$$\begin{aligned}
||Q(L) - Q(\bar{L})|| = & \max_{t\in[0,t_1]} \left| \int_0^{t_1} \mathcal{N}(x, L(x))dx - \int_0^{t_1} \mathcal{N}(x, \bar{L}(x)dx \right| \\
\leq & \max_{t\in[0,t_1]} \int_0^{t_1} \mathcal{K}_N |L - \bar{L}| dx \\
\leq & \parallel L - \bar{L} \parallel \mathcal{K}_N t_1
\end{aligned}$$

$$||Q(L) - Q(\bar{L})|| \leq \theta_1 \parallel L - \bar{L} \parallel$$

Next we consider the interval (t_1,T].

$$\begin{aligned}
|Q(L) - Q(\bar{L})| \leq & \frac{1-w}{M(w)} |\mathcal{N}(t, L) - \mathcal{N}(t, \bar{L})| + \frac{w}{M(w)} \int_{t_1}^{t} |\mathcal{N}(x, L) - \mathcal{N}(x, \bar{L})| dx \\
\leq & \frac{1-w}{M(w)} \mathcal{K}_N |L - \bar{L}| + \frac{w}{M(w)} \mathcal{K}_N |L - \bar{L}|(t - t_1)
\end{aligned}$$

$$||Q(L) - Q(\bar{L})|| \leq \mathcal{K}_N \frac{1-w+w(t-t_1)}{M(w)} ||L - \bar{L}||$$

$$||Q(L) - Q(\bar{L})|| \leq \theta_2 ||L - \bar{L}||$$

So clearly

$$||Q(L) - Q(\bar{L})|| \leq \begin{cases} \theta_1 ||L - \bar{L}||;\ if\ t \in [0, t_1] \\ \theta_2 ||L - \bar{L}|;\ if\ t \in (t_1, T] \end{cases}$$

According to the Banach contraction theorem, Q has a single fixed point since it is a contraction. As a result, the suggested model (5.1) has a unique solution.. □

7. Stability Analysis of the Proposed Model

We investigate the Ulam-Hyers stability of the suggested model in this section.

Definition 7.1. *The considered model is said to be Ulam-Hyers stable if for all $\xi > 0$ and*

$$|^{PCF}\mathrm{D}_t^w L(t) - \mathcal{N}(t, L(t))| < \xi, 0 < w \leq 1.$$

There exist a unique solution $\bar{L}$ and a constant $\mathcal{H} > 0$ such that

$$||L - \bar{L}|| < \mathcal{H}\xi.$$

Further, if there exist a non decreasing function $\phi : [0, \infty) \to R^+$, with $\phi(0)$=0 such that

$$||L - \bar{L}|| < \mathcal{H}\phi(\xi),$$

then the proposed model is said to be generalized Ulam-Hyers stable.

Remark 7.1. *Consider a function ϕ:[0, T] → R is independent of K∈ Z, such that ϕ(0)=0, then*

$$|\phi(t)| \leq \xi; t \in [0, T] \tag{7.1}$$

$$^{PCF}\mathrm{D}_t^w L(t) = \mathcal{N}(t, L(t)) + \phi(t) \tag{7.2}$$

The solution of the above perturbed problrm is computed as;

$$L(t) = \begin{cases} L(0) + \int_0^{t_1} [\mathcal{N}(x, L(x)) + \phi(x)]dx; \; if \; t \in \beta_1, \\ L(t_1) + \frac{1-w}{M(w)}[\mathcal{N}(t, L(t)) + \phi(t)] + \frac{w}{M(w)} \int_{t_1}^{t} [\mathcal{N}(x, L(x)) + \phi(x)]dx; \; if \; t \in \beta_2. \end{cases}$$

Theorem 7.1. *Consider the remark (7.1) the solution of the proposed model (5.1), is Ulam-Hyers stable if the following condition is satisfied,*

$$max\{\frac{t_1}{1-\mathcal{K}_{\mathcal{N}} t_1}, \frac{1-w+w(t-t_1)}{M((w)-\mathcal{K}_{\mathcal{N}}[1-w+w(t-t_1)]}\} < 1.$$

Proof. **Case(1)**: When t $\in [0, t_1]$,

$$\begin{aligned} ||L-\bar{L}|| =& \sup_{t\in\beta_1} |L - (L(0) + \int_0^{t_1} [\mathcal{N}(x, \bar{L}(x)) + \phi(x)]dx)|, \\ \leq& \sup_{t\in\beta_1} |L - (L(0) + \int_0^{t_1} [\mathcal{N}(x, \bar{L}(x)) + \phi(x)]dx)| + \sup_{t\in\beta_1} |\int_0^{t_1} \mathcal{N}(x, L)dx| - \sup_{t\in\beta_1} |\int_0^{t_1} \mathcal{N}(x, L)dx|, \\ \leq& t_1\xi + \mathcal{K}_{\mathcal{N}} t_1 ||L-\bar{L}||, \end{aligned}$$

$$||L-\bar{L}|| \leq \frac{t_1}{1-\mathcal{K}_{\mathcal{N}} t_1}\xi \tag{7.3}$$

Case(2): When t $\in (t_1, T]$,

$$\begin{aligned} ||L-\bar{L}|| =& \sup_{t\in\beta_2} |L - L(t_1) + \frac{1-w}{M(w)}[\mathcal{N}(t, \bar{L}(t)) + \phi(t)] + \frac{w}{M(w)} \int_{t_1}^{t} [\mathcal{N}(x, \bar{L}(x)) + \phi(x)]dx|, \\ &+ \sup_{t\in\beta_2} |\frac{1-w}{M(w}\mathcal{N}(t, L) + \frac{w}{M(w)} \int_{t_1}^{t} \mathcal{N}(x, L)dx|, \\ &- \sup_{t\in\beta_2} |\frac{1-w}{M(w}\mathcal{N}(t, L) + \frac{w}{M(w)} \int_{t_1}^{t} \mathcal{N}(x, L)dx|, \end{aligned}$$

After simplification and $\theta = \frac{1-w+w(t-t_1)}{M(w)}$, we have

$$||L-\bar{L}|| \leq \theta\xi + \mathcal{K}_{\mathcal{N}}\theta||L-\bar{L}||,$$

$$||L-\bar{L}|| \leq \frac{\theta}{1-\mathcal{K}_{\mathcal{N}}\theta}\xi \tag{7.4}$$

by using

$$\mathcal{H} = max\left\{\frac{t_1}{1-\mathcal{K}_{\mathcal{N}} t_1}, \frac{\theta}{1-\mathcal{K}_{\mathcal{N}}\theta}\right\}$$

From (7.3) and (7.4), we get

$$\|L - \bar{L}\| \leq \mathcal{H}\xi$$

Hence, the solution is Ulam-Hyers stable, further by replacing ξ by $\phi(\xi)$ with $\phi(0) = 0$, we have

$$\|L - \bar{L}\| \leq \mathcal{H}\ \phi(\xi)$$

Thus the solution is also generalized Ulam-Hyers stable. □

8. Numerical Scheme Under Piecewise Derivative

In this section, we present numerical scheme for Cauchy type problem under piecewise derivative. Since by lemma (2.1), we have

$$\begin{cases} \mathcal{R}(t) = \begin{cases} \mathcal{R}_0 + \int_0^{t_1} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx; & \text{if } t \in \beta_1, \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} F_1(t, \mathcal{R}(t), \mathcal{S}(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx; & \text{if } t \in \beta_2. \end{cases} \\ \mathcal{S}(t) = \begin{cases} \mathcal{S}_0 + \int_0^{t_1} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx; & \text{if } t \in \beta_1, \\ \mathcal{S}(x_1) + \frac{1-\sigma}{M(\sigma)} F_2(t, \mathcal{R}(t), \mathcal{S}(t)) + \frac{\sigma}{M(\sigma)} \int_{t_1}^{t} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx; & \text{if } t \in \beta_2. \end{cases} \end{cases}$$

Putting $t = t_{n+1}$

$$\begin{cases} \mathcal{R}(t_{n+1}) = \begin{cases} \mathcal{R}_0 + \sum_{k=0}^{i} \int_{t_k}^{t_{k+1}} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx, \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} [F_1(t_n, \mathcal{R}^n(t), \mathcal{S}^n(t)) - F_1(t_{n-1}, \mathcal{R}^{n-1}(t), \mathcal{S}^{n-1}(t))] + \frac{\sigma}{M(\sigma)} \sum_{k=i+1}^{n} \int_{t_k}^{t_{k+1}} F_1(x, \mathcal{R}(x), \mathcal{S}(x))dx. \end{cases} \\ \mathcal{S}(t_{n+1}) = \begin{cases} \mathcal{S}_0 + \sum_{k=0}^{i} \int_{t_k}^{t_{k+1}} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx, \\ \mathcal{S}(x_1) + \frac{1-\sigma}{M(\sigma)} [F_2(t_n, \mathcal{R}^n(t), \mathcal{S}^n(t)) - F_2(t_{n-1}, \mathcal{R}^{n-1}(t), \mathcal{S}^{n-1}(t))] + \frac{\sigma}{M(\sigma)} \sum_{k=i+1}^{n} \int_{t_k}^{t_{k+1}} F_2(x, \mathcal{R}(x), \mathcal{S}(x))dx. \end{cases} \end{cases}$$

Replacing it by Newton polynomial interpolation formula, the following scheme can be obtain.

$$\begin{cases} \mathcal{R}(t_{n+1}) = \begin{cases} \mathcal{R}_0 + \sum_{k=2}^{i} \Big[\frac{5}{12} F_1(t_{k-2}, \mathcal{R}^{k-2}(t), \mathcal{S}^{k-2}(t)) - \frac{4}{3} F_1(t_{k-1}, \mathcal{R}^{k-1}(t), \mathcal{S}^{k-1}(t)) + \frac{23}{12} F_1(t_k, \mathcal{R}^k(t), \mathcal{S}^k(t))\Big]\delta t \\ \mathcal{R}(t_1) + \frac{1-\sigma}{M(\sigma)} [F_1(t_n, \mathcal{R}^n(t), \mathcal{S}^n(t)) - F_1(t_{n-1}, \mathcal{R}^{n-1}(t), \mathcal{S}^{n-1}(t))] \\ + \frac{\sigma}{M(\sigma)} \sum_{k=i+3}^{n} \Big[\frac{5}{12} F_1(t_{k-2}, \mathcal{R}^{k-2}(t), \mathcal{S}^{k-2}(t)) - \frac{4}{3} F_1(t_{k-1}, \mathcal{R}^{k-1}(t), \mathcal{S}^{k-1}(t)) + \frac{23}{12} F_1(t_k, \mathcal{R}^k(t), \mathcal{S}^k(t))\Big]\delta t. \end{cases} \\ \mathcal{S}(t_{n+1}) = \begin{cases} \mathcal{S}_0 + \sum_{k=2}^{i} \Big[\frac{5}{12} F_2(t_{k-2}, \mathcal{R}^{k-2}(t), \mathcal{S}^{k-2}(t)) - \frac{4}{3} F_2(t_{k-1}, \mathcal{R}^{k-1}(t), \mathcal{S}^{k-1}(t)) + \frac{23}{12} F_2(t_k, \mathcal{R}^k(t), \mathcal{S}^k(t))\Big]\delta t, \\ \mathcal{S}(x_1) + \frac{1-\sigma}{M(\sigma)} [F_2(t_n, \mathcal{R}^n(t), \mathcal{S}^n(t)) - F_2(t_{n-1}, \mathcal{R}^{n-1}(t), \mathcal{S}^{n-1}(t))] \\ + \frac{\sigma}{M(\sigma)} \sum_{k=i+3}^{n} \Big[\frac{5}{12} F_2(t_{k-2}, \mathcal{R}^{k-2}(t), \mathcal{S}^{k-2}(t)) - \frac{4}{3} F_2(t_{k-1}, \mathcal{R}^{k-1}(t), \mathcal{S}^{k-1}(t)) + \frac{23}{12} F_2(t_k, \mathcal{R}^k(t), \mathcal{S}^k(t))\Big]\delta t. \end{cases} \end{cases}$$

9. Examples of Cauchy Type Problems

In this part, we present several examples of Cauchy type problems utilizing piecewise Caputo-Fabrizio derivative.

Example 9.1. *Consider the following Cauchy type problem with piecewise derivatives as*

$$\begin{cases} {}^{PCF}\mathrm{D}^{\sigma}\mathcal{R}(t) = \frac{\sin|\mathcal{R}(t)|+|\mathcal{S}(t)|}{t^2+50},\ \sigma \in (0,1],\ t \in [0,10], \\ {}^{PCF}\mathrm{D}^{\sigma}\mathcal{S}(t) = \frac{|\mathcal{R}(t)|+|\sin\mathcal{S}(t)|}{t^4+50},\ \sigma \in (0,1],\ t \in [0,10], \\ \mathcal{R}(0) = 0.3, \\ \mathcal{S}(0) = 0.4. \end{cases} \tag{9.1}$$

Then $L_{F_1} = L_{F_2} = \frac{1}{50}$, *taking* t_1=1 *and* T=10.

$$K_1 = t_1(L_{F_1} + L_{F_2}) = \frac{1}{25}$$
$$K_2 = (L_{F_1} + L_{F_2})\left(\frac{1-\sigma+\sigma(T-t_1)}{M(\sigma)}\right) = \frac{9}{25}$$

Therefore $\max\{K_1, K_2\}$=$\max\{\frac{1}{25}, \frac{9}{25}\} < 1$, *also* $\max\{L_{F_1} + L_{F_2}, \frac{1-\sigma}{M(\sigma)}(L_{F_1} + L_{F_2})\}$=$\max\{\frac{1}{25}, \frac{1-\sigma}{25}\} < 1$. *So by Theorem 3.2 at has at least one solution,and by Theorem3.1 the solution is unique. Since the condition of Ulam-Hyers stability is satisfied so the solution is Ulam-Hyers stable, Moreover the solution is generalized Ulam-Hyers stable.*

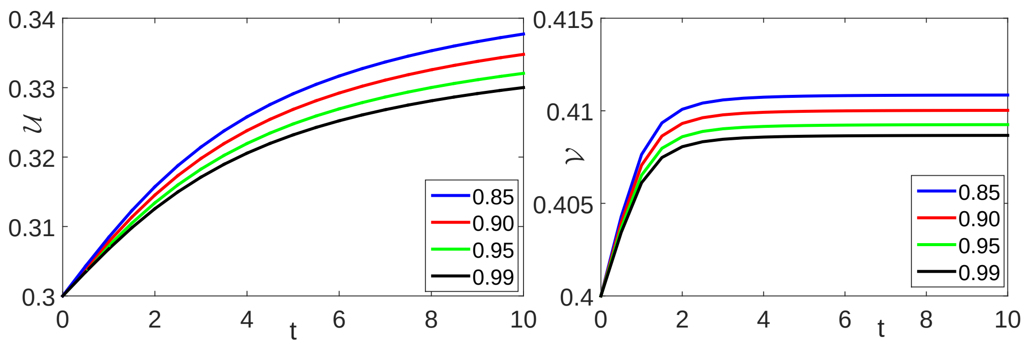

FIGURE 1. Numerical illustration of solutions for Example 9.1 at different fractional order $\sigma \in (0.80, 1.0)$.

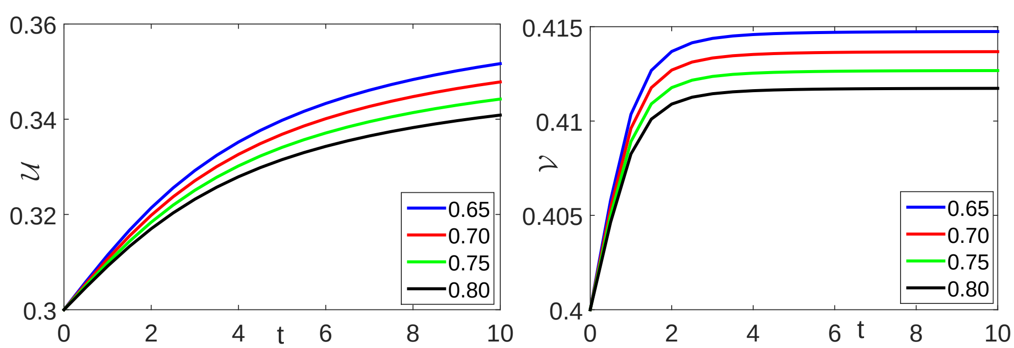

FIGURE 2. Numerical illustration of solutions for Example 9.1 at different fractional order $\sigma \in (0.60, 0.80]$.

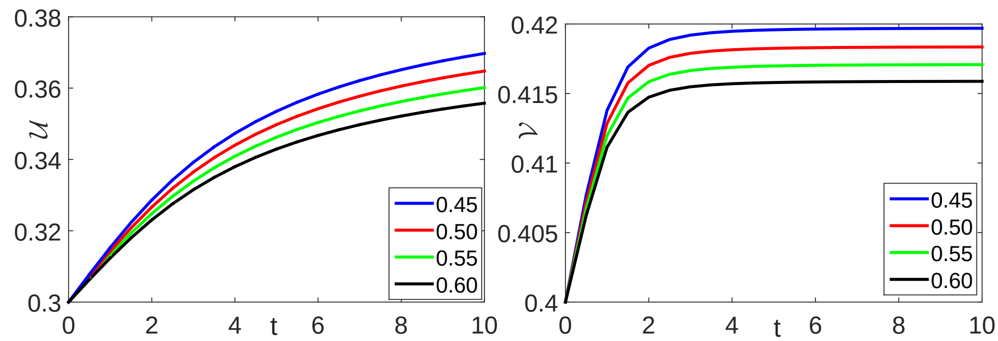

FIGURE 3. Numerical illustration of solutions for Example 9.1 at different fractional order $\sigma \in (0.40, 0.60]$.

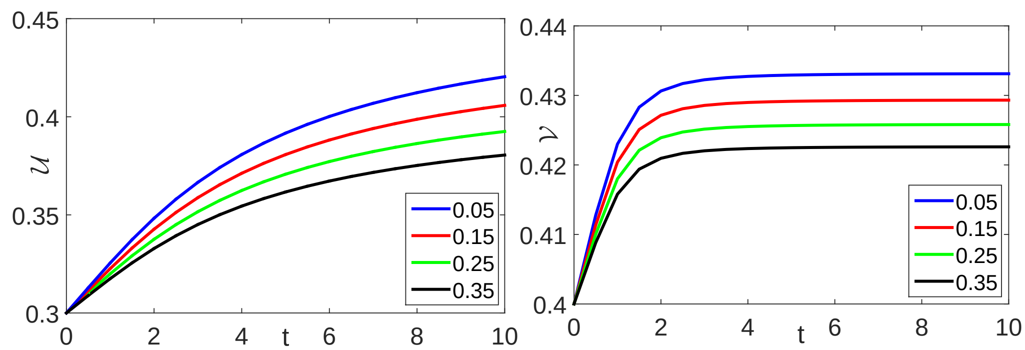

FIGURE 4. Numerical illustration of solutions for Example 9.1 at different fractional order $\sigma \in (0, 0.40)$.

In figures 1-4, we have presented the numerical results for the given problems by using different fractional orders to understand the piecewise dynamics of the problems.

10. Graphical Solution of the Proposed Prey-predator Model

In this section, the graphical solution of the proposed model [22]. is presented.

Parameters	Description of Parameters	Numerical values
a_1	Per capita growth rate of the prey	0.001187
b_1	The impact of predators on the prey growth rate	0.00225
a_2	Per capita death rate of the predators	0.00225
b_2	The impact of prey on the predators growth rate	0.000375
γ_1	The population of the prey at t=0	5
γ_2	The population of the predators at t=0	10

Table 1. Values of parameters for the considered model

.

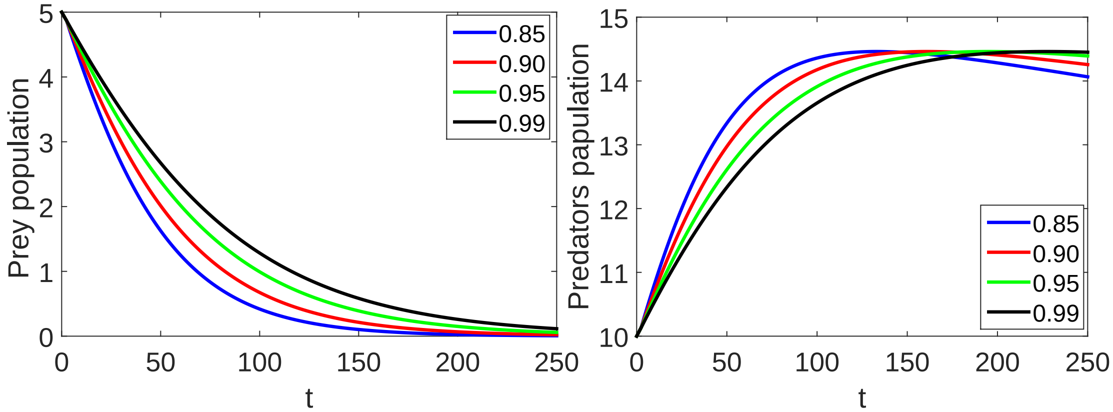

Figure 5. Graphical illustration of approximate solutions of system 5.1 for different fractional orders $\sigma \in (0.80, 0.99)$.

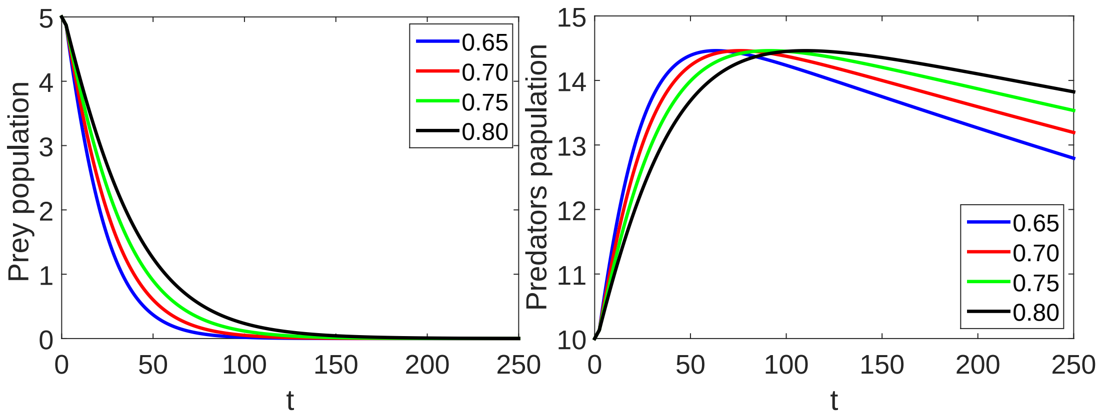

Figure 6. Graphical illustration of approximate solutions of system (5.1) for different fractional orders $\sigma \in (0.60, 0.80]$.

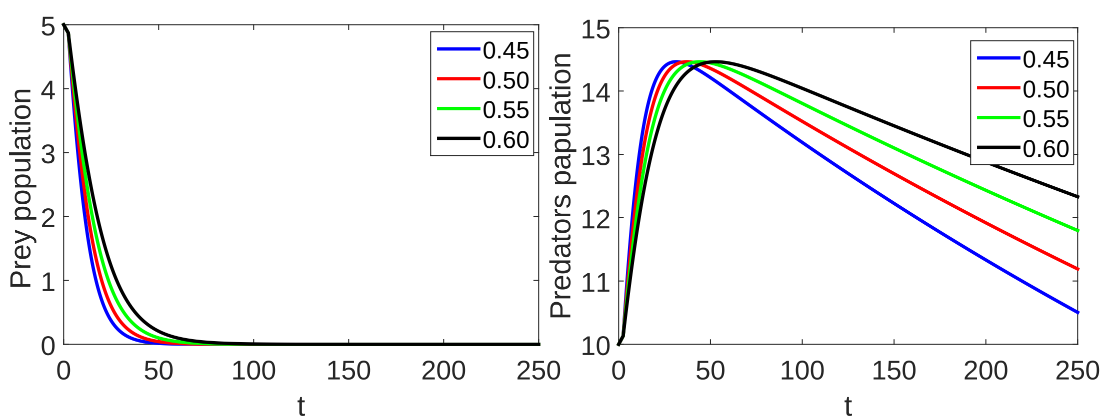

Figure 7. Graphical illustration of approximate solutions of system (5.1) for different fractional orders $\sigma \in (0.40, 0.60]$.

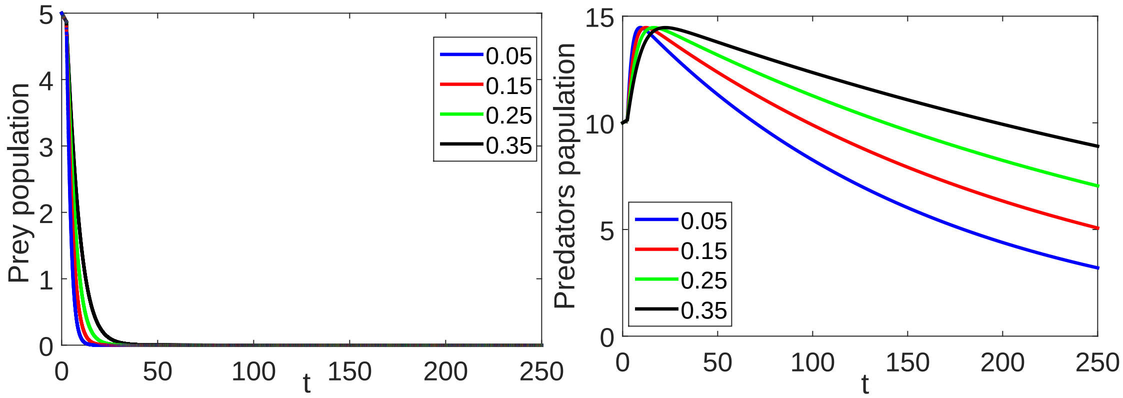

Figure 8. Graphical illustration of approximate solutions of system (5.1) for different fractional orders $\sigma \in (0, 0.40)$.

Here in figures 5-8, we have presented the illustration of approximate solutions of system (5.1) for different fractional orders graphically to investigate the concerned dynamics.

11. Conclusion

This study introduces some new ideas on piecewise equations under CFD. We have developed certain conclusions pertaining to the existence, uniqueness, and stability analysis of Cauchy type problems, keeping in mind the significance of fractional calculus in the recent past. Both nonlinear functional analysis and the fixed point approach have been used to establish the relevant results. Sufficient conditions have been established to ensure that the suggested Cauchy issue has at least one solution and that it is unique. Moreover, nonlinear analysis tools have been used to infer its stability. To illusterate the rsults, a Biological Prey-Predator model and one arbitrery example are given. There have also been certain graphical presentations. It is evident that these derivatives more effectively convey the abrupt shift in behavior.
Future research will address how to handle piecewise equation boundary value problems under different fractional order derivatives.

Acknowledgment: The authors M. Sarwar and K.Abodayeh are thankful to Prince Sultan University for the support of this manuscript through TAS research lab.

Authors' Contributions: All authors contribute equally to the writing of this manuscript. All authors reviewed the results and approved the final version of the manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H.M. Srivastava, Y. Ling, G. Bao, Some Distortion Inequalities Associated with the Fractional Derivatives of Analytic and Univalent Functions, J. Inequal. Pure Appl. Math. 2 (2021), 23.

[2] M. Darus, R.W. Ibrahim, Radius Estimates of a Subclass of Univalent Functions, Mat. Vesnik 63 (2011), 55–58. https://eudml.org/doc/253329.

[3] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, V. Feliu, Fractional-Order Systems and Controls: Fundamentals and Applications, Springer, 2010. https://doi.org/10.1007/978-1-84996-335-0.

[4] R. Caponetto, G. Dongola, L. Fortuna, I. Petras, Fractional Order Systems: Modeling and Control Applications, World Scientific, 2010.

[5] G. Mani, A.J. Gnanaprakasam, L. Guran, R. George, Z.D. Mitrović, Some Results in Fuzzy B-Metric Space with B-Triangular Property and Applications to Fredholm Integral Equations and Dynamic Programming, Mathematics 11 (2023), 4101. https://doi.org/10.3390/math11194101.

[6] G. Mani, A. Gnanaprakasam, O. Ege, A. Aloqaily, N. Mlaiki, Fixed Point Results in $C^\star$-Algebra-Valued Partial B-Metric Spaces with Related Application, Mathematics 11 (2023), 1158. https://doi.org/10.3390/math11051158.

[7] S. Westerlund, Dead Matter Has Memory!, Phys. Scr. 43 (1991), 174–179. https://doi.org/10.1088/0031-8949/43/2/011.

[8] M. Caputo, F. Mainardi, A New Dissipation Model Based on Memory Mechanism, Pure Appl. Geophys. 91 (1971), 134–147. https://doi.org/10.1007/bf00879562.

[9] W. Wyss, The Fractional Diffusion Equation, J. Math. Phys. 27 (1986), 2782–2785. https://doi.org/10.1063/1.527251.

[10] P. Agarwal, J.J. Nieto, M. Luo, Extended Riemann-Liouville Type Fractional Derivative Operator with Applications, Open Math. 15 (2017), 1667–1681. https://doi.org/10.1515/math-2017-0137.

[11] M. Caputo, M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl. 1 (2015), 73–85.

[12] J. Losada, J.J. Nieto, Properties of a New Fractional Derivative without Singular Kernel, Progr. Fract. Differ. Appl. 1 (2015), 87–92.

[13] J.T. Machado, V. Kiryakova, F. Mainardi, Recent History of Fractional Calculus, Commun. Nonlinear Sci. Numer. Simul. 16 (2011), 1140–1153. https://doi.org/10.1016/j.cnsns.2010.05.027.

[14] L. Byszewski, Strong Maximum Principles for Parabolic Nonlinear Problems with Nonlocal Inequalities Together with Arbitrary Functionals, J. Math. Anal. Appl. 156 (1991), 457–470. https://doi.org/10.1016/0022-247x(91)90409-s.

[15] L. Byszewski, V. Lakshmikantham, Theorem About the Existence and Uniqueness of a Solution of a Nonlocal Abstract Cauchy Problem in a Banach Space, Appl. Anal. 40 (1991), 11–19. https://doi.org/10.1080/00036819008839989.

[16] G. Wu, D. Zeng, D. Baleanu, Fractional Impulsive Differential Equations: Exact Solutions, Integral Equations and Short Memory Case, Fract. Calc. Appl. Anal. 22 (2019), 180–192. https://doi.org/10.1515/fca-2019-0012.

[17] G. Wu, M. Luo, L. Huang, S. Banerjee, Short Memory Fractional Differential Equations for New Memristor and Neural Network Design, Nonlinear Dyn. 100 (2020), 3611–3623. https://doi.org/10.1007/s11071-020-05572-z.

[18] A. Atangana, S. İğret Araz, New Concept in Calculus: Piecewise Differential and Integral Operators, Chaos, Solitons Fractals 145 (2021), 110638. https://doi.org/10.1016/j.chaos.2020.110638.

[19] G. Mani, S. Haque, A.J. Gnanaprakasam, O. Ege, N. Mlaiki, The Study of Bicomplex-Valued Controlled Metric Spaces with Applications to Fractional Differential Equations, Mathematics 11 (2023), 2742. https://doi.org/10.3390/math11122742.

[20] K. Shah, T. Abdeljawad, B. Abdalla, M.S. Abualrub, Utilizing Fixed Point Approach to Investigate Piecewise Equations with Non-Singular Type Derivative, AIMS Math. 7 (2022), 14614–14630. https://doi.org/10.3934/math.2022804.

[21] K. Shah, T. Abdeljawad, A. Ali, Mathematical Analysis of the Cauchy Type Dynamical System Under Piecewise Equations with Caputo Fractional Derivative, Chaos Solitons Fractals 161 (2022), 112356. https://doi.org/10.1016/j.chaos.2022.112356.

[22] M.A. Alqudah, T. Abdeljawad, Eiman, K. Shah, F. Jarad, et al., Existence Theory and Approximate Solution to Prey–Predator Coupled System Involving Nonsingular Kernel Type Derivative, Adv. Differ. Equ. 2020 (2020), 520. https://doi.org/10.1186/s13662-020-02970-w.

[23] S. Muhammad, O.J. Algahtani, S. Saifullah, A. Ali, Theoretical and Numerical Aspects of the Malaria Transmission Model with Piecewise Technique, AIMS Math. 8 (2023), 28353–28375. https://doi.org/10.3934/math.20231451.