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Bi-Starlike and Bi-Convex Function Classes Connected to Shell-Like Curves and the q-Analogue of Fibonacci Numbers

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Abstract. Using the subordination principle, this study explores two subclasses of bi-univalent functions associated with shell-like curves via the q-analogue of Fibonacci numbers, namely the starlike and convex classes. We derive coefficient bounds for the initial terms of these function classes and establish the corresponding Fekete- Szegö inequalities. Our findings contribute to the advancement of biunivalent function theory and its interaction with special function spaces.

1. Introduction, Preliminaries and Definitions

Let $\mathcal{A}$ denote the family of all analytic functions defined on the open unit disk $\mathbb{U}$, where $\mathbb{U}$ is the set of all complex numbers $z = a + ib$ (with $a, b \in \mathbb{R}$) satisfying $|z| < 1$. Geometrically, $\mathbb{U}$ represents the collection of all points in the complex plane that lie strictly inside the unit circle centered at the origin.

The functions $f \in \mathcal{A}$ are normalized to satisfy the following initial conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These normalization conditions ensure that the functions are uniquely determined and facilitate the study of their properties within the unit disk. For every function $f \in \mathcal{A}$, the Taylor-Maclaurin

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series expansion can be expressed in the following form:

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

An analytic function f that satisfies $|f(z)| < 1$ and $f(0) = 0$ within the domain $\mathbb{U}$ is called a Schwartz functions. When considering two functions f_1 and f_2 from $\mathcal{A}$, f_1 is referred to as subordinate to f_2, denoted by $f_1 \prec f_2$, if a Schwarz function η exists such that $f_1(z) = f_2(\eta(z))$ for all $z \in \mathbb{U}$.

Additionally, examine the class S, which includes all functions $f \in \mathcal{A}$ that are univalent (injective) in the unit disk $\mathbb{U}$. Let P represent the collection of functions within $\mathcal{A}$ that possess positive real parts, defined as follows:

$$\mathsf{p}(z) = 1 + \sum_{n=1}^{\infty} \mathsf{p}_n z^n = 1 + \mathsf{p}_1 z + \mathsf{p}_2 z^2 + \mathsf{p}_3 z^3 + \ldots, \tag{1.2}$$

where

$$|\mathsf{p}_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{1.3}$$

This is in accordance with the renowned Carathéodory's Lemma (for more details, see [1]). Essentially, $\varphi \in \mathsf{P}$ if and only if $\varphi(z) \prec (1+z)(1-z)^{-1}$ for $z \in \mathbb{U}$.

The class of starlike functions, represented as S^*, can be defined by multiple methodologies employing the principle of subordination. Ma and Minda [2] introduced the class

$$\mathsf{S}^*(\Omega) = \left\{ f \in \mathscr{A} : \frac{z f'(z)}{f(z)} \prec \Omega(z), \quad \text{where} \quad \Omega \in \mathsf{P} \ \text{ and } \ z \in \mathbb{U} \right\}.$$

In this formulation, Ω is an analytic function in $\mathcal{A}$ with positive real parts. Table 1 delineates numerous categories of starlike functions, exemplifying the diverse methodologies employed by writers in establishing supplementary subclasses through the selection of particular forms of Ω.

As the foundation upon which many important subclasses of analytic functions are built, the class P is crucial to the study of analytic functions. For any function f in the subfamily S of $\mathcal{A}$, there exists an inverse function denoted as f^{-1} and defined by

$$z = f^{-1}(f(z)) \ \text{ and } \ \xi = f(f^{-1}(\xi)), \quad (\mathsf{r}_0(f) \geq 0.25; \ |\xi| < \mathsf{r}_0(f); z \in \mathbb{U}). \tag{1.4}$$

where

$$\chi(\xi) = f^{-1}(\xi) = \xi - \alpha_2 \xi^2 + \left(2\alpha_2^2 - \alpha_3\right) \xi^3 - \left(5\alpha_2^3 + \alpha_4 - 5\alpha_3\alpha_2\right) \xi^4 + \cdots. \tag{1.5}$$

function $f \in \mathsf{S}$ is said to be bi-univalent if its inverse function $f^{-1} \in \mathsf{S}$. The subclass of S denoted by $\sum$ contains all bi-univalent functions in $\mathbb{U}$. A table illustrating certain functions within the class $\sum$ and their inverse functions is provided below.

Recently, quantum calculus, also known as q-calculus, has garnered significant interest in the fields of physics and mathematics. Its historical origins date back to the 19th century when Jackson [17, 18] introduced the q-difference operator and integral. Subsequently, Aral and

TABLE 1. Lists several of the starlike classes defined by the subordination principle.

The family of starlike functions	Author/s	Ref.
$n^*\left(\frac{1+z}{1-z}\right) = \left\{ f \in \mathscr{A} : \dfrac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$	Janowski	[10,11]
$n^*(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \dfrac{1+(1-2\vartheta)z}{1-z}, 0 \leq \vartheta < 1 \right\}$	Robertson	[15]
$\mathsf{SL}(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \dfrac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}, \vartheta = \frac{1-\sqrt{5}}{2} \right\}$	Sokół	[12]
$\mathcal{SK}(\vartheta) = \left\{ f \in \mathscr{A} : \dfrac{zf'(z)}{f(z)} \prec \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2}, \vartheta \in (-3,1] \right\}$	Sokół	[13]

TABLE 2. Lists several of the starlike classes defined by the subordination principle.

f	f^{-1}
$f_1(z) = \dfrac{z}{1+z}$	$f_1^{-1}(z) = \dfrac{z}{1-z}$
$f_2(z) = -\log(1-z)$	$f_2^{-1}(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$
$f_3(z) = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$	$f_3^{-1}(z) = \dfrac{e^z-1}{e^z}$

Gupta [19–21] expanded upon this foundational research by examining q analogues of many operators, particularly in the realm of geometric function theory. Quantum calculus, grounded in q-differences, extends classical calculus and offers a robust framework for exploring specific subclasses of analytic functions, such as starlike and convex functions. In this context, the parameter q is anticipated to satisfy $0 < q < 1$, ensuring the requisite convergence and essential attributes necessary for these analyses.

Definition 1.1. *[22] The q-bracket $\lceil \lambda \rfloor_q$ is defined as follows:*

$$\lceil \lambda \rfloor_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & 0<q<1,\ \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ 1, & q \mapsto 0^+,\ \lambda \in \mathbb{C}^* \\ \lambda, & q \mapsto 1^-,\ \lambda \in \mathbb{C}^* \\ q^{\gamma-1} + q^{\gamma-2} + \cdots + q + 1 = \sum\limits_{n=0}^{\gamma-1} q^n, & 0<q<1,\ \lambda = \gamma \in \mathbb{N}, \end{cases}$$

with the useful identity $\lceil \kappa + 1 \rfloor_q = \lceil \kappa \rfloor_q + q^\kappa$.

Definition 1.2. *[22] The q−derivative, also known as the q−difference operator, of a function f is defined by*

$$ð_q\langle f(z)\rangle = \begin{cases} (f(z) - f(qz))(z - qz)^{-1}, & \text{if} \quad 0<q<1,\ z \neq 0, \\ f'(0), & \text{if} \quad z = 0, \\ f'(z), & \text{if} \quad q \mapsto 1^-,\ z \neq 0. \end{cases} .$$

Remark 1.1. *For $f \in \mathcal{A}$ of the form (1.1), it is straightforward to verify that*

$$ð_q\langle f(z)\rangle = ð_q\left\langle z + \sum_{n=2}^{\infty} \lceil n \rfloor_q \alpha_n z^n \right\rangle = 1 + \sum_{n=2}^{\infty} \lceil n \rfloor_q \alpha_n z^{n-1}, \quad (z \in \mathbb{U}),$$

and for the inverse function f^{-1} of the form (1.4), we have

$$ð_q\langle f^{-1}(\xi)\rangle = 1 - \lceil 2 \rfloor_q \alpha_2 \xi + \lceil 3 \rfloor_q \left(2\alpha_2^2 - \alpha_3\right) \xi^2 - \lceil 4 \rfloor_q \left(5\alpha_2^3 + \alpha_4 - 5\alpha_3\alpha_2\right) \xi^3 + \cdots .$$

More recently, by employing the q-Jackson difference operators, Alsoboh et al. [3] introduced a significant class of functions, termed q-starlike functions and denoted by SL_q. This class is formally defined as

$$\mathsf{SL}_q = \left\{ f \in \mathscr{A} : \quad \frac{zð_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q) \quad (z \in \mathbb{U}) \right\}, \tag{1.6}$$

where the function $\Upsilon(z;q)$ is explicitly given by

$$\Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2}, \tag{1.7}$$

and

$$\vartheta_q = \frac{1 - \sqrt{4q+1}}{2q} \tag{1.8}$$

denotes the q-analogue of the Fibonacci numbers. Furthermore, Alsoboh et al. [3] established a fundamental link between the q-analogue of Fibonacci numbers, denoted by ϑ_q, and their associated Fibonacci polynomials $\varphi_{\mathsf{s}}(q)$. In particular, they showed that if

$$\Upsilon(z;q) = 1 + \sum_{\mathsf{s}=1}^{\infty} \widehat{\mathsf{p}_{\mathsf{s}}}\, z^{\mathsf{s}},$$

then the coefficients $\widehat{\mathsf{p}_\mathsf{s}}$ satisfy the recurrence relation:

$$\widehat{\mathsf{p}_\mathsf{s}} = \begin{cases} \vartheta_q, & \text{for } \mathsf{s} = 1, \\ (2q+1)\vartheta_q^2, & \text{for } \mathsf{s} = 2, \\ (3q+1)\vartheta_q^3, & \text{for } \mathsf{s} = 3, \\ (\varphi_{\mathsf{s}+1}(q) + q\varphi_{\mathsf{s}-1}(q))\,\vartheta_q^{\mathsf{s}}, & \text{for } \mathsf{s} \geq 4. \end{cases} \tag{1.9}$$

where the q-Fibonacci polynomials $\varphi_\mathsf{s}(q)$ are defined as

$$\varphi_\mathsf{s}(q) = \frac{\left(1 - q\vartheta_q\right)^\mathsf{s} - \left(\vartheta_q\right)^\mathsf{s}}{\sqrt{4q+1}}, \quad \mathsf{s} \in \mathbb{N}. \tag{1.10}$$

This result provides a comprehensive framework for analyzing the interplay between q-deformed Fibonacci numbers and their polynomial counterparts.

The initial terms of the q-Fibonacci numbers, whichserve as a generalization of the classical Fibonacci numbers in the limit as approaches $q \mapsto 1^-$ (see [28, 29]), are provided in Table 2.

Table 3. The first initial terms of the sequence q-Fibonacci.

The q-analogue of Fibonacci numbers	The classical Fibonacci numbers
$\varphi_0(q) = 0$	$\varphi_0 = 0$
$\varphi_1(q) = 1$	$\varphi_1 = 1$
$\varphi_2(q) = 1$	$\varphi_2 = 1$
$\varphi_3(q) = 1 + q$	$\varphi_3 = 2$
$\varphi_4(q) = 1 + 2q$	$\varphi_4 = 3$

It is worth noting that the function $\Upsilon(z;q)$ is not univalent in the domain $\mathbb{U}$. Specifically, it attains the same value at two distinct points:

$$\Upsilon(0;q) = 1 \quad \text{and} \quad \Upsilon\left(-\frac{1}{2q\vartheta_q};q\right) = 1.$$

Example 1.1. *If $q \to 1^-$ in Definition 2.1, we obtain the class $\mathsf{SL} = \lim_{q\to 1^-} \mathsf{SL}_q$ defined as follows:*
A function f belongs to the class SL if and only if

$$\frac{z f'(z)}{f(z)} \prec \Upsilon(z),$$

where

$$\Upsilon(z;1) = \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}, \tag{1.11}$$

and $\vartheta = \frac{1-\sqrt{5}}{2}$ corresponds to the classical Fibonacci numbers.

In addition, Alsoboh et al. [4] introduced a q-convex class and denoted by KSL_q if and only if

$$1+\frac{z\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q) \qquad (z \in \mathbb{U}), \tag{1.12}$$

where the function $\Upsilon(z;q)$ is explicitly given by (1.7) and ϑ_q given by (1.8).

The advent of q-calculus has significantly advanced the study of analytic function theory by enabling the discovery of novel subclasses with intricate geometric and algebraic properties. These developments underscore the versatility of q-calculus, demonstrating its potential to enrich classical function theory and uncover new mathematical phenomena. The relevance of these findings extends to both theoretical and applied settings, providing a robust foundation for future research and innovation in the field [5–9, 23, 24, 30–36].

2. Definition and Example

Motivated by q-Fibonacci numbers, this section will now look at a novel subclasses of bi-univalent functions related to shell-like curves.

Definition 2.1. *A bi-univalent function f of the form* (1.1) *belongs to the class $\mathsf{SL}_\Sigma(\Upsilon(z;q))$ if and only if*

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q\, z-q\vartheta_q^2 z^2}, \tag{2.1}$$

and

$$\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} \prec \Upsilon(\xi;q) = \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q\, \xi-q\vartheta_q^2 \xi^2}, \tag{2.2}$$

where ϑ_q is given by (1.8)*.*

Definition 2.2. *A bi-univalent function f of the form* (1.1) *belongs to the class $\mathsf{KL}_\Sigma(\Upsilon(z;q))$ if and only if*

$$1+\frac{z\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q\, z-q\vartheta_q^2 z^2}, \tag{2.3}$$

and

$$1+\frac{\xi\eth_q^2\langle \chi(\xi)\rangle}{\eth_q\langle \chi(\xi)\rangle} \prec \Upsilon(\xi;q) = \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q\, \xi-q\vartheta_q^2 \xi^2}, \tag{2.4}$$

where ϑ_q is given by (1.8)*.*

Definition 2.3. *A bi-univalent function f of the form* (1.1) *belongs to the class* $\mathsf{SL}_{\Sigma}(\Upsilon(z))$ *if and only if*

$$\frac{zf'(z)}{f(z)} \prec \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-z-\vartheta^2 z^2},$$

and

$$\frac{\xi\chi'(\xi)}{\chi(\xi)} \prec \Upsilon(\xi) = \frac{1+q\vartheta^2\xi^2}{1-\vartheta\,\xi-\vartheta^2\xi^2},$$

where $\vartheta = \frac{1-\sqrt{5}}{2}$.

Definition 2.4. *A bi-univalent function f of the form* (1.1) *belongs to the class* $\mathsf{KL}_{\Sigma}(\Upsilon(z))$ *if and only if*

$$1+\frac{zf''(z)}{f'(z)} \prec \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-z-\vartheta^2 z^2},$$

and

$$1+\frac{\xi\chi''(\xi)}{\chi'(\xi)} \prec \Upsilon(\xi) = \frac{1+q\vartheta^2\xi^2}{1-\vartheta\,\xi-\vartheta^2\xi^2},$$

where $\vartheta = \frac{1-\sqrt{5}}{2}$.

3. Main Results

In this section, we obtain the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_3|$ for the bi-univalent starlike and convex subclasses $\mathsf{SL}_{\Sigma}(\Upsilon(z;q))$ and $\mathsf{KL}_{\Sigma}(\Upsilon(z;q))$, respectively.

Firstly, let $\mathsf{p}(z) = 1+\mathsf{p}_1 z+\mathsf{p}_2 z^2+\mathsf{p}_3 z^3+\dots$, and $\mathsf{p}(z) \prec \Upsilon(z;q)$. Then there exist $\varphi \in \mathsf{P}$ such that $|\varphi(z)| < 1$ in $\mathbb{U}$ and $\mathsf{p}(z) = \Upsilon(\varphi(z);q)$, we have

$$\hbar(z) = (1+\varphi(z))(1-\varphi(z))^{-1} = 1+\ell_1 z+\ell_2 z^2+\dots \in \mathsf{P} \qquad (z \in \mathbb{U}). \tag{3.1}$$

It follows that

$$\varphi(z) = \frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots, \tag{3.2}$$

and

$$\begin{aligned}
\Upsilon(\varphi(z);q) &= 1+\widehat{\mathsf{p}_1}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right] \\
&\quad + \widehat{\mathsf{p}_2}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^2 \\
&\quad + \widehat{\mathsf{p}_3}\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right]^3 + \cdots \\
&= 1+\frac{\widehat{\mathsf{p}_1}\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right]z^2 \\
&\quad + \frac{1}{2}\left[\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right)\widehat{\mathsf{p}_1} + \ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_2} + \frac{\ell_1^3}{4}\widehat{\mathsf{p}_3}\right]z^3 + \cdots.
\end{aligned} \tag{3.3}$$

And similarly, there exists an analytic function ν such that $|\nu(\xi)| < 1$ in $\mathbb{U}$ and $\mathsf{p}(\xi) = \Upsilon(\nu(\xi);q)$. Therefore, the function

$$\kappa(\xi) = (1+\nu(\xi))(1-\nu(\xi))^{-1} = 1 + \tau_1\xi + \tau_2\xi^2 + \cdots \in \mathsf{P}. \tag{3.4}$$

It follows that

$$\nu(\xi) = \frac{\tau_1\xi}{2} + \left(\tau_2 - \frac{\tau_1^2}{2}\right)\frac{\xi^2}{2} + \left(\tau_3 - \tau_1\tau_2 - \frac{\tau_1^3}{4}\right)\frac{\xi^3}{2} + \cdots, \tag{3.5}$$

and

$$\begin{aligned}\Upsilon(\nu(\xi);q) = 1 &+ \frac{\widehat{\mathsf{p}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right]\xi^2 \\ &+ \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\widehat{\mathsf{p}_1} + \tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_2} + \frac{\tau_1^3}{4}\widehat{\mathsf{p}_3}\right]\xi^3 + \cdots.\end{aligned} \tag{3.6}$$

In the following theorem we determine the initial Taylor coefficients $|\alpha_2|$ and $|\alpha_2|$ for the subclasses $\mathsf{SL}_\sum(\Upsilon(z);q)$ and $\mathsf{KL}_\sum(\Upsilon(z);q)$. Later we will reduce these bounds to other classes for special cases.

Theorem 3.1. *Let f given by* (1.1) *be in the class* $\mathsf{SL}_\sum(\Upsilon(z);q)$. *Then*

$$\left|\alpha_2\right| \leq \frac{\left|\vartheta_q\right|}{q\sqrt{1-2q\vartheta_q}}. \tag{3.7}$$

$$\left|\alpha_3\right| \leq \frac{\left|\vartheta_q\right|\left(q-(1+q+2q^2)\vartheta_q\right)}{q^2(1+q)\left(1-2q\vartheta_q\right)}. \tag{3.8}$$

Proof. Let $f \in \mathsf{SL}_\sum(\Upsilon(z))$ and $\xi = f^{-1}$. Considering (2.1) and (2.2) we have

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)} = \Upsilon(\varphi(z);q), \qquad (z \in \mathbb{U}), \tag{3.9}$$

and

$$\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} = \Upsilon(\nu(\xi);q), \qquad (\xi \in \mathbb{U}). \tag{3.10}$$

Since

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)} = 1 + q\alpha_2 z + q\left((1+q)\alpha_3 - \alpha_2^2\right)z^2 + \cdots, \tag{3.11}$$

and

$$\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} = 1 - q\alpha_2\xi + q\left((1+2q)\alpha_2^2 - (1+q)\alpha_3\right)\xi^2 + \cdots, \tag{3.12}$$

By comparing (3.9) and (3.11), along (3.3), yields

$$q\alpha_2 z + q\left((1+q)\alpha_3 - \alpha_2^2\right)z^2 + \cdots = \frac{\widehat{\mathsf{p}_1}\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right]z^2 + \cdots. \tag{3.13}$$

Besied that, by comparing (3.10) and (3.12), along (3.6), yields

$$-q\alpha_2\xi + q\Big((1+2q)\alpha_2^2 - (1+q)\alpha_3\Big)\xi^2 + \cdots = \frac{\widehat{\mathsf{p}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right]\xi^2 + \cdots. \tag{3.14}$$

Equating the pertinent coefficient in (3.13) and (3.14), we obtain

$$q\alpha_2 = \frac{\widehat{\mathsf{p}_1}\ell_1}{2} \tag{3.15}$$

$$-q\alpha_2 = \frac{\widehat{\mathsf{p}_1}\tau_1}{2} \tag{3.16}$$

$$q\Big((1+q)\alpha_3 - \alpha_2^2\Big) = \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right] \tag{3.17}$$

$$q\Big((1+2q)\alpha_2^2 - (1+q)\alpha_3\Big) = \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1} + \frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right] \tag{3.18}$$

From (3.15) and (3.16), we have

$$\ell_1 = -\tau_1 \quad \Longleftrightarrow \quad \ell_1^2 = \tau_1^2, \tag{3.19}$$

and

$$\alpha_2^2 = \frac{\vartheta_q^2}{8q^2}\Big(\ell_1^2 + \tau_1^2\Big) \quad \Longleftrightarrow \quad \ell_1^2 + \tau_1^2 = \frac{8q^2}{\vartheta_q^2}\alpha_2^2. \tag{3.20}$$

Now, by summing (3.17) and (3.18), we obtain

$$\begin{aligned} 2q^2\alpha_2^2 &= \frac{(\ell_2+\tau_2)\vartheta_q}{2} - \frac{\Big(\ell_1^2+\tau_1^2\Big)\vartheta_q}{4} + \frac{(2q+1)\Big(\ell_1^2+\tau_1^2\Big)\vartheta_q^2}{4} \\ &= \frac{(\ell_2+\tau_2)\vartheta_q}{2} + \left[\frac{(2q+1)\vartheta_q^2}{4} - \frac{\vartheta_q}{4}\right]\Big(\ell_1^2+\tau_1^2\Big). \end{aligned} \tag{3.21}$$

By putting (3.20) in (3.21), we obtain

$$\alpha_2^2 = \frac{(\ell_2+\tau_2)\vartheta_q^2}{4q^2\Big(1-2q\vartheta_q\Big)}. \tag{3.22}$$

Using (1.3) for (3.22), we have

$$\Big|\alpha_2\Big| \le \frac{\Big|\vartheta_q\Big|}{q\sqrt{1-2q\vartheta_q}}. \tag{3.23}$$

Now, so as to find the bound on $|\alpha_3|$, let's subtract from (3.17) and (3.18) along (3.20), we obtain

$$\alpha_3 = \alpha_2^2 + \frac{\Big(\ell_2-\tau_2\Big)\vartheta_q}{4q(1+q)}. \tag{3.24}$$

Hence, we get

$$\Big|\alpha_3\Big| \le \Big|\alpha_2\Big|^2 + \frac{|\vartheta_q|}{q(1+q)}. \tag{3.25}$$

Then, in view of (3.23), we obtain

$$\left|\alpha_3\right| \leq \frac{\left|\vartheta_q\right|\left(q-(1+q+2q^2)\vartheta_q\right)}{q^2(1+q)\left(1-2q\vartheta_q\right)}. \tag{3.26}$$

□

Theorem 3.2. *Let f given by* (1.1) *be in the class* $\mathsf{KL}_{\Sigma}(\Upsilon(z);q)$. *Then*

$$\left|\alpha_2\right| \leq \frac{\left|\vartheta_q\right|}{\sqrt{\lceil 2\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)}}, \tag{3.27}$$

and

$$\left|\alpha_3\right| \leq \frac{\left|\vartheta_q\right|\left(\lceil 2\rfloor_q-2(\lceil 3\rfloor_q+q)\vartheta_q\right)}{\lceil 2\rfloor_q\lceil 3\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)}. \tag{3.28}$$

Proof. Let $f \in \mathsf{SL}_{\Sigma}(\Upsilon(z))$ and $\xi = f^{-1}$. Considering (2.3) and (2.4) we have

$$1+\frac{z\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} = \Upsilon(\varphi(z);q), \qquad (z \in \mathbb{U}), \tag{3.29}$$

and

$$1+\frac{\xi\eth_q^2\langle \chi(\xi)\rangle}{\eth_q\langle \chi(\xi)\rangle} = \Upsilon(\nu(\xi);q), \qquad (\xi \in \mathbb{U}). \tag{3.30}$$

Since

$$1+\frac{z\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} = 1+\lceil 2\rfloor_q\alpha_2 z+\lceil 2\rfloor_q\left(\lceil 3\rfloor_q\alpha_3-\lceil 2\rfloor_q\alpha_2^2\right)z^2+\cdots, \tag{3.31}$$

and

$$1+\frac{\xi\eth_q^2\langle \chi(\xi)\rangle}{\eth_q\langle \chi(\xi)\rangle} = 1-\lceil 2\rfloor_q\alpha_2\xi+\lceil 2\rfloor_q\left(\left(2\lceil 3\rfloor_q-\lceil 2\rfloor_q\right)\alpha_2^2-\lceil 3\rfloor_q\alpha_3\right)\xi^2+\cdots, \tag{3.32}$$

By comparing (3.29) and (3.31), along (3.3), yields

$$\lceil 2\rfloor_q\alpha_2 z+\lceil 2\rfloor_q\left(\lceil 3\rfloor_q\alpha_3-\lceil 2\rfloor_q\alpha_2^2\right)z^2+\cdots = \frac{\widehat{\mathsf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widehat{\mathsf{p}_2}\right]z^2+\cdots. \tag{3.33}$$

Besied that, By comparing (3.10) and (3.12), along (3.6), yields

$$-\lceil 2\rfloor_q\alpha_2\xi+\lceil 2\rfloor_q\left(\left(2\lceil 3\rfloor_q-\lceil 2\rfloor_q\right)\alpha_2^2-\lceil 3\rfloor_q\alpha_3\right)\xi^2+\cdots = \frac{\widehat{\mathsf{p}_1}\tau_1}{2}\xi+\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{\mathsf{p}_1}+\frac{\tau_1^2}{2}\widehat{\mathsf{p}_2}\right]\xi^2+\cdots. \tag{3.34}$$

Equating the pertinent coefficient in (3.13) and (3.14), we obtain

$$\lceil 2 \rfloor_q \alpha_2 = \frac{\widehat{p_1}\ell_1}{2} \tag{3.35}$$

$$-\lceil 2 \rfloor_q \alpha_2 = \frac{\widehat{p_1}\tau_1}{2} \tag{3.36}$$

$$\lceil 2 \rfloor_q \Big(\lceil 3 \rfloor_q \alpha_3 - \lceil 2 \rfloor_q \alpha_2^2 \Big) = \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2} \right) \widehat{p_1} + \frac{\ell_1^2}{2}\widehat{p_2} \right] \tag{3.37}$$

$$\lceil 2 \rfloor_q \Big(\big(2\lceil 3 \rfloor_q - \lceil 2 \rfloor_q \big) \alpha_2^2 - \lceil 3 \rfloor_q \alpha_3 \Big) = \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2} \right) \widehat{p_1} + \frac{\tau_1^2}{2}\widehat{p_2} \right] \tag{3.38}$$

From (3.35) and (3.36), we have

$$\ell_1 = -\tau_1 \quad \Longleftrightarrow \quad \ell_1^2 = \tau_1^2, \tag{3.39}$$

and

$$\alpha_2^2 = \frac{\vartheta_q^2}{8\lceil 2 \rfloor_q^2} \Big(\ell_1^2 + \tau_1^2 \Big) \quad \Longleftrightarrow \quad \ell_1^2 + \tau_1^2 = \frac{8\lceil 2 \rfloor_q^2}{\vartheta_q^2} \alpha_2^2. \tag{3.40}$$

Now, by summing (3.37) and (3.38), we obtain

$$\begin{aligned} 2q^2 \lceil 2 \rfloor_q \alpha_2^2 &= \frac{(\ell_2 + \tau_2)\vartheta_q}{2} - \frac{\Big(\ell_1^2 + \tau_1^2\Big)\vartheta_q}{4} + \frac{(2q+1)\Big(\ell_1^2 + \tau_1^2\Big)\vartheta_q^2}{4} \\ &= \frac{(\ell_2 + \tau_2)\vartheta_q}{2} + \left[\frac{(2q+1)\vartheta_q^2}{4} - \frac{\vartheta_q}{4} \right] \Big(\ell_1^2 + \tau_1^2 \Big). \end{aligned} \tag{3.41}$$

By putting (3.40) in (3.41), we obtain

$$\alpha_2^2 = \frac{(\ell_2 + \tau_2)\vartheta_q^2}{4\lceil 2 \rfloor_q \Big(\lceil 2 \rfloor_q - \Big(\lceil 3 \rfloor_q + 2q \Big) \vartheta_q \Big)}. \tag{3.42}$$

Using (1.3) for (3.42), we have

$$\Big| \alpha_2 \Big| \leq \frac{\Big| \vartheta_q \Big|}{\sqrt{\lceil 2 \rfloor_q \Big(\lceil 2 \rfloor_q - \Big(\lceil 3 \rfloor_q + 2q \Big) \vartheta_q \Big)}}. \tag{3.43}$$

Now, so as to find the bound on $|\alpha_3|$, let's subtract from (3.17) and (3.18) along (3.20), we obtain

$$\alpha_3 = \alpha_2^2 + \frac{\Big(\ell_2 - \tau_2 \Big) \vartheta_q}{4\lceil 2 \rfloor_q \lceil 3 \rfloor_q}. \tag{3.44}$$

Hence, we get

$$\Big| \alpha_3 \Big| \leq \Big| \alpha_2 \Big|^2 + \frac{|\vartheta_q|}{\lceil 2 \rfloor_q \lceil 3 \rfloor_q}. \tag{3.45}$$

Then, in view of (3.23), we obtain

$$\left|\alpha_3\right| \leq \frac{\left|\vartheta_q\right|\left(\lceil 2\rfloor_q - 2(\lceil 3\rfloor_q + q)\vartheta_q\right)}{\lceil 2\rfloor_q\lceil 3\rfloor_q\left(\lceil 2\rfloor_q - \left(\lceil 3\rfloor_q + 2q\right)\vartheta_q\right)}. \tag{3.46}$$

□

In the following theorem, we find the Fekete-Szegö functional for $f \in \mathsf{SL}_{\sum}(\Upsilon(z);q)$.

Theorem 3.3. *Let f given by* (1.1) *be in the class $\mathsf{SL}_{\sum}(\Upsilon(z);q)$ and $\mu \in \mathbb{R}$. Then we have*

$$\left|\alpha_3 - \mu\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{q(1+q)}, & \left|1-\mu\right| \leq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \\ \frac{|1-\mu|\vartheta_q^2}{q^2\left(1-2q\vartheta_q\right)}, & \left|1-\mu\right| \geq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \end{cases} \tag{3.47}$$

Proof. Let $f \in \mathsf{SL}_{\sum}(\Upsilon(z);q)$, from (3.22) and (3.24) we have

$$\begin{aligned} \alpha_3 - \mu\alpha_2^2 &= \frac{(1-\mu)\vartheta_q^2}{4q^2\left(1-2q\vartheta_q\right)}(\ell_2+\tau_2) + \frac{\vartheta_q}{4q(1+q)}\left(\ell_2-\tau_2\right) \\ &= \left(\mathscr{K}(\mu) + \frac{\vartheta_q}{4q(1+q)}\right)\ell_2 + \left(\mathscr{K}(\mu) - \frac{\vartheta_q}{4q(1+q)}\right)\tau_2, \end{aligned} \tag{3.48}$$

where

$$\mathscr{K}(\mu) = \frac{(1-\mu)\vartheta_q^2}{4q^2\left(1-2q\vartheta_q\right)}. \tag{3.49}$$

Then, by taking modulus of (3.48), we conclude that

$$\left|\alpha_3 - \mu\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{q(1+q)}, & 0 \leq \left|\mathscr{K}(\mu)\right| \leq \frac{|\vartheta_q|}{4q(1+q)} \\ 4\left|\mathscr{K}(\mu)\right|, & \left|\mathscr{K}(\mu)\right| \geq \frac{|\vartheta_q|}{4q(1+q)} \end{cases}$$

□

In the last theorem of this section, we find the Fekete-Szegö functional for functions belong to the class $\mathsf{KL}_{\sum}(\Upsilon(z);q)$.

Theorem 3.4. *Let f given by* (1.1) *be in the class $\mathsf{KL}_{\sum}(\Upsilon(z);q)$ and £ $\in \mathbb{R}$. Then we have*

$$\left|\alpha_3 - £\alpha_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{\lceil 2\rfloor_q\lceil 3\rfloor_q}, & \left|1-£\right| \leq \frac{\lceil 2\rfloor_q - \left(\lceil 3\rfloor_q + 2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \\ \frac{|1-£|\vartheta_q^2}{\lceil 2\rfloor_q\left(\lceil 2\rfloor_q - \left(\lceil 3\rfloor_q + 2q\right)\vartheta_q\right)}, & \left|1-£\right| \geq \frac{\lceil 2\rfloor_q - \left(\lceil 3\rfloor_q + 2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \end{cases} \tag{3.50}$$

Proof. Let $f \in \mathsf{SL}_{\sum}(\Upsilon(z);q)$, from (3.22) and (3.24) we have

$$\begin{aligned}\alpha_3 - £\alpha_2^2 &= \frac{(1-£)\vartheta_q^2}{4\lceil 2\rfloor_q\Big(\lceil 2\rfloor_q - \Big(\lceil 3\rfloor_q + 2q\Big)\vartheta_q\Big)}(\ell_2+\tau_2) + \frac{\vartheta_q}{4\lceil 2\rfloor_q\lceil 3\rfloor_q}\Big(\ell_2-\tau_2\Big) \\ &= \left(\mathscr{L}(£) + \frac{\vartheta_q}{4\lceil 2\rfloor_q\lceil 3\rfloor_q}\right)\ell_2 + \left(\mathscr{L}(£) - \frac{\vartheta_q}{4\lceil 2\rfloor_q\lceil 3\rfloor_q}\right)\tau_2,\end{aligned} \tag{3.51}$$

where

$$\mathscr{L}(£) = \frac{(1-£)\vartheta_q^2}{4\lceil 2\rfloor_q\Big(\lceil 2\rfloor_q - \Big(\lceil 3\rfloor_q + 2q\Big)\vartheta_q\Big)}. \tag{3.52}$$

Then, by taking modulus of (3.51), we conclude that

$$\left|\alpha_3 - £\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta_q|}{\lceil 2\rfloor_q\lceil 3\rfloor_q}, & 0 \le \left|\mathscr{L}(£)\right| \le \frac{|\vartheta_q|}{4\lceil 2\rfloor_q\lceil 3\rfloor_q} \\ 4\left|\mathscr{L}(£)\right|, & \left|\mathscr{L}(£)\right| \ge \frac{|\vartheta_q|}{4\lceil 2\rfloor_q\lceil 3\rfloor_q} \end{cases}$$

□

Taking $q \mapsto 1^-$, we have the following corollaries.

Corollary 3.1. *[27] Let f given by* (1.1) *be in the class* $\mathsf{SL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\alpha_2\right| \le \frac{\left|\vartheta\right|}{\sqrt{1-2\vartheta}}, \qquad \left|\alpha_3\right| \le \frac{\left|\vartheta\right|\Big(1-4\vartheta\Big)}{2\Big(1-2\vartheta\Big)}.$$

and

$$\left|\alpha_3 - \mu\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta|}{2}, & \left|1-\mu\right| \le \frac{1-2\vartheta}{2|\vartheta|} \\ \frac{(1-\mu)\vartheta^2}{1-2\vartheta}, & \left|1-\mu\right| \ge \frac{1-2\vartheta}{2|\vartheta|} \end{cases}$$

Corollary 3.2. *[27] Let f given by* (1.1) *be in the class* $\mathsf{KL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\alpha_2\right| \le \frac{\left|\vartheta_q\right|}{\sqrt{4-10\,\vartheta}}, \qquad \left|\alpha_3\right| \le \frac{\left|\vartheta_q\right|\Big(1-4\,\vartheta_q\Big)}{3\Big(1-2\,\vartheta\Big)}.$$

and

$$\left|\alpha_3 - £\alpha_2^2\right| \le \begin{cases} \frac{|\vartheta|}{6}, & \left|1-£\right| \le \frac{2-5\vartheta}{3|\vartheta|} \\ \frac{|1-£|\vartheta^2}{2\left(2-5\vartheta\right)}, & \left|1-£\right| \ge \frac{2-5\vartheta}{3|\vartheta|} \end{cases}$$

4. Conclusion

In this work, we investigated two subclasses of bi-univalent functions associated with shell-like curves through the q-analogue of Fibonacci numbers, namely the starlike and convex classes. Utilizing the subordination principle, we established coefficient bounds for the initial terms of these function classes and derived the corresponding Fekete-Szegö inequalities. These results enhance the theoretical framework of bi-univalent function theory and elucidate its deeper connections with special function spaces.

Future research could extend these findings by exploring higher-order coefficient estimates, refining the structural characteristics of these subclasses, and examining their geometric properties. Moreover, investigating upper bounds related to the Zalcman conjecture and analyzing Hankel determinants of orders two and three within these subclasses could provide new insights and open further avenues in the study of analytic and bi-univalent function theory.

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