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A Modified Inertial Subgradient Extragradient Method for Solving Pseudomonotone Equilibrium Problem

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Abstract. In this paper, we introduce and study a new modified inertial subgradient extragradient method that includes a self-adaptive step size and viscosity technique for approximating the solution of pseudomonotone equilibrium and fixed point problems in the framework of real Hilbert space. We obtain a strong convergence result of the proposed method under mild conditions. Furthermore, we apply our results to solve variational inequality. Finally, we present some numerical experiments for our proposed method in comparison with existing methods in the literature. Our result improves, extends and generalizes several existing results in the literature.

1. Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space H. The Equilibrium Problem (EP) introduced and studied by Blum and Oettli [12] is the problem of finding a point $x^* \in C$ such that

$$g(x^*, y) \geq 0 \; \forall y \in C, \tag{1.1}$$

where $g : C \times C \to \mathbb{R}$ is a bifunction. For any point $x^* \in C$ that solves EP is called an equilibrium point of g. We denote $SOL(g, C)$ the solution set of problem (1.1). Researchers have paid close attention to the EP (1.1) because it unifies a good number of mathematical models,

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including optimization problem, fixed point problem, convex minimization problem, Nash equilibrium, variational inequality problem, saddle point problem, and many more (see [1–4] and the references therein). Regularization methods, proximal point methods, extragradient methods, projection methods, gap-function methods, the auxiliary problem principle method, and the Bregman distance method are among the fundamental techniques for solving EP (1.1). In particular, Moudafi [13,14] used the proximal point method to solve monotone equilibrium problem. However, the proximal point method fail to solve the pseudomonotone equilibrium problem. In the light of this drawback Flam [10] and Tran et al. [15] introduced and studied a proximal-like method known as the Extragradient Method (EGM) solve the pseudomonotone equilibrium problem. They gave the following iterative method

$$\begin{cases} x_0 \in C \\ y_m = \text{argmin}_{y\in C}\{\lambda g(x_m, y) + \frac{1}{2}\|y - x_m\|^2\} \\ x_{m+1} = \text{argmin}_{y\in C}\{\lambda g(y_m, y) + \frac{1}{2}\|y - x_m\|^2\}, \end{cases} \tag{1.2}$$

where λ is a constant satisfying suitable assumptions. It is worth mentioning that when using this method, one needs to solve two strongly convex optimization problems in the feasible set C per iteration, and λ depend on bifunctional Lipschitz-type constants. In addition, the sequence $\{x_m\}$ converges weakly to the solution set. The concept of inertial extrapolation was first introduced and studied in 1964 by Polyak in [16] as a technique for accelerating the process of solving the smooth convex minimization problem. Since then, researchers have used this technique to improve the rate of convergence of different iterative processes. Numerous authors have improved, expanded, and generalized the inertial extrapolation method since its inception, see [5–9,23–25,29–31] and the references therein. In this area of research, relaxation techniques have proven to be a good technique for better rate of convergence. It is well-known that combining the inertial and the relaxation technique improve and give a better rate or convergence when compared with just relaxation technique or the inertial technique. For example, Vinh and Muu [33] proposedthe following iterative algorithm

$$\begin{cases} x_0, x_1 \in C, \\ w_m = x_m + \theta_m(x_m - x_{m-1}), \\ y_m = \text{argmin}_{y\in C}\{\lambda g(w_m, y) + \frac{1}{2}\|y - w_m\|^2\}, \\ z_m = \text{argmin}_{y\in C}\{\lambda g(y_m, y) + \frac{1}{2}\|y - w_m\|^2\}. \\ \text{If } w_m = y_n, \text{ then stop and } y_n \text{ is a solution. Otherwise.} \\ x_{m+1} = (1 - \eta_m - \alpha_m)x_m + \alpha_m z_m. \end{cases} \tag{1.3}$$

where λ is a constant satisfying suitable assumptions. As mentioned above one needs to solve two strongly convex optimization problems in the feasible set C per iteration, and λ depend on bifunctional Lipschitz-type constants. These drawbacks limit the application of the above method. Motivated by the above drawbacks, Lyashko et. al. [22] introduced and studied the following iterative process

$$\begin{cases} x_1 \in H, \\ y_m = P_C(x_m - \lambda_m \partial g(x_m, \cdot)(x_m)), \\ x_{m+1} = P_{T_m}(x_m - \lambda_n \partial g(y_m, \cdot)(y_m)), \\ T_m - \{z \in H : \langle x_m - \lambda_m \partial g(x_m, \cdot)(x_m) - y_m, z - y_m \rangle \leq 0\}, \end{cases} \tag{1.4}$$

where λ_n are positive numbers and $\partial g(x, \cdot)(y)$ is the Gâteaux derivative of the functional $g(x, \cdot)$ at a point y. It was established that the sequence generated by (1.4) converges weakly to the solution set of the aforementioned problem. It is easy to see that, the iterative process (1.4) reduces the computation time of the iterative process (1.3). Since in iterative process (1.4) only one metric projection into the feasible set is required. However, it is well-known in this area of research that strong convergence is more desirable. The main benefit of the Subgradient Extragradient Method (SEM) over the extragradient method is the fact that the second convex optimization problem is onto a half-space and has a closed form solution. So, compared to the extragradient approach, its computational complexity is less expensive. In the light of this development, so many authors have modified the SEM, see ([17–21, 27, 28, 32, 37, 38] and the references therein). In this area of research, the notion of improving iterative algorithm to get a better rate of convergence is highly sort after.

Motivated by the above literature and the research work in this direction, we introduce and study a modified pseudomonotone equilibrium problem, and a new inertial-viscosity condition for the subgradient extragradient method with self-adaptive step size for approximating a solution of the pseudomonotne equilibrium problem in a real Hilbert space. We prove strong convergence result for our proposed iterative method and present some numerical experiments to show the efficiency and applicability of our proposed method in comparison with existing methods in the literature.

2. Preliminaries

In this section, we present some existing results which will be useful in the sequel. Let H be a real Hilbert space. The fixed point of the self mapping $T : H \to H$ is repented $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. Let "$\to$" and "$\rightharpoonup$" stand for weak and strong convergence, respectively. For any $\bar{x}, \bar{y} \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$||\bar{x} - \bar{y}||^2 = ||\bar{x}||^2 - 2\langle \bar{x}, \bar{y} \rangle + ||\bar{y}||^2. \tag{2.1}$$

$$||\bar{x} + \bar{y}||^2 = ||\bar{x}||^2 + 2\langle \bar{x}, \bar{y} \rangle + ||\bar{y}||^2. \tag{2.2}$$

$$||\bar{x} - \bar{y}||^2 \leq ||\bar{x}||^2 + 2\langle \bar{y}, \bar{x} - \bar{y} \rangle. \tag{2.3}$$

$$||\alpha\bar{x} + (1 - \alpha)\bar{y}||^2 = \alpha||\bar{x}||^2 + (1 - \alpha)||\bar{y}||^2 - \alpha(1 - \alpha)||\bar{x} - \bar{y}||^2. \tag{2.4}$$

Definition 2.1. *Let $T : H \to H$ be an operator. Then the operator T is called*

(a) *L-Lipschitz continuous if there exists $L > 0$ such that*

$$\|T\bar{x} - T\bar{y}\| \leq L\|\bar{x} - \bar{y}\|,$$

for all $\bar{x}, \bar{y} \in H$. If $L = 1$, then T is called nonexpansive;

(b) *monotone if*

$$\langle T\bar{x} - T\bar{y}, \bar{x} - \bar{y}\rangle \geq 0, \ \forall \bar{x}, \bar{y} \in H;$$

(c) *pseudomonotone if*

$$\langle T\bar{x}, \bar{y} - \bar{x}\rangle \geq 0 \Rightarrow \langle T\bar{y}, \bar{y} - \bar{x}\rangle \geq 0, \ \forall \bar{x}, \bar{y} \in H;$$

(d) *β- strongly monotone if there exists $\beta > 0$, such that*

$$\langle T\bar{x} - T\bar{y}, \bar{x} - \bar{y}\rangle \geq \beta\|\bar{x} - \bar{y}\|^2, \ \forall \ \bar{x}, \bar{y} \in H;$$

(e) *firmly nonexpansive*

$$\|T\bar{x} - T\bar{y}\|^2 \leq \langle T\bar{x} - T\bar{y}, \bar{x} - \bar{y}\rangle \ \forall \ \bar{x}, \bar{y} \in H;$$

or equivalently

$$\|T\bar{x} - T\bar{y}\|^2 \leq \|\bar{x} - \bar{y}\|^2 - \|(I - T)\bar{x} - (I - T)\bar{y}\|^2 \ \forall \ \bar{x}, \bar{y} \in H;$$

(f) *directed (also called to be firmly quasi-nonexpansive) if $F(T) \neq \emptyset$ and*

$$\langle \bar{x} - \bar{y}, T\bar{x} - T\bar{y}\rangle \geq \|T\bar{x} - p\|^2 \ \forall \ x \in H \text{ and } p \in F(T);$$

(g) *sequentially weakly continuous if for each sequence $\{\bar{x}_m\}$, we obtain $\{\bar{x}_m\}$ converges weakly to $\bar{x}$ implies that $T\bar{x}_m$ converges weakly to $T\bar{x}$.*

Definition 2.2. *Let $g : C \times C \to \mathbb{R}$ is said to be:*

(a) *Strongly monotone on C if there exists a constant $\tau > 0$ such that*

$$g(x, y) + g(y, x) \leq -\tau\|x - y\|^2 \tag{2.5}$$

for all $x, y \in C$;

(b) *Monotone on C if*

$$g(x, y) + g(y, x) \leq 0 \tag{2.6}$$

for all $x, y \in C$;

(c) *Strongly pseudomonote on C if there exists a constant $\gamma > 0$ such that*

$$g(x, y) \geq 0 \Rightarrow g(y, x) \leq -\gamma\|x - y\|^2, \ \forall x, y \in C;$$

(d) *Psuedomonotone on C if*

$$g(x, y) \geq 0 \Rightarrow g(y, x) \leq 0, \ \forall \ x, y \in C;$$

(e) *Satisfying a Lipschitz-like condition if there exist two positive constant L_1, L_2 such that*

$$g(x, y) + g(y, z) \geq g(x, z) - L_1\|x - y\| - L_2\|y - z\|^2 \ \forall \ x, y, z \in C. \tag{2.7}$$

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \ \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.8}$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.9}$$

for all $x \in H$ and $y \in C$. A subset C of H is called proximal if for each $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C).$$

The normal cone N_C to C at a point $x \in C$ is defined by $N_C(x) = \{z \in H : \langle z, x - y\rangle \geq 0 \ \forall y \in C\}$.

Lemma 2.1. *Let C be a convex subset of a real Hilbert space H and $\phi : C \to \mathbb{R}$ be a subdifferential function on C. Then x^* is a solution to the convex problem: minimize$\{\phi(x) : x \in C\}$ if and only if $0 \in \partial\phi(x^*) + N_C(x^*)$, where $\partial\phi(x^*)$ denotes the subdifferential of ϕ and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.2. *[34] Let $\{a_m\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{m=1}^{\infty} \alpha_m = \infty$ and $\{d_m\}$ be a sequence of real numbers. Suppose that*

$$a_{m+1} \leq (1 - \alpha_m)a_m + \alpha_m d_m, m \geq 1.$$

If $\limsup_{k\to\infty} d_{m_k} \leq 0$ for all subsequences $\{a_{m_k}\}$ of $\{a_m\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{m_k+1} - a_{m_k}\} \geq 0,$$

then, $\lim\limits_{m\to\infty} a_m = 0$.

3. Proposed Algorithm

In this section, we present our proposed method for solving the problem (1.1).

Assumption 3.1.

Condition A. *Let C be a nonempty, closed convex subset of a real Hilbert space H. Suppose $g : H \times H \to \mathbb{R}$ satisfies the following conditions:*

(1) *g is pseudomonotone on C;*

(2) *g satisfies the Lipschitz-type condition* (2.7) *on H and the constants L_1, L_2 do not necessary need to be known;*

(3) *$g(\cdot, y)$ is sequentially weakly upper semi-continuous on C for each fixed $y \in C$;*

(4) *$g(x, \cdot)$ is convex, lower semi-continuous on H for every fixed $x \in H$;*

(5) *$f : H \to H$ is a contraction with constant $k \in [0, 1)$;*

(6) *$SOL(g, C)$ is non empty.*

Condition B. *Suppose that $\{\alpha_n\}, \{\eta_n\}, \{\beta\}, \{\epsilon_n\}$ and $\{\eta_n\}$ are positive sequences such that*

(1) $\{\eta_m\}, \{\alpha_m\} \subset (0, 1)$, $\lim_{m\to\infty} \alpha_m = 0$, *such that* $\sum_{m=1}^{\infty} \alpha_m = \infty$, $\lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0$.

(2) $\lambda_1 > 0, \mu \in (0, 1)$, $\theta > 0$, $0 \leq \theta_m \leq \overline{\theta}_m$.

We present the following iterative algorithm.

Algorithm 3.1. *An Hybrid Viscosity Iterative Method*

Step 1: *Choose $x_0, x_1 \in H_1$, given the iterates x_{m-1} and x_m for all $m \in \mathbb{N}$.*

$$\overline{\theta}_m = \begin{cases} \min\left\{\theta, \frac{\epsilon_m}{\|x_m - x_{m-1}\|}\right\}, & \text{if } x_m \neq x_{m-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: *Compute*

$$w_n = \alpha_m f(x_m) + (1 - \alpha_m)(x_m + \theta_m(x_m - x_{m-1})), \tag{3.2}$$

$$y_m = argmin_{y \in C}\{\lambda_m g(w_m, y) + \frac{1}{2}\|y - w_m\|^2\}. \tag{3.3}$$

If $w_m = y_m$, then stop and y_m is a solution. Otherwise, go to Step 3.

Step 3: *Select $z_m \in \partial g(w_m, \cdot)(y_m)$ and $q_m \in N_C(y_m)$ such that*

$$q_m = w_m - \lambda_m \beta z_m - y_m, \tag{3.4}$$

and construct a half-space

$$T_m = \{z \in H : \langle w_m - \lambda_m \beta z_m - y_m, z - y_m\rangle \leq 0\}, \tag{3.5}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\lambda_m, \frac{\mu[\|w_m - y_m\|^2 + \|u_m - y_m\|^2]}{2(g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m))}\right\}, & \text{if } g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m) > 0 \\ \\ \lambda_m, & \text{otherwise.} \end{cases} \tag{3.6}$$

compute

$$u_m = argmin_{y \in T_m}\{\lambda_m g(y_m, y) + \frac{1}{2}\|y - w_m\|^2\}. \tag{3.7}$$

Step 4: *Compute*

$$x_{m+1} = (1 - \eta_m)w_m + \eta_m u_m. \tag{3.8}$$

Remark 3.1. *If $f = I$ (identity mapping), we obtain a relaxed type inertia.*

Lemma 3.1. *The sequence $\{\lambda_m\}$ generated by* (3.6) *is monotonically non-increasing and*

$$\lim_{m\to\infty} \lambda_m = \lambda \geq \frac{\mu}{2\max\{L_1, L_2\}}$$

Proof. Clearly, $\{\lambda_n\}$ is monotonically non-increasing. Also, since g satisfies condition A (2), we have

$$\frac{\mu[\|w_m - y_m\|^2 + \|u_m - y_m\|^2]}{2(g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m))} \geq \frac{\mu[\|w_m - y_m\|^2 + \|u_m - y_m\|^2]}{2(L_1\|w_m - y_m\|^2 + L_2\|y_m - u_m\|^2)} \geq \frac{\mu}{2\max\{L_1, L_2\}}. \quad (3.9)$$

Hence $\{\lambda_m\}$ is bounded below by $\frac{\mu}{2\max\{L_1,L_2\}}$. This implies that there exists

$$\lim_{m\to\infty} \lambda_m = \lambda \geq \frac{\mu}{2\max\{L_1, L_2\}}.$$

□

Lemma 3.2. *Let $\{x_m\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then, $\{x_m\}$ is bounded.*

Proof. Let $p \in SOL(g, C)$ using (3.1) and the fact that $0 \leq \theta_m \leq \overline{\theta}_m$,, we have

$$\theta_m\|x_m - x_{m-1}\| \leq \overline{\theta}_m\|x_m - x_{m-1}\| \leq \epsilon_m.$$

Therefore, it follows from $\lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0$, that

$$\lim_{m\to\infty} \frac{\theta_m}{\alpha_m}\|x_m - x_{m-1}\| \leq \lim_{m\to\infty} \frac{\epsilon_m}{\alpha_m} = 0. \quad (3.10)$$

It follows that the sequence $\{\frac{\theta_m}{\alpha_m}\|x_m - x_{m-1}\|\}$ is bounded. Hence, there exists $N_2 > 0$ such that $\frac{\theta_m}{\alpha_m}\|x_m - x_{m-1}\| \leq N_2$, for all $m \in \mathbb{N}$. Then using Algorithm 3.1, we have

$$\begin{aligned}
\|w_m - p\| &= \|\alpha_m f(x_m) + (1 - \alpha_m)(x_m + \theta_m(x_m - x_{m-1})) - p\| \\
&\leq \alpha_m\|f(x_m) - p\| + (1 - \alpha_m)\|x_m - p\| + \theta_m(1 - \alpha_m)\|x_m - x_{m-1}\| \\
&\leq \alpha_m\|f(x_m) - f(p)\| + \alpha_m\|f(p) - p\| + (1 - \alpha_m)\|x_m - p\| + \theta_n\|x_m - x_{m-1}\| \\
&\leq \alpha_m k\|x_m - p\| + \alpha_m\|f(p) - p\| + (1 - \alpha_m)\|x_m - p\| + \theta_n\|x_m - x_{m-1}\| \\
&= (1 - \alpha_m(1 - k))\|x_m - p\| + \alpha_m\|f(p) - p\| + \alpha_m\frac{\theta_m}{\alpha_n}\|x_m - x_{m-1}\| \\
&\leq (1 - \alpha_m(1 - k))\|x_m - p\| + \alpha_m(1 - k)\left[\frac{\|f(p) - p\| + N_2}{(1 - k)}\right] \\
&\leq \max\{\|x_m - p\|, \frac{\|f(p) - p\| + N_2}{(1 - k)}\}. \qquad (3.11)
\end{aligned}$$

From Algorithm 3.1, we have $u_m = \text{argmin}_{y\in T_m}\{\lambda_m\beta g(y_m, y) + \frac{1}{2}\|y - w_m\|^2\}$, and using Lemma 2.1, we get

$$\beta\lambda_m[g(y_m, y) - g(y_m, u_m)] \geq \langle w_m - u_m, y - u_m\rangle, \ \forall y \in T_m. \quad (3.12)$$

Since $p \in \mathrm{SOL}(g, C) \subset C \subset T_m$, and taking $y := p$, we get

$$\beta\lambda_m(g(y_m, p) - g(y_m, u_m)) \geq \langle w_m - u_m, p - u_m\rangle. \tag{3.13}$$

Since $y_m \in C$, we have $g(p, y_m) \geq 0$, and by the pseudomonotonicity of g, we get $g(y_m, p) \leq 0$. Thus, (3.13) becomes

$$-\beta\lambda_m g(y_m, u_m) \geq \langle w_m - u_m, p - u_m\rangle. \tag{3.14}$$

In addition, from $z_m \in \partial g(w_m, \cdot)(y_m)$, we obtain

$$\beta\lambda_m(g(w_m, u_m) - g(w_m, y_m)) \geq \beta\lambda_m\langle z_m, y - y_m\rangle. \tag{3.15}$$

Using the definition of T_m, we get

$$\langle w_m - \lambda_m\beta z_m - y_m, u_m - y_m\rangle \leq 0, \tag{3.16}$$

it follows that

$$\beta\lambda_m\langle z_m, u_m - y_m\rangle \geq \langle w_m - y_m, u_m - y_m\rangle. \tag{3.17}$$

From (3.17) and (3.15), we get

$$\beta\lambda_m(g(w_m, u_m) - g(w_m, y_m)) \geq \langle w_m - y_m, u_m - y_m\rangle. \tag{3.18}$$

Also, adding (3.18) and (3.14), we obtain

$$\begin{aligned}
&\beta\lambda_m(g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m)) \geq \langle w_m - y_m, u_m - y_m\rangle + \langle w_m - u_m, p - u_m\rangle \\
\Rightarrow &2\beta\lambda_m(g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m)) \geq 2\langle w_m - y_m, u_m - y_m\rangle + 2\langle w_m - u_m, p - u_m\rangle \\
&= \|w_m - y_m\|^2 + \|u_m - y_m\|^2 - \|u_m - w_m\|^2 + \|w_m - u_m\|^2 + \|u_m - p\|^2 - \|w_m - p\|^2 \\
&= \|w_m - y_m\|^2 + \|u_m - y_m\|^2 + \|u_m - p\|^2 - \|w_m - p\|^2.
\end{aligned} \tag{3.19}$$

It follows that

$$\|u_m - p\|^2 \leq \|w_m - p\|^2 - \|w_m - y_m\|^2 - \|u_m - y_m\|^2 + 2\beta\lambda_m(g(w_m, u_m) - g(w_m, y_m) - g(y_m, u_m)). \tag{3.20}$$

and using the step size (3.6), we obtain

$$\|u_m - p\|^2 \leq \|w_m - p\|^2 - (1 - \frac{\mu\lambda_m}{\lambda_{m+1}})[\|w_m - y_m\|^2 + \|u_m - y_m\|^2]. \tag{3.21}$$

Since $\mu \in (0, 1)$ and by Lemma 3.1, $\lim_{m\to\infty} \lambda_m = \lambda$, we have

$$\lim_{m\to\infty}\left(1 - \mu\frac{\lambda_m}{\lambda_{m+1}}\right) > 0. \tag{3.22}$$

Thus, from (3.21) and (3.22)

$$\|u_m - p\| \leq \|w_m - p\|. \tag{3.23}$$

In addition, using Algorithm 3.1, (3.23) and (3.11), we have

$$\begin{aligned}||x_{m+1}-p||^2 &= ||(1-\eta_m)w_m+\eta_m u_m-p||^2\\ &= (1-\eta_m)||w_m-p||^2+\eta_m||u_m-p||^2-\eta_m(1-\eta_m)||w_m-u_m||^2\\ &\leq (1-\eta_m)||w_m-p||^2+\eta_m||u_m-p||^2\\ &\leq (1-\eta_m)||w_m-p||^2+\eta_m||w_m-p||^2\\ &= ||w_m-p||^2. \end{aligned} \tag{3.24}$$

This implies

$$||x_{m+1}-p|| \leq ||w_m-p||.$$

Thus, we have

$$\begin{aligned}||x_{m+1}-p|| &\leq ||w_m-p||\\ &\leq \max\{||x_m-p||, \frac{||f(p)-p||+N_2}{(1-k)}\}\\ &\leq \vdots\\ &\leq \max\{||x_0-p||, \frac{||f(p)-p||+N_2}{(1-k)}\}. \end{aligned} \tag{3.25}$$

Thus, $\{x_m\}$ is bounded. □

Theorem 3.1. *Let $\{x_m\}$ be the sequence generated by Algorithm 3.1. Then, under the Assumption 3.1, $\{x_m\}$ converges strongly to $p \in SOL(g, C)$.*

Proof. Let $p \in sol(g, C)$ and $s_m = x_m + \theta_m(x_m - x_{m-1})$, thus, w_m in Algorithm 3.1 becomes $w_m = \alpha_m f(x_m) + (1-\alpha_m)s_m$. We have

$$\begin{aligned}||s_m-p||^2 &= ||x_m+\theta_m(x_m-x_{m-1})-p||^2\\ &= ||x_m-p||^2+2\theta_n\langle x_m-p, x_m-x_{m-1}\rangle+\theta_m^2||x_m-x_{m-1}||^2\\ &\leq ||x_m-p||^2+2\theta_m||x_m-x_{m-1}||||x_m-p||+\theta_m^2||x_m-x_{m-1}||^2\\ &\leq ||x_m-p||^2+\theta_m||x_m-x_{m-1}||[2||x_m-p||+\theta_m||x_m-x_{m-1}||]\\ &= ||x_m-p||^2+\theta_m||x_m-x_{m-1}||[2||x_m-p||+\alpha_n\frac{\theta_m}{\alpha_m}||x_m-x_{m-1}||]\\ &\leq ||x_m-p||^2+\theta_m||x_m-x_{m-1}||[2||x_m-p||+\alpha_m N_2]\\ &\leq ||x_m-p||^2+\theta_m||x_m-x_{m-1}||N_3, \end{aligned} \tag{3.26}$$

where $N_3 > 0$. Furthermore, using (2.1) and (3.26), we get

$$\begin{aligned}||w_m-p||^2 &= ||\alpha_m f(x_m)+(1-\alpha_m)s_m-p||^2\\ &= \alpha_m^2||f(x_m)-p||^2+(1-\alpha_m)^2||s_m-p||^2+2\alpha_m(1-\alpha_m)\langle f(x_m)-p, s_m-p\rangle\\ &= \alpha_m^2||f(x_m)-p||^2+(1-\alpha_m)^2||s_m-p||^2\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_m(1-\alpha_m)[\langle f(x_m)-f(p), s_m-p\rangle + \langle f(p)-p, s_m-p\rangle]\\
&\le \alpha_m^2\|f(x_m)-p\|^2 + (1-\alpha_m)^2\|s_m-p\|^2 + 2\alpha_m(1-\alpha_m)[\frac{1}{2}\|f(x_m)-f(p)\|^2 + \frac{1}{2}\|s_m-p\|^2\\
&+\langle f(p)-p, s_m-p\rangle]\\
&\le \alpha_m^2\|f(x_m)-p\|^2 + (1-\alpha_m)^2\|s_m-p\|^2 + \alpha_m(1-\alpha_m)k^2\|x_m-p\|^2\\
&+2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle\\
&\le \alpha_m^2\|f(x_m)-p\|^2 + (1-\alpha_m)\|x_m-p\|^2 + \theta_m\|x_m-x_{m-1}\|N_2 + \alpha_m k\|x_m-p\|^2\\
&+2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle\\
&=(1-\alpha_m(1-k))\|x_m-p\|^2 + \alpha_m^2\|f(x_m)-p\|^2\\
&+\theta_m\|x_m-x_{m-1}\|N_2 + 2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle. \qquad (3.27)
\end{aligned}
$$

In addition, we get

$$
\begin{aligned}
\|x_{m+1}-p\|^2 &= \|(1-\eta_m)w_m + \eta_m u_m - p\|^2\\
&\le (1-\eta_m)\|w_m-p\|^2 + \eta_m\|u_m-p\|^2 - \eta_m(1-\eta_m)\|w_m-u_m\|^2\\
&\le (1-\eta_m)\|w_m-p\|^2 + \eta_m\|w_m-p\|^2 - \eta_m(1-\eta_m)\|w_m-u_m\|^2\\
&= \|w_m-p\|^2 - \eta_m(1-\eta_n)\|w_m-u_m\|^2\\
&\le (1-\alpha_m(1-k))\|x_m-p\|^2 + \alpha_m^2\|f(x_m)-p\|^2 + \theta_m\|x_m-x_{m-1}\|N_2\\
&+2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle - \eta_m(1-\eta_m)\|w_m-u_m\|^2 \qquad (3.28)\\
&= (1-\alpha_m(1-k))\|x_m-p\|^2 + \alpha_m(1-k)\Bigg[\frac{\alpha_m}{(1-k)}\|f(x_s)-p\|^2 + \frac{\theta_m}{\alpha_m}\frac{\|x_m-x_{m-1}\|N_2}{(1-k)}\\
&+\frac{2(1-\alpha_m)}{(1-k)}\langle f(p)-p, s_m-p\rangle - \frac{\alpha_m\eta_m(1-\eta_m)}{(1-k)}\|w_m-u_m\|^2\Bigg]\\
&= (1-\alpha_m(1-k))\|x_m-p\|^2 + \alpha_m(1-k)\Psi_m,
\end{aligned}
$$

where $\Psi_m = \Bigg[\frac{\alpha_m}{(1-k)}\|f(x_m)-p\|^2 + \frac{\theta_m}{\alpha_m}\frac{\|x_m-x_{m-1}\|N_2}{(1-k)} + \frac{2(1-\alpha_m)}{(1-k)}\langle f(p)-p, s_m-p\rangle - \frac{\alpha_m\eta_m(1-\eta_m)}{(1-k)}\|w_m-u_m\|^2\Bigg]$. By Lemma 2.2, next, it is suffices to show that $\limsup_{k\to\infty}\Psi_{m_k} \le 0$ for every subsequence $\{\|x_{m_k}-p\|\}$ of $\{\|x_m-p\|\}$ fulfilling the condition:

$$\liminf_{k\to\infty}\{\|x_{m_k+1}-p\| - \|x_{m_k}-p\|\} \ge 0. \qquad (3.29)$$

From (3.28), we have

$$
\begin{aligned}
\|x_{m+1}-p\|^2 &\le \|x_m-p\|^2 + \alpha_m^2\|f(x_m)-p\|^2 + \theta_m\|x_m-x_{m-1}\|N_2 + 2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle\\
&-\eta_m(1-\eta_m)\|w_m-u_m\|^2, \qquad (3.30)
\end{aligned}
$$

which implies

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\eta_{m_k}(1-\eta_{m_k})\|w_{m_k}-u_{m_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\Big[\|x_{m_k}-p\|^2+\alpha_{m_k}^2\|f(x_{m_k})-p\|^2+\alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k}-x_{m_k-1}\|N_2 \\
&+2\alpha_{m_k}(1-\alpha_{m_k})\langle f(p)-p, s_{m_k}-p\rangle-\|x_{m_k+1}-p\|^2\Big]\\
&\leq -\liminf_{k\to\infty}[\|x_{m_k+1}-p\|^2-\|x_{m_k}-p\|^2]\leq 0.
\end{aligned} \tag{3.31}$$

Thus, we have

$$\lim_{k\to\infty}\|w_{m_k}-u_{m_k}\|=0. \tag{3.32}$$

From 3.28 and (3.23), we have

$$\begin{aligned}
\|x_{m+1}-p\|^2 &= \|(1-\eta_m)w_m+\eta_m u_m-p\|^2\\
&\leq (1-\eta_m)\|w_m-p\|^2+\eta_m\|u_m-p\|^2-\eta_m(1-\eta_m)\|w_m-u_m\|^2\\
&\leq (1-\eta_m)\|w_m-p\|^2+\eta_m[\|w_m-p\|^2-(1-\frac{\mu\lambda_m}{\lambda_{m+1}})[\|w_m-y_m\|^2+\|u_m-y_n\|^2]]\\
&-\eta_m(1-\eta_n)\|w_m-u_m\|^2\\
&= \|w_m-p\|^2-\eta_n(1-\frac{\mu\lambda_m}{\lambda_{n+1}})[\|w_m-y_m\|^2+\|u_m-y_m\|^2]]-\eta_m(1-\eta_n)\|w_m-u_m\|^2\\
&\leq \|x_m-p\|^2+\alpha_m^2\|f(x_m)-p\|^2+\theta_m\|x_m-x_{m-1}\|N_2+2\alpha_m(1-\alpha_m)\langle f(p)-p, s_m-p\rangle\\
&-\eta_m(1-\frac{\mu\lambda_m}{\lambda_{m+1}})[\|w_m-y_m\|^2+\|u_m-y_m\|^2]]-\eta_m(1-\eta_m)\|w_m-u_m\|^2,
\end{aligned} \tag{3.33}$$

which implies

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\eta_{m_k}(1-\frac{\mu\lambda_{m_k}}{\lambda_{m_k+1}})[\|w_{m_k}-y_{m_k}\|^2+\|u_{m_k}-y_{m_k}\|^2]]\right)\\
&\leq \limsup_{k\to\infty}\Big[|x_{m_k}-p\|^2+\alpha_{m_k}^2\|f(x_{m_k})-p\|^2+\alpha_{m_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{m_k}-x_{m_k-1}\|N_2-\eta_{m_k}(1-\eta_{m_k})\|w_{m_k}-u_{n_k}\|^2\\
&+2\alpha_{m_k}(1-\alpha_{m_k})\langle f(p)-p, s_{m_k}-p\rangle\|x_{m_k+1}-p\|^2\Big]\\
&\leq -\liminf_{k\to\infty}[\|x_{m_k+1}-p\|^2-\|x_{m_k}-p\|^2]\leq 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|w_{m_k}-y_{m_k}\|=0=\lim_{k\to\infty}\|u_{m_k}-y_{m_k}\|^2. \tag{3.34}$$

It is also, easy to see that

$$\|w_{m_k}-s_{m_k}\|\leq \alpha_{m_k}\|f(x_{m_k})-s_{m_k}\|\to 0 \text{ as } k\to\infty. \tag{3.35}$$

$$\|s_{m_k} - x_{m_k}\| = \alpha_{m_k}\frac{\theta_{m_k}}{\alpha_{m_k}}\|x_m - x_{m_k-1}\| \to 0 \text{ as } k \to \infty. \tag{3.36}$$

$$\|w_{m_k} - x_{m_k}\| \le \|w_{m_k} - s_{m_k}\| + \|s_{m_k} - x_{m_k}\| \to 0 \text{ as } k \to \infty. \tag{3.37}$$

$$\|u_{m_k} - x_{m_k}\| \le \|u_{m_k} - w_{m_k}\| + \|w_{m_k} - x_{m_k}\| \to 0 \text{ as } k \to \infty. \tag{3.38}$$

Thus, we obtain

$$\|x_{m_k+1} - x_{m_k}\| \le (1-\eta_{m_k})\|u_{m_k} - w_{m_k}\| + \eta_{m_k}\|w_{m_k} - x_{m_k}\| \to 0 \text{ as } k \to \infty. \tag{3.39}$$

Now, since $\{x_{m_k}\}$ is bounded, then, there exists a subsequence $\{x_{m_{k_j}}\}$ of $\{x_{m_k}\}$ such that $\{x_{m_{k_j}}\}$ converges weakly to $x^* \in H$. From the fact that $\liminf_{k\to\infty}\|w_{m_k} - x_{n_k}\| = 0$, and $\liminf_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0$, then $\{w_{m_{k_j}}\}$, and $\{u_{n_{k_j}}\}$ converges weakly to x^*. From, (3.34), we obtain that $\{y_{n_{k_j}}\}$ converges weakly to x^*. Next, we claim that $x^* \in SOL(g, C)$. To see this, from the definition of y_{m_k} and Lemma 2.1, we get

$$0 \in \partial\left(\beta\lambda_{m_k} g(w_{m_k}, y) + \frac{1}{2}\|w_{m_k} - y\|^2\right)(y_{m_k}) + N_C(y_{m_k}).$$

Thus, there exists $q \in N_C(y_{m_k})$ and $z_{m_k} \in \partial g(w_m, \cdot)(y_{m_k})$ such that

$$\beta\lambda_m z_{m_k} + y_{m_k} - w_{m_k} + q = 0. \tag{3.40}$$

Since $q \in N_C(y_{m_k})$, it follows that $\langle q, y - y_{m_k}\rangle \le 0$, for all $y \in C$. From (3.40), we have

$$\lambda_{m_k}\beta\langle z_{m_k}, y - y_{m_k}\rangle \ge \langle w_{m_k} - y_{m_k}, y - y_{m_k}\rangle \ \forall\ y \in C. \tag{3.41}$$

Also, since $z_{m_k} \in \partial g(w_{m_k}, \cdot)(y_{m_k})$, we have

$$g(w_{m_k}, y) - g(w_{m_k}, y_{m_k}) \ge \langle z_{m_k}, y - y_{m_k}\rangle \ \forall\ y \in C. \tag{3.42}$$

Thus,

$$\beta\lambda_{m_k}\left[g(w_{m_k}, y) - g(w_{m_k}, y_{m_k})\right] \ge \langle w_{m_k} - y_{m_k}, y - y_{m_k}\rangle \ \forall\ y \in C. \tag{3.43}$$

Taking limit as $k \to \infty$, and using conditions and (3.34), we have

$$g((x^*, y)) \ge 0 \ \forall\ y \in C.$$

Thus, $x^* \in SOL(g, C)$.
Furthermore, we get

$$\limsup_{k\to\infty}\langle f(p) - p, x_{m_k} - p\rangle = \limsup_{j\to\infty}\langle f(p) - p, x_{m_{k_j}} - p\rangle = \langle f(p) - p, x^* - p\rangle \le 0, \tag{3.44}$$

using (3.39) and (3.44), we have

$$\limsup_{k\to\infty}\langle f(p) - p, x_{m_k+1} - p\rangle = \limsup_{k\to\infty}\langle f(p) - p, x_{m_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(p) - p, x_{m_k} - p\rangle = \langle f(p) - p, x^* - p\rangle \le 0. \tag{3.45}$$

which implies that

$$\limsup_{k\to\infty}\langle f(p) - p, x_{m_k+1} - p\rangle \le 0. \tag{3.46}$$

Also, using (3.36) and (3.44), we get

$$\limsup_{k\to\infty}\langle f(p)-p, s_{m_k}-p\rangle = \limsup_{k\to\infty}\langle f(p)-p, s_{m_k}-x_{m_k}\rangle + \limsup_{k\to\infty}\langle f(p)-p, x_{m_k}-p\rangle = \langle f(p)-p, x^*-p\rangle \leq 0. \tag{3.47}$$

Thu, we have

$$\limsup_{k\to\infty}\langle f(p)-p, s_{m_k}-p\rangle = \langle f(p)-p, x^*-p\rangle \leq 0. \tag{3.48}$$

Using our Condition B (1) and the above inequality, we have that $\limsup_{k\to\infty}\Psi_{m_k} = \limsup_{k\to\infty}\left[\frac{\alpha_{m_k}}{(1-k)}\|f(x_{m_k})-p\|^2 + \frac{\theta_{m_k}}{\alpha_{m_k}}\frac{\|x_{m_k}-x_{m_k-1}\|N_2}{(1-k)} + \frac{2(1-\alpha_{m_k})}{(1-k)}\langle f(p)-p, s_{m_k}-p\rangle - \frac{\alpha_{n_k}\eta_{m_k}(1-\eta_{m_k})}{(1-k)}\|w_{m_k}-u_{m_k}\|^2\right] \leq 0$. Thus, by Lemma 2.2, we have $\lim_{n\to\infty}\|x_m-p\| = 0$. Thus, $\{x_m\}$ converges strongly to $p \in SOL(g,C)$. □

4. Application to Variational Inequality Problem

In this section, we will apply our results to variational inequality problem (VIP). The classical VIP for an operator $B : H \to H$ is formulated as follows: find $w^* \in C$ such that

$$\langle Bw^*, y-w^*\rangle \geq 0, \forall y \in H. \tag{4.1}$$

The solution set of the VIP (4.1) is denoted by $VI(C,B)$. Now, we consider the following condition for solving the VIP (4.1):

(A_1) $B : H \to H$ is a pseudoonotone operator, i.e.

$$\langle Bw, y-w\rangle \geq 0 \implies \langle By, w-y\rangle \leq 0, \forall w, y \in H.$$

(A_2) $B : H \to H$ is a L–Lipschitz continuous operator, i.e. there exist $L > 0$ such that

$$\|Bw-By\| \leq L\|w-y\|, \forall w, y \in H.$$

(A_3) $B : H \to H$ is a sequentially weakly continuous operator.

Set $g(w,y) = \langle Bw, y-w\rangle$, $\forall w, y \in C$, then the (EP) becomes the (VIP) with $L = 2L_1 = 2L_2$. Moreover, we have

$$y_m = \text{argmin}_{y\in C}\{\beta\lambda_m g(w_m, y) + \frac{1}{2}\|y-w_m\|^2\} = P_C(w_m - \beta\lambda_m Bw_m).$$

where P_C is called the metric projection of H onto C. Hence, we obtain the following result:

Corollary 4.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Assume that condition B and assumption (A_1)–(A_3) hold. Let $f : H \to H$ be a contraction mapping with contraction constant $k \in [0,1)$ and the solution set $VI(C,B) \neq \emptyset$. Then the sequence generated by Algorithm 4.1 strongly converges to an element $p \in VI(C,B)$.*

Algorithm 4.1. *An Hybrid Viscosity Iterative Method*

Step 1: *Choose $x_0, x_1 \in H_1$, given the iterates x_{m-1} and x_m for all $m \in \mathbb{N}$.*

$$\overline{\theta}_m = \begin{cases} \min\left\{\theta, \frac{\epsilon_m}{\|x_m - x_{m-1}\|\}}\right\}, & \text{if } x_m \neq x_{m-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{4.2}$$

Step 2: *Compute*

$$w_n = \alpha_m f(x_m) + (1 - \alpha_m)(x_m + \theta_m(x_m - x_{m-1})), \tag{4.3}$$

$$y_m = P_C(w_m - \beta\lambda_m B w_m). \tag{4.4}$$

If $w_m = y_m$, then stop and y_m is a solution. Otherwise, go to Step 3.

Step 3: *Construct a half-space*

$$T_m = \{z \in H : \langle w_m - \lambda_m \beta B w_m - y_m, z - y_m\rangle \leq 0\}, \tag{4.5}$$

and

$$\lambda_{m+1} = \begin{cases} \min\left\{\lambda_m, \frac{\mu[\|w_m - y_m\|^2 + \|u_m - y_m\|^2]}{2\beta\langle Bw_m - By_m, u_m - y_m\rangle}\right\}, & \text{if } \langle Bw_m - By_m, u_m - y_m\rangle > 0 \\ \\ \lambda_m, & \text{otherwise.} \end{cases} \tag{4.6}$$

compute

$$u_m = P_{T_m}(w_m - \beta\lambda_m B y_m). \tag{4.7}$$

Step 4: *Compute*

$$x_{m+1} = (1 - \eta_m) w_m + \eta_m u_m. \tag{4.8}$$

5. Numerical Example

In this section, we present a numerical example to further test the computational advantage of the proposed Algorithm 3.1 with Algorithm 3 of Xie et al. [35] (shortly, XCT Alg. 3), Algorithm 3.1 of Yang and Liu [36] (shortly, YL Alg.3.1) and Algorithm 2.1 of Yekini el al. [19] (shortly, SSTT Alg. 2.1). For Algorithm 4.1, we choose the following parameters: $fx = \frac{x}{2}$, $\epsilon_m = \frac{1}{(2m+1)^3}$, $\alpha_m = \eta_m = \frac{1}{(2m+1)}$, $\mu = 0.6$, $\theta = 0.4$ and $\lambda_1 = 2.5$. For XCT Alg. 3, choose $\alpha_m = \frac{1}{(2m+1)}$, $\beta_m = \frac{1}{2}(1 - \alpha_m)$, $\mu = 0.6$, $\lambda_1 = 2.5$, $Ts = \frac{s}{2}$, $f(x) = \frac{x}{3}$ and $k = 0.8$. For YL Alg. 3.1, choose $\lambda_1 = 2.5$, $\mu = 0.6$, $Ss = \frac{s}{2}$, $\beta_m = \frac{1}{2}$. For SSTT Alg. 2.1, choose $\lambda_1 = 2.5$, $\mu = 0.6$, $\alpha = 0.1$.

Example 5.1. *Let the feasible set C be defined by $C = \{x \in \mathbb{R}^n : -5 \leq x_j \leq 5, j = 1, 2 \cdots, n\}$, and $g : C \times C \to \mathbb{R}$ be a bifunction defined by*

$$g(x, y) = \langle Px + Qy + r, y - x\rangle, \ \forall x, y \in C,$$

where $r \in \mathbb{R}^n$ and $P, Q \in \mathbb{R}^{n \times n}$. The matrix P is symmetric positive semi-definite and the matrix $(Q - P)$ is symmetric negative semi-definite with Lipschitz constant $L_1 = L_2 = \frac{\|P-Q\|}{2}$ (for more details, see [35]). In this experiment. We consider the stopping criterion $E_m = \|x_{m+1} - x_m\| < 10^{-7}$ and for $n = 40, 80$, we obtain the following table and figure.

Table 1. Results of the Numerical Simulations for Different Dimensions

Numerical Results for $n = 40, 80, 120$ in Example 5.1								
	MOAAN Alg. 3.2		*XCT Alg. 3*		*YL Alg.3.1*		*SSTT Alg. 2.1*	
n	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*	*Iter*	*CPU time (sec.)*
$n = 40$	*10*	*0.0010*	*19*	*0.0021*	*29*	*0.0032*	*92*	*0.0092*
$n = 80$	*17*	*0.0015*	*20*	*0.0022*	*22*	*0.0025*	*110*	*0.0213*

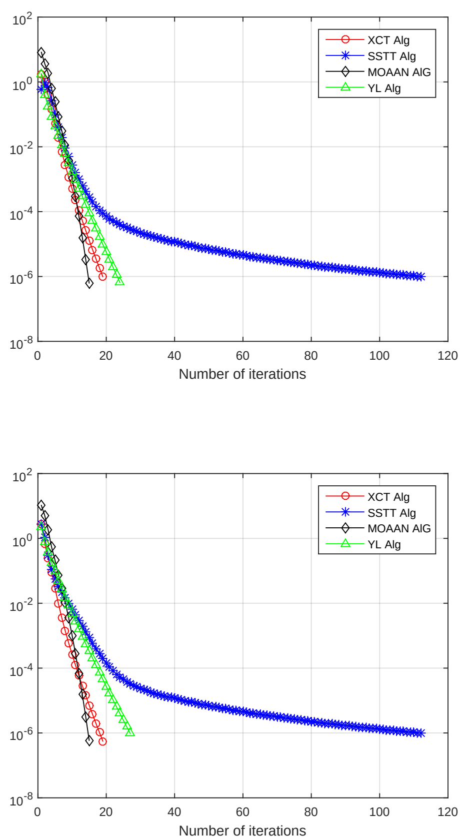

Figure 1. Example 5.1, **n=40** (top); **n=80** (bottom).

6. Conclusion

In this work, we have introduced an efficient method for approximating the solution of equilibrium problem. The strong convergence result of the proposed method is achieved under some mild conditions on the control parameters. Furthermore, we apply our results to solve variational inequality problem and image recovery problem. We carried out some numerical experiments to show the applicability of our method and also illustrate that our method outperforms several existing methods.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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