Int. J. Anal. Appl. (2025), 23:195

International Journal of Analysis and Applications

Applications of Fuzzy Sets on Essential Ideals of n-Ary Semigroups with Zero

Pattarawan Singavananda[1], **Ronnason Chinram**[2,*]

[1]*Mathematics Program, Faculty of Science and Technology, Songkhla Rajabhat University, Songkhla 90000, Thailand*

[2]*Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*

**Corresponding author:* ronnason.c@psu.ac.th

Abstract. An n-ary semigroup is an algebraic system that consists of a non-empty set and an associative n-ary operation. In this paper, we introduce the concepts of essential ideals and essential fuzzy ideals of n-ary semigroups. Furthermore, we investigate certain relationships between essential ideals and essential fuzzy almost ideals in n-ary semigroups.

1. Introduction

In 1904, Kasner [4] first introduced the generalization of classical algebraic structures to n-ary structures. An n-ary semigroup is a non-empty set equipped with an associative n-ary operation. The cases $n = 2$ and $n = 3$ correspond to classical semigroups and ternary semigroups, respectively, making them special instances of n-ary semigroups. Many properties of semigroups and ternary semigroups have been extended to n-ary semigroups. However, an n-ary semigroup does not necessarily reduce to a semigroup or a ternary semigroup. A fuzzy subset of a universal set is defined as a membership function that maps each element of the universal set to a value in the closed interval $[0, 1]$. The concept of fuzzy subsets was first introduced by Zadeh in 1965 [7]. His ideas have been applied in a wide range of fields. The concepts of essential ideals and fuzzy essential ideals have been defined and studied in semigroups [1]. Subsequently, various kinds of essential ideals and their fuzzifications within semigroups were defined and explored [2, 3, 5]. The introduction and study of essential ideals and their fuzzifications in ternary semigroups were carried out in [6].

Received: Jun. 30, 2025.

2020 *Mathematics Subject Classification.* 20N15, 03E72.

Key words and phrases. n-ary semigroups; essential ideals; essential fuzzy ideals; minimal; prime; semiprime.

https://doi.org/10.28924/2291-8639-23-2025-195

ISSN: 2291-8639

This paper aims to extend the results of essential ideals and essential fuzzy ideals in semigroups and ternary semigroups to the setting of n-ary semigroups with zero. Some basic notations and definitions are presented in Section 2. Section 3 presents the main results. We introduce essential ideals and fuzzy essential ideals of n-ary semigroups, provide their properties, and explore some relationships between essential ideals and fuzzy essential ideals.

2. Preliminaries

The aim of this section is to review some notations and definitions related to n-ary semigroups and fuzzy sets.

2.1. **n-ary semigroups.** First, we recall the definition of n-ary semigroups, where n is a positive integer such that $n \geq 2$.

A nonempty set A, together with an n-ary operation given by $f : A^n \to A$, where $n \geq 2$, is called an *n-ary groupoid* and denoted by (A, f). According to the general convention used in the symbols of n-ary groupoids, we denote the sequence of elements $a_i, a_{i+1}, \ldots, a_j$ by a_i^j. In the case $j < i$, the symbol a_i^j is the empty symbol. If $a_{i+1} = a_{i+2} = \cdots = a_{i+t} = a$, then we denote a^t instead of a_{i+1}^{i+t}. Under this convention, we write $f(a_1, a_2, \ldots, a_n) = f(a_1^n)$ and

$$f(a_1, \ldots, a_i, \underbrace{a, \ldots, a}_{t}, a_{i+t+1}, \ldots, a_n) = f(a_1^i, a^t, a_{i+t+1}^n).$$

An n-ary groupoid (A, f) is called *(i, j)-associative* if

$$f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1})$$

holds for all $a_1, a_2, \ldots, a_{2n-1} \in A$. The n-ary operation f is called *associative* if the above identity holds for every $1 \leq i \leq j \leq n$. In the case of the n-ary operation f is associative, (A, f) is called an *n-ary semigroup*. An element z of an n-ary semigroup A is called a *zero* of A if $f(a_1^{i-1}, z, a_{i+1}^n) = z$ for all $1 \leq i \leq n$ and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, , a_n \in A$. The zero element is denoted by 0. A nonempty subset S of an n-ary semigroup (A, f) is called an *n-ary subsemigroup* of A if $f(a_1^n) \in S$ for all $a_1, a_2, \ldots, a_n \in S$. For nonempty subsets $S_1, S_2, \ldots, S_n$ of A, define

$$f(S_1^n) := \{f(a_1^n) \mid a_i \in S_i \text{ for all } i \in \{1, 2, \ldots, n\}\}.$$

If $S_1 = \{a_1\}$, we write $f(\{a_1\}, S_2^n)$ as $f(a_1, S_2^n)$. Similarly, for other cases, we write $f(\{a_1\}, S_2^{n-1}, \{a_n\})$ as $f(a_1, S_2^{n-1}, a_n)$, and so on. For any subset S of A, we define

$$f(S^n) = \{f(a_1^n) \mid a_1, a_2, \ldots, a_n \in S\}$$

and

$$f(A^n) = \{f(a_1^n) \mid a_1, a_2, \ldots, a_n \in A\}.$$

A nonempty subset I of an n-ary semigroup A is called an *i-ideal* of A if for every $a_1, ..., a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and $x \in I$, we have $f(a_1^{i-1}, x, a_{i+1}^n) \in I$. A nonempty subset I of A is called an *ideal* of A if I is an i-ideal for every $1 \leq i \leq n$. If an n-ary semigroup A contains a zero, then every ideal of A also contains a zero.

2.2. **Fuzzy subsets.** A fuzzy subset of a set A is a membership function from A into the closed unit interval $[0,1]$. Next, we recall some notations in fuzzy sets. Let g and h be any two fuzzy subsets of a nonempty set A.

1. (The intersection of g and h) $g \cap h$ is a fuzzy subset of A defined for $a \in A$ by $(g \cap h)(a) = \min\{g(a), h(a)\}$.
2. (The union of g and h) $g \cup h$ is a fuzzy subset of A defined for $a \in A$ by $(g \cup h)(x) = \max\{g(a), h(a)\}$.
3. (Fuzzy subset) $g \subseteq h$ if $g(a) \leq h(a)$ for all $a \in A$.

For a fuzzy subset g of A, the *support* of g is a subset of A denoted by $supp(g)$ and defined by

$$supp(g) = \{a \in A \mid f(a) \neq 0\}.$$

The *characteristic mapping* of a subset S of A is a fuzzy subset χ_S of A defined by

$$\chi_S(a) = \begin{cases} 1 & a \in S, \\ 0 & a \notin S. \end{cases}$$

A fuzzy subset g of an n-ary semigroup A is called a *fuzzy ideal* of A if $g(a_1^n) \geq \max\{g(a_1), g(a_2), \ldots, g(a_n)\}$ for all $a_1, a_2, \ldots, a_n \in A$.

Theorem 2.1. *A non-empty subset I of a n-ary semigroup A is an ideal of A if and only if χ_I is a fuzzy ideal of A.*

Theorem 2.2. *If g is a nonzero fuzzy ideal of A, then $supp(g)$ is an ideal of A.*

3. Main Results

Throughout this section, we let A be an n-ary semigroup with zero 0. We begin by introducing the concept of essential ideals in A.

Definition 3.1. A nonzero ideal I of A is called an *essential ideal* of A if $I \cap N \neq \{0\}$ for every nonzero ideal N of A.

We consider the n-ary semigroup $\mathbb{Z}_{10}$ under the usual n-ary multiplication of integers modulo 10. Let $I = \{\overline{0}, \overline{5}\}$ and $J = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$. Then $I \cap J = \{\overline{0}\}$. Hence I and J are not essential ideals of $\mathbb{Z}_{10}$.

We have that the n-ary semigroup $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ under the usual n-ary multiplication of integers. Every nonzero ideal of $\mathbb{N}_0$ is essential.

Theorem 3.1. *If an ideal J of A contains an essential ideal I of A, then J is also an essential ideal of A.*

Proof. Assume that an ideal J of A contains an essential ideal I of A. Let N be any nonzero ideal of A. Then $I \cap N \neq \{0\}$. Since I is a subset of J, $I \cap N$ is also a subset of $J \cap N$. Since I is essential, $I \cap N \neq \{0\}$, and also $J \cap N \neq \{0\}$. Hence, J is an essential ideal of A. □

Corollary 3.1. *Let I and J be essential ideals of A. Then $I \cup J$ is also an essential ideal of A.*

Proof. Since $I \subseteq I \cup J$ and $I \cup J$ is clearly an ideal of A, by Theorem 3.1, we have that $I \cup J$ is an essential ideal of A. □

Theorem 3.2. *Let I and J be essential ideals of of A. Then $I \cap J$ is also an essential ideal of A.*

Proof. Assume that I and J are essential ideals of A. Thus $I \cap J \neq \{0\}$ and then $I \cap J$ is an ideal of A. Let N be any nonzero ideal of A. Therefore $I \cap N \neq \{0\}$. So there exists a nonzero element in A such that $a \in I \cap N$. Let (a) be an ideal of A generated by a. Then $(a) \neq \{0\}$. So $J \cap (a) \neq \{0\}$, thus there exists a nonzero element $b \in (a) \cap J$. Then $b \in (I \cap J) \cap N$. This implies that $(I \cap J) \cap N \neq \{0\}$. Hence, $I \cap J$ is an essential ideal of A. □

A fuzzy ideal g of A is called a *nontrivial fuzzy ideal* of A if there exists an element $a \in A \setminus \{0\}$ such that $g(a) \neq 0$. Based on this definition, we obtain the following theorem.

Theorem 3.3. *A fuzzy ideal g of A is nontrivial if and only if $supp(g) \neq \{0\}$.*

Definition 3.2. A fuzzy ideal g of A is said to be an *essential fuzzy ideal* of A if $supp(g \cap k) \neq \{0\}$ for every nontrivial fuzzy ideal k of A.

We consider the n-ary semigroup $\mathbb{N}_0$ under the usual n-ary multiplication of integers. Define a fuzzy subset g of $\mathbb{N}_0$ by $g(0) = 1$ and $g(n) = \dfrac{n-1}{n}$ for all $n \in \mathbb{N}$. We have that g is an essential fuzzy ideal of $\mathbb{N}_0$.

Theorem 3.4. *Let g be an essential fuzzy ideal of A. If h is a fuzzy ideal of A such that $g \subseteq h$, then h is also an essential fuzzy ideal of A.*

Proof. Let g be an essential fuzzy ideal of A and let h be any fuzzy ideal of A such that $g \subseteq h$. Let k be any nontrivial fuzzy ideal of A. Since g is essential, we have $supp(g \cap k) \neq \{0\}$. Because $g \subseteq h$, it follows that $supp(g \cap k) \subseteq supp(h \cap k)$. Therefore, $supp(h \cap k) \neq \{0\}$, and thus h is also an essential fuzzy ideal of A. □

Corollary 3.2. *Let g and h be essential fuzzy ideals of A. Then $g \cup h$ is also an essential fuzzy ideal of A.*

Proof. Since $g \subseteq g \cup h$ and $g \cup h$ is a fuzzy ideal of A, it follows from Theorem 3.4 that $g \cup h$ is also an essential fuzzy ideal of A. □

The following two theorems describe the relationship between essential ideals and essential fuzzy ideals of A.

Theorem 3.5. *A nonzero ideal I of A is essential if and only if χ_I is an essential fuzzy ideal of A.*

Proof. Assume that I is an essential ideal of A. By Theorem 2.1, we have that χ_I is a fuzzy ideal of A. To show that χ_I is essential, let k be any nontrivial fuzzy ideal of A. By Theorem 2.2 and Theorem 3.3, we have that $supp(k)$ is a nonzero ideal of A. Thus $I \cap supp(k) \neq \{0\}$. Then there exists a nonzero element a of A such that $a \in I \cap supp(k)$, this implies that $(\chi_I \cap k)(a) \neq 0$. So $a \in supp(\chi_I \cap k)$. Thus $supp(\chi_I \cap k) \neq \{0\}$. Therefore χ_I is an essential fuzzy ideal of A. To prove the converse, we assume that χ_I is an essential fuzzy ideal of A. Let N be a nonzero ideal of A. By Theorem 3.3, we have that χ_N is a nontrivial fuzzy ideal of A. Then $supp(\chi_I \cap \chi_N) \neq \{0\}$. So $\chi_{I\cap N} \neq C_{\{0\}}$. Thus $I \cap N \neq \{0\}$. Hence I is essential. □

Theorem 3.6. *A nontrivial fuzzy ideal g of A is essential if and only if $supp(g)$ is an essential ideal of A.*

Proof. Let g be a essential fuzzy ideal of A. By Theorem 2.2, we have that $supp(g)$ is an ideal of A. Let J be any nonzero ideal of A. By Theorem 3.3, χ_J is a nontrivial fuzzy ideal of A. Since g is essential, $supp(g \cap \chi_J) \neq \{0\}$. Thus there exists a nonzero element a in A such that $(g \cap \chi_J)(a) \neq 0$. Then $g(a) \neq 0$ and $\chi_J(a) \neq 0$. This implies that $a \in supp(g) \cap J$. Therefore $supp(g) \cap J \neq \{0\}$. We conclude that $supp(g)$ is an essential ideal of A. Conversely, assume that $supp(g)$ is an essential ideal of A. Let k be any nontrivial fuzzy ideal of A. By Theorem 3.3, we have $supp(k)$ is a nonzero ideal of A. So $supp(g) \cap supp(k) \neq \{0\}$. Thus there exists a nonzero element a of A such that $a \in supp(g) \cap supp(k)$. So $g(a) \neq 0$ and $k(a) \neq 0$. Therefore, $(g \cap k)(a) \neq 0$. Hence, $supp(g \cap k) \neq \{0\}$. This implies that g is an essential fuzzy ideal of A. □

Theorem 3.7. *Let g and h be essential fuzzy ideals of A. Then $g \cap h$ is also an essential fuzzy ideal of A.*

Proof. Assume that g and h are essential fuzzy ideals of A. It is clear that g and h are nontrivial and $g \cap h$ is also a nontrivial fuzzy ideal of A. By Theorem 3.6, we have that $supp(g)$ and $supp(h)$ are essential ideals of A. By Theorem 3.2, $supp(g \cap h) = supp(g) \cap supp(h)$ is an essential ideal of A. Also, Theorem 3.6 implies that $g \cap h$ is an essential fuzzy ideal of A. □

In the remainder of this section, we will investigate the relationships between minimal, prime, and semiprime essential ideals and their corresponding fuzzy counterparts: minimal, prime, and semiprime essential fuzzy ideals.

An essential ideal I of A is *minimal* if for any essential ideal J of A such that $I \subseteq J$, we have $I = J$. Next, we examine the minimality of essential fuzzy ideals.. An essential fuzzy ideal g of A is called *minimal* if for any essential fuzzy ideal h of A contained in g, implies that $supp(g) = supp(h)$. Now, we provide the relationship between minimal essential ideals and their fuzzifications.

Theorem 3.8. *A nonempty subset I of A is a minimal essential ideal of A if and only if χ_I is a minimal essential fuzzy ideal of A.*

Proof. Suppose that I is a minimal essential ideal of A. By Theorem 3.5, we have that χ_I is a fuzzy essential ideal of A. We let g is an essential fuzzy ideal of A such that $g \subseteq \chi_I$. By Theorem 3.6, we have that $supp(g)$ is an essential ideal of A. Since $g \subseteq \chi_S$, it follows that $supp(g) \subseteq supp(\chi_I) = I$. Since I is minimal, we have $supp(g) = I = supp(\chi_I)$. Therefore, χ_I is minimal. To prove the converse, we suppose that χ_I is a minimal essential fuzzy ideal of A and let J be an essential ideal of A such that $I \subseteq J$. It is implies from Theorem 3.5, we have χ_I is an essential fuzzy ideal of A. Since $I \subseteq J$, we have that $\chi_I \subseteq \chi_J$. Thus, $J = supp(\chi_J) = supp(\chi_I) = I$. We conclude that I is minimal. □

An essential ideal I of A is called *prime* if for all $a_1, a_2, \ldots, a_n \in A$, $f(a_1^n) \in I$ implies $a_i \in I$ for some $i \in \{1, 2, \ldots, n\}$. Similarly, an essential fuzzy ideal g of A is called *prime* if for all $a_1, a_2, \ldots, a_n \in A$, we have $g(f(a_1^n)) \leq \max\{g(a_1), g(a_2), \ldots, g(a_n)\}$. Next, we investigate the relationship between prime essential ideals and their fuzzifications.

Theorem 3.9. *A nonempty subset I of A is a prime essential ideal of A if and only if χ_I is a prime essential fuzzy ideal of A.*

Proof. Suppose that I is a prime essential ideal of A. By Theorem 3.5, we have that χ_I is an essential fuzzy ideal of A. Let $a_1, a_2, \ldots, a_n$ be any n elements of A. We consider two cases:

Case 1: $f(a_1^n) \in I$. So $a_i \in I$ for some $i \in \{1, 2, \ldots, n\}$ because I is prime. This implies that $\chi_I(a_i) = 1$ for some $i \in \{1, 2, \ldots, n\}$. Therefore $\chi_I(f(a_1^n)) = 1 \leq \max\{\chi_I(a_1), \chi_I(a_2), \ldots, \chi_I(a_n)\}$.

Case 2: $f(a_1^n) \notin I$. Then

$$\chi_I(f(a_1^n) = 0 \leq \max\{\chi_I(a_1), \chi_I(a_2), \ldots, \chi_I(a_n)\}.$$

In both two cases, we can conclude that

$$\chi_I(f(a_1^n) \leq \max\{\chi_I(a_1), \chi_I(a_2), \ldots, \chi_I(a_n)\}.$$

Therefore, χ_I is a prime essential fuzzy ideal of A. To prove the converse, assume that χ_I is a prime essential fuzzy ideal of A. By Theorem 3.5, we have that I is an essential ideal of A. Let $a_1, a_2, \ldots, a_n$ be any n elements in A such that $f(a_1^n) \in I$. Thus, $\chi_I(f(a_1^n)) = 1$. By assumption, we have that

$$1 = \chi_I(f(a_1^n)) \leq \max\{\chi_I(a_1), \chi_I(a_2), \ldots, \chi_I(a_n)\}.$$

Hence, $\chi_I(a_i) = 1$ for some $i \in \{1, 2, \ldots, n\}$. Thus $a_i \in I$ for some $i \in \{1, 2, \ldots, n\}$. Therefore, I is a prime essential ideal of A. □

An essential ideal I of A is called *semiprime* if for $a \in A$, $f(a^n) \in I$ implies $a \in I$. A fuzzy essential ideal g of A is called *semiprime* if for $a \in A$, we have $g(f(a^n)) \leq g(a)$. Every prime essential ideal of A is semiprime. It is also that every prime essential fuzzy ideal of A is semiprime. Finally, we present a relationship between semiprime essential ideals and their fuzzifications.

Theorem 3.10. *A nonempty subset I of A is a semiprime essential ideal of A if and only if χ_I is a semiprime essential fuzzy ideal of A.*

Proof. Assume that I is a semiprime essential ideal of A. By Theorem 3.5, we have that χ_I is an essential fuzzy ideal of A. Let a be an element in A. If $f(a^n) \in I$, then $a \in I$ because I is semiprime. Therefore $\chi_I(a) = 1$. Hence, $\chi_I(f(a^n) \leq \chi_I(a)$. If $f(a^n) \notin I$, then $\chi_I(f(a^n)) = 0 \leq \chi_I(a)$. In both cases, we can conclude that $\chi_I(f(a^n)) \leq \chi_I(a)$ for all $a \in A$. Therefore, χ_I is a semiprime essential fuzzy ideal of A. To show the converse, we assume that χ_I is a semiprime essential fuzzy ideal of A. By Theorem 3.5, we have that I is an essential ideal of A. Let a be any element of A such that $f(a^n) \in I$. Then $\chi_I(f(a^n)) = 1$. Because of χ_I is semiprime, this implies that $\chi_I(f(a^n)) \leq \chi_I(a)$. It follows that $\chi_I(a) = 1$. Therefore, $a \in I$. Consequently, I is a semiprime essential ideal of A. □

4. Conclusion

In this paper, we introduce the concepts of essential ideals and essential fuzzy ideals of A. We prove that the union and intersection of essential ideals is also an essential ideal. Additionally, we show that both the union and intersection of essential fuzzy ideals are always essential fuzzy ideals. Furthermore, we establish the relationships between essential ideals and their corresponding essential fuzzy ideals.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Baupradist, B. Chemat, K. Palanivel, R. Chinram, Essential Ideals and Essential Fuzzy Ideals in Semigroups, J. Discret. Math. Sci. Cryptogr. 24 (2020), 223–233. https://doi.org/10.1080/09720529.2020.1816643.

[2] R. Chinram, T. Gaketem, Essential (m,n)-Ideal and Essential Fuzzy (m,n)-Ideals in Semigroups, ICIC Express Lett. 15 (2021), 1037–1044. https://doi.org/10.24507/icicel.15.10.1037.

[3] T. Gaketem, A. Iampan, On Essential Bi-Ideals and Essential Fuzzy Bi-Ideals in Semigroups, Int. J. Appl. Math. 36 (2023), 483–496. https://doi.org/10.12732/ijam.v36i4.4.

[4] R. Kasner, An Extension of the Group Concepts, Bull. Am. Math. Soc. 10 (1904), 290–291.

[5] N. Panpetch, T. Muangngao, T. Gaketem, Some Essential Bi-Ideals and Essential Fuzzy Bi-Ideals in a Semigroup, J. Math. Comput. Sci. 28 (2022), 326–334. https://doi.org/10.22436/jmcs.028.04.02.

[6] R. Rittichuai, A. Iampan, R. Chinram, P. Singavananda, Essential Ideals and Their Fuzzifications of Ternary Semigroups, ICIC Express Lett. 17 (2023), 191–199. http://doi.org/10.24507/icicel.17.02.191.

[7] L. A. Zadeh, L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.