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Solvability of a Nonlocal Integral Problem of a Mixed Type Derivatives Functional Integro-Differential Equation

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Abstract. In this work, we study a nonlocal integral problem of a mixed-type (integer and fractional-order derivatives) functional integro-differential equation. The existence of absolutely continuous nondecreasing solutions will be proved. The sufficient conditions for the uniqueness of the solution will be given. The continuous dependence of the unique solution, on the parameters of the problem will be proved. Finally, Hyers-Ulam stability of the problem itself will be studied.

1. Introduction

The nonlocal integral problems of the differential equations have gained significant attention in the recent years. These problems naturally arise in control theory, population dynamics and thermal conduction. The mathematical analysis of such problems is to study the existence of solutions and some of its properties (boundedness, monotonicity continuous dependence) and the stability of the problem it self (Hyers-Ulam) (see [10] - [13]).

Here we are concerning with the initial value problem of the functional integral equation.

$$\frac{dx}{dt} = f\left(t, \lambda \int_0^{\phi(t)} g(s, D^\alpha x(s)ds)\right), \ a.e., \ t \in (0, T] \tag{1.1}$$

with the non-local internal condition

$$x(0) + \int_\tau^{T-\tau} h\left(s, x(s), D^\gamma x(s)\right) ds = x_0, \qquad \tau \in [0, T] \tag{1.2}$$

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where D^{α}, D^{γ} ares the Caputo fractional derivative of orders α, $\gamma \in (0,1]$.

Remark:

(1) If $\tau = 0$, then the nonlocal integral condition will be of the form

$$x(0) + \int_0^T h\Big(s, x(s), D^{\gamma} x(s)\Big) ds \ = \ x_0$$

(2) If $\tau = T$, we obtain (2) in the form

$$x(0) - \int_0^T h\Big(s, x(s), D^{\gamma} x(s)\Big) ds \ = \ x_0$$

(3) If $\tau = \frac{T}{2}$, the (2) will be the initial value

$$x(0) \ = \ x_0.$$

Here, we study the existence of at least one absolutely continuous nondecreasing solution $x \in AC[0,T]$ of the problem (1.1)-(1.2). The sufficient condition for the uniqueness of the solution will be given.

The continuous dependence of the solution $x \in AC[0,T]$ on the parameter $\lambda > 0$, the delay function ϕ and on its derivative $\frac{dx}{dt} = y$ will be studied. Finally the Hyers-Ulam stability of the problem (1.1)-(1.2) itself will be proved.

The structure of the paper is as follows: In Section 2, we present the problem formulation and preliminary definitions. Section 3 is devoted to the existence and uniqueness results. Section 4 examines the continuous dependence and Hyers-Ulam stability of the solutions. An illustrative example is provided in Section 5, and concluding remarks are given in Section 6.

2. Problem formulation

Let $\frac{dx}{dt} = y$, then we obtain

$$x(t) = x(0) + \int_0^t y(s) ds$$

and from the properties of the fractional order derivative [19], we can get

$$D^{\alpha} x(t) = I^{1-\alpha} y(t) \ \text{ and } D^{\gamma} x(t) = I^{1-\gamma} y(t). \tag{2.1}$$

Then the solution of the problem (1.1)-(1.2) will be given by

$$x(t) = x_0 + \int_{\tau}^{T-\tau} h(s, x(s), I^{1-\gamma} y(s)) ds + \int_0^t y(s) ds \tag{2.2}$$

where y is the solution of the functional integral equation

$$y(t) = f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s) ds)). \tag{2.3}$$

Conversely, differentiating (2.2) and using (2.1) we obtain

$$\frac{d}{dt} x(t) \ = \ y(t) \ = \ f(t, \lambda \int_0^{\phi(t)} g(s, D^{\alpha} x(s) ds))$$

and from (2.2) with $t = 0$ we obtain (1.2).
So, we have proved thefollowing equivalent lemma

Lemma 2.1. *The problem (1.1)-(1.2) is equivalent to the problem (2.2)-(2.3).*

Now, our problem will be considered under the following assumptions
(i) $\phi : [0,T] \to [0,T], \qquad \phi(t) \le t$ is continuous on $I = [0,T]$.

(ii) $f : I \times R \to R^+$ is measurable in $t \in I$, $\forall\, x \in R$, and continuous in $x \in R$, $\forall\, t \in I$ and there exists a $a_1 \in L_1(I)$ and a constant $b_1 > 0$ such that

$$f(t,x) \le |a_1(t)| + b_1|x|$$

(iii) $g : I \times R \to R$ is measurable in $t \in I$, $\forall\, x \in R$, and continuous in $x \in R$, $\forall\, t \in I$ and there exists $a_2 \in L_1(I)$ and a constant $b_2 > 0$ such that

$$|g(t,x)| \le |a_2(t)| + b_2|x|$$

(iv) $h : I \times R \times R \to R$ is measurable in $t \in I$, $\forall\, x, y \in R$, and continuous in $x, y \in R$, $\forall\, t \in I$ and there exists $a_3 \in L_1(I)$ and a constant $b_3 > 0$ such that

$$|h(t,x,y)| \le |a_3(t)| + b_3(|x| + |y|).$$

(v) $b^2\, \lambda\, T^{2-\alpha} < 1$, $b = max\{ b_1, b_2, b_3 \}$, $a = Max\, \{ \|a_1\|_1, \|a_2\|_1, \|a_3\|_1 \}$ and

$$\|a_i\|_1 = \sup \int_0^T |a_i(t)| dt, \ i = 1,2,3.$$

3. Existence of solution

3.1. **Solutions of the functional integral equation (2.3).** Now we have the following theorem.

Theorem 3.1. *Let the assumptions $(i) - (iii)$ and (v) be satisfied. Then the functional integral equation (2.3) has at least one solution $y \in L_1(I)$.*

Proof. Let Q_{r_1} be the closed ball of positive integrable functions

$$Q_{r_1} = \{ y \in L_1(I) : \|y\|_1 \le r_1 \} \subset L_1(I), \ r_1 = \frac{a + bTa\,\lambda}{1 - b^2\, \lambda T^{2-\alpha}}$$

and define the operator F by

$$Fy(t) = f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds).$$

Let $y \in L_1(I)$, then we have

$$\begin{aligned} |y(t)| &= |f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds)| \\ &\leq |a_1(t)| + b_1 \lambda \int_0^t (|a_2(s)| + b_2 |I^{1-\alpha} y(s)|) ds \\ &\leq |a_1(t)| + b_1 \lambda \, \|a_2\|_1 + b_1 b_2 \lambda \, I^{1-\alpha} \int_0^t |y(s)|) ds \end{aligned}$$

and

$$|y(t)| \leq |a_1(t)| + b\lambda \, a + b^2 \lambda \, \|y\|_1 \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{3.1}$$

then

$$\begin{aligned} \|Fy\|_1 &= \int_0^T |f(t, \lambda \int_0^t g\big(s, I^{1-\alpha} y(s)) ds)| dt \\ &\leq a + bT \, \lambda \, a + b^2 T \, \lambda \, \|y\|_1 T^{2-\alpha} = r_1. \end{aligned}$$

This proves that $F : Q_{r_1} \to Q_{r_1}$ and the class $\{Fy\}$ is uniformly bounded on Q_{r_1}.
Now, let $\Omega \in Q_{r_1}, \; y \in \Omega$, then

$$\begin{aligned} |Fy_h(t) - Fy(t)| &= |\frac{1}{h} \int_t^{t+h} Fy(\theta) d\theta - Fy(t)| \\ &\leq \frac{1}{h} \int_t^{t+h} |Fy(\theta) - Fy(t)| d\theta \end{aligned}$$

and

$$\|Fy_h - Fy|_1| \leq \int_0^T \frac{1}{h} \int_t^{t+h} |Fy_h(\theta) - Fy(t)| d\theta \, dt.$$

But $F \in L_1(I)$, then from the properties of the Lebesgue point [18] we have

$$|Fy_h(\theta) - Fy(t)| \to 0 \qquad as \qquad h \to 0.$$

Hence,

$$\|Fy_h - Fy\|_1 \to 0$$

this means that $Fy(t)_h \to Fy$ uniformly in $L_1(I)$. Thus the class of functions $\{Fy\}$ is relatively compact [18].
Now, let $y_n \subset Q_{r_1}$, and $y_n \to y$, then

$$Fy_n(t) = f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_n(s)) ds)$$

and

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f(t,\lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_n(s))ds)$$
$$= f(t,\lambda \lim_{n\to\infty} \int_0^{\phi(t)} g(s, I^{1-\alpha} y_n(s))ds)$$

Applying the Lebesgue Dominated Convergence Theorem [4], then from our assumptions we get

$$\lim_{n\to\infty} Fy_n(t) = f(t,\lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} \lim_{n\to\infty} y_n(s)ds)$$
$$= f(t,\lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s))ds) = Fy(t)$$

This means that $Fy_n(t) \to Fy(t)$.
Hence the operator F is continuous and by Schauder fixed point Theorem [4] there exist at least one positive solution $y \in L_1(I)$ of the functional integral equation (2.3).

Now, from the first part of the proof of Theorem 1, the following corollary can be proved.

Corollary 3.1. *Let $y \in L_1(I)$ be a solution of (2.3), then from(3.1) we have*

$$I^{1-\gamma}|y(t)| \le A + b^2 r_1 \frac{t^{2-\alpha-\gamma}}{\Gamma(3-\alpha-\gamma)}$$

where

$$A = \sup_{t\in I} I^{1-\gamma}(|a_1(t)| + ab\lambda).$$

3.1.1. *Uniqueness of the solution.* Consider the assumptions
(ii)* $f : I \times R \to R^+$ is measurable in $t \in I$, $\forall\, x \in R$, and satisfies the Lipschitz condition

$$|f(t,x) - f(t,x^*)| \le b_1|x - x^*|, \; f(t,0) = a_1(t) \in L_1(I)$$

from which we can deduce that

$$|f(t,x(t)| \le |a_1(t)| + b_1|x(t)|.$$

(iii)* $g : I \times R \to R$ is measurable in $t \in (I)$, $\forall\, x \in R$, and is measurable in $t \in [0,T]$, $\forall\, x \in R$, and satisfies the Lipschitz condition

$$|g(t,x) - g(t,x^*)| \le b_2|x - x^*|, \; g(t,0) = a_2(t) \in L_1(I)$$

from which we can deduce that

$$|g(t,x(t)| \le |a_2(t)| + b_2|x(t)|.$$

Theorem 3.2. *Let the assumptions (i), (ii)*, (iii)* and (iv) be satisfied, then the solution $y \in L_1(I)$ of the functional integral equation (2.3) is unique.*

Proof. From the assumptions $(i), (ii)^*$, $(iii)^*$ and (iv) we deduce that the assumptions of Theorem 3.1 are satisfied. Then the functional integral equation (2.3) has at least one solution $y \in L_1(I)$.

Let y_1, y_2 be two solution of (2.3), then

$$\begin{aligned}|y_2(t) - y_1(t)| = & \ |f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_2(s)) ds) - f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y_1(s)) ds)| \\ & \leq b_1 \lambda \int_0^{\phi(t)} |g(s, I^{1-\alpha} y_2(s)) - g(s, I^{1-\alpha} y_1(s))| ds \\ & \leq b_1 b_2 \lambda \int_0^{\phi(t)} |I^{1-\alpha} y_2(s) - I^{1-\alpha} y_2(s)| ds \\ & \leq b_1 b_2 \lambda \int_0^{\phi(t)} I^{1-\alpha} |y_2(s) - y_1(s)| ds \\ & \leq b_1 b_2 \lambda \int_0^{t} I^{1-\alpha} |y_2 - y_1| ds \\ & \leq b^2 \lambda \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|y_2 - y_1\|_1,\end{aligned}$$

then

$$\|y_2 - y_1\|_1 \leq b^2 \lambda \ T^{2-\alpha} \|y_2 - y_1\|_1, \ (1 - b^2 \lambda \ T^{2-\alpha}) \|y_2 - y_1\|_1 \ \leq \ 0$$

and

$$\|y_2 - y_1\|_1 \leq \ 0.$$

Which implies that $y_1(t) = y_2(t)$ and the solution of (2.3) is unique.

3.2. **Solution of functional integral equation (2.2).**

Theorem 3.3. *Let the assumptions (i)-(v) be satisfied. If $b\ T < 1$ then the functional integral equation (2.2) has at least one nondecreasing solution $x \in AC(I)$.*

Proof. Let Q_{r_2} be the closed ball

$$Q_{r_2} = \ \{x \in AC(I) : \|x\| \leq r_2\}, \ r_2 = \frac{|x_0| + a + A\ b\ T + b^3\ \lambda\ T^3\ r_1 + r_1}{1 - b\ T}$$

and define the operator F by

$$Fx(t) = x_0 + \int_\tau^{\tau - T} h(s, x(s), I^{1-\gamma} y(s)) ds + \int_0^t y(s) ds.$$

Now, let $x \in Q_{r_2}$, then

$$\begin{aligned}Fx(t) &= x_0 + \int_\tau^{T-\tau} h(s, x(s), I^{1-\gamma} y(s)) ds + \int_0^t y(s) ds \\ |Fx(t)| &= |x_0 + \int_\tau^{T-\tau} h(s, x(s), I^{1-\gamma} y(s)) ds + \int_0^t y(s) ds| \\ & \leq |x_0| + \int_\tau^{T-\tau} (|a_3(t)| + b_3 |x(t)| + b_3 |I^{1-\gamma} y(s)|) ds + \int_0^t |y(s)| ds\end{aligned}$$

$$\leq |x_0| + \int_0^T (|a_3(s)| + b_3\,|x(s)| + b_3\,|I^{1-\gamma}y(s)|)ds + ||y||_1$$

and from Corollary 1, we obtain

$$\leq |x_0| + a + b\,||x||_c T + b\int_0^T (A\ +\ b^2\,\lambda r_1\,\frac{t^{2-\alpha-\gamma}}{\Gamma(3-\alpha-\gamma)})dt + r_1$$
$$\leq |x_0| + a + b||x||_c T\ + b\,A\,T\ +\ b^3\lambda\,r_1\,T^3\ + r_1.$$

Then

$$||Fx||_C \leq |x_0| + a + b\,T\,r_2 +\ Ab\,T\ +\ b^3\lambda\,T^3\,r_1 + r_1 = r_2.$$

This proves that $F : Q_{r_2} \to Q_{r_2}$ and the class $\{\ Fx\ \}$ is uniformly bounded on Q_{r_2}.

Now, let $x \in Q_{r_2}$ and $t_1, t_2 \in I$, such that $t_2 > t_1$, and $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |x_0 + \int_\tau^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds + \int_0^{t_2} y(s)ds \\
&\quad - x_0 - (\int_\tau^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds + \int_0^{t_1} y(s)ds)| \\
&= |x_0 + \int_\tau^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds + \int_0^{t_2} y(s)ds \\
&\quad - x_0 - \int_\tau^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds - \int_0^{t_1} y(s)ds)| \\
&= |\int_0^{t_2} y(s)ds - \int_0^{t_1} y(s)ds| \\
&\leq \int_{t_1}^{t_2} |y(s)|ds = \epsilon.
\end{aligned}$$

Then the class $\{Fx\}$ is equicontinuous on Q_{r_2} and by the Arzela-Ascoli Theorem [4], the class $\{Fx\}$ is relatively compact. Now, let $x \in Q_{r_2},\ x_n(t) \to x(t) \in Q_{r_2}$.

$$\lim_{n\to\infty} Fx_n(t) = \lim_{n\to\infty}(x_0 + \int_\tau^{T-\tau} h(s,x_n(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds)$$

Applying Lebesgue dominated convergence Theorem [4], then from our assumptions we get

$$\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= x_0 + \int_\tau^{T-\tau} h(s,\lim_{n\to\infty} x_n(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds \\
&= x_0 + \int_\tau^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds \\
&= Fx(t).
\end{aligned}$$

This means that $Fx_n(t) \to Fx(t)$. Hence the operator F is continuous. Now, by the Schauder fixed point theorem [18], there exists at least one fixed point $x \in Q_{r_2} \subset AC(I)$ of the integral equation (2.2).

Then there exists at least one solution $x\,AC(I)$ of the functional integral equation (2.2).
Now, from the integral equation (2.2) and the positivity of the solution y we deduce that for $t_1 < t_2$

$$x(t_1) - x(t_2) \ = \ \int_{t_1}^{t_2} y(s)ds > 0$$

and the solution $x\,AC(I)$ of the functional integral equation (2.2) is nondecreasing.

Now, from Theorem 3.3, the following corollary can be proved.

Corollary 3.2. *Let the assumptions of Theorem 3.3 be satisfied. Then the problem (1.1)-(1.2) has at least one nondecreasing solution $x \in Q_{r_2} \subset AC(I)$.*

3.3. **Uniqueness of the solution.** Consider the assumptions
(iv)* $h : I \times R \times R \to R$ is measurable in $t \in I$, $\forall\, x \in R$, and satisfies the Lipschitz condition

$$|h(t,x,y) - h(t,x^*,y^*)| \le b_3(|x - x^*| + |y - y^*|),\ f(t,0,0) \ = \ a_3(t) \in L_1(I)$$

from which we can deduce that

$$|f(t,x(t),y(t)| \le |a_3(t)| + b_3(|x(t)| + |y(t)|).$$

Theorem 3.4. *Let the assumptions $(iv)^*$ and (v) be satisfied, then the solution $x \in AC(I)$ of the functional integral equation (2.2) is unique.*

Proof. From the assumptions $(iv)^*$ and (v) we deduce that the assumptions of Theorem 3.3 are satisfied. Then the functional integral equation (2.2) has at least one solution $x \in AC(I)$.
Let x_1, x_2 be two solution of (2.2), then

$$\begin{aligned}
|x_2(t) - x_1(t)| &= |x_0 + \int_{\tau}^{T-\tau} h(s,x_2(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds \\
&\quad - x_0 - \int_{\tau}^{T-\tau} h(s,x_1(s),I^{1-\gamma}y(s))ds - \int_0^t y(s)ds| \\
&= |\int_{\tau}^{T-\tau} h(s,x_2(s),I^{1-\gamma}y(s))ds - \int_{\tau}^{T-\tau} h(s,x_1(s),I^{1-\gamma}y(s))ds| \\
&\le \int_{\tau}^{T-\tau} |h(s,x_2(s),I^{1-\gamma}y(s)) - h(s,x_1(s),I^{1-\gamma}y(s))|ds \\
&\le b_3 \int_0^T |x_2(s) - x_1(s)|ds \\
&\le b\, T\, ||x_2 - x_1||_C,
\end{aligned}$$

then

$$||x_2 - x_1||_C \le b\, T\, ||x_2 - x_1||_C\ ,\ (1 - b\, T)\, ||x_2 - x_1||_C \ \le \ 0$$

and

$$||x_2 - x_1||_C \ \le \ 0.$$

Which implies that $x_1(t) \ = \ x_2(t)$ and the solution of (2.2) is unique.

Remark 3.1. *Let y^* be the unique solution of (2.2)*

$$y^*(t) \ = \ f(t, \lambda \int_0^{\phi(t)} \ g(s, I^{1-\alpha} y^*(s) ds),$$

then we get

$$|y(t) - y^*(t)| \le \lambda \, b^2 \, \|y - y^*\|_1 \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{3.2}$$

4. Continuous dependence

Here, we study the continuous dependence of the solutions $y \in (L_I), \ x \in AC(I)$.

Theorem 4.1. *Let the assumptions of Theorem 3.2 be satisfied. Then the unique solution $y \in L_1(I)$ of (2.3) depends continuously on the parameter λ and the delay function ϕ.*

Proof. (1) Let $\delta > 0$ be given such that $|\lambda - \lambda^*| < \delta, \ y$ be the solution of (2.3) and y^* be the unique solution of

$$y^*(t) \ = \ f(t, \lambda^* \int_0^{\phi(t)} \ g(s, I^{1-\alpha} y^*(s)) ds).$$

Then

$$\begin{aligned}
|y(t) - y^*(t)| &= |f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds) - f(t, \lambda^* \int_0^{\phi(t)} g(s, I^{1-\alpha} y^*(s)) ds)| \\
&\le b_1 | \, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds - \lambda^* \int_0^{\phi(t)} g(s, I^{1-\alpha} y^*(s)) ds| \\
&\le b_1 | \, (\lambda - \lambda^*) \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds + \lambda^* \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds \\
&- \lambda^* \int_0^{\phi(t)} g(s, I^{1-\alpha} y^*(s)) ds| \\
&\le b_1 \delta \int_0^{\phi(t)} |g(s, I^{1-\alpha} y(s))| ds \ + \ \lambda^* \int_0^{\phi(t)} |g(s, I^{1-\alpha} y(s)) - g(s, I^{1-\alpha} y^*(s))| ds \\
&\le b_1 \delta \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds \ + \ b_1 b_2 \lambda^* \int_0^{\phi(t)} I^{1-\alpha} |y(s)) - y^*(s)| ds \\
&\le b_1 \delta \int_0^{T} g(s, I^{1-\alpha} y(s)) ds \ + \ b_1 b_2 \lambda^* \, \|y - y^*\| \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}$$

and

$$\|y - y^*\|_1 \le b\delta \int_0^T g(s, I^{1-\alpha} y(s)) ds + b^2 T^{2-\alpha} \lambda^* \, \|y - y^*\|_1$$

which implies that

$$\|y - y^*\|_1 \le \frac{\delta_1}{1 - b^2 T^{2-\alpha} \lambda^*} \ = \ \epsilon$$

where $\delta_1 = b\delta \int_0^T g(s, I^{1-\alpha} y(s)) ds$.

(2) Let $\delta > 0$ be given such that $|\phi(t) - \phi^*(t)| \le \delta$, y be the solution of (2.3) and y^* be the unique solution of

$$y^*(t) = f(t, \lambda \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s) ds).$$

Then

$$\begin{aligned}
|y(t) - y^*(t)| &= |f(t, \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds) - f(t, \lambda \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s)) ds)| \\
&\le b_1 | \lambda \int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds - \lambda \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s)) ds| \\
&\le b_1 | \lambda (\int_0^{\phi(t)} g(s, I^{1-\alpha} y(s)) ds - \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y(s)) ds) \\
&+ \lambda (\int_0^{\phi^*(t)} g(s, I^{1-\alpha} y(s)) ds - \int_0^{\phi^*(t)} g(s, I^{1-\alpha} y^*(s)) ds)| \\
&\le b_1 \lambda \, \epsilon_1 - b_1 b_2 \lambda \int_0^{\phi^*(t)} I^{1-\alpha} |y(s)) - y^*(s))| ds \\
&\le b_1 \lambda \epsilon_1 \ + \ b_1 b_2 \lambda \int_0^{\phi^*(t)} I^{1-\alpha} |y(s)) - y^*(s))| ds \\
&\le b \lambda \epsilon_1 \ + \ b^2 \lambda^* \ \|y - y^*\| \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}$$

and

$$\|y - y^*\|_1 \le b \lambda \epsilon_1 \ + + b^2 T^{2-\alpha} \lambda^* \ \|y - y^*\|_1$$

which implies that

$$\|y - y^*\|_1 \le \frac{b \lambda \epsilon_1}{1 - b^2 \ T^{2-\alpha} \ \lambda^*} \ = \ \epsilon$$

where

$$\epsilon_1 \ = \ \int_{\phi^*(t)}^{\phi(t)} |g(s, I^{1-\alpha} y(s)) ds.$$

Theorem 4.2. *Let the assumptions of Theorem 3.4 be satisfied, the solution $x \in AC(I)$ of (2.2) depends continuously on the solution y and the internal condition x_0.*

Proof. (1) Let $\delta > 0$ be given such that $\|y - y^*\|_1 < \delta$, x be the solution of (2.2) and x^* be the solution of

$$x^*(t) \ = \ x_0 + \int_\tau^{T-\tau} h(s, x^*(s), I^{1-\gamma} y^*(s)) ds + \int_0^t y^*(s) ds,$$

then

$$\begin{aligned}
|x(t) - x^*(t)| = \ &|x_0 + \int_\tau^{T-\tau} h(s, x(s), I^{1-\gamma} y(s)) ds \ + \ \int_0^t y(s) ds \\
&- x_0 - \int_\tau^{T-\tau} h(s, x^*(s), I^{1-\gamma} y^*(s)) d\, s - \int_0^t y^*(s) ds|
\end{aligned}$$

$$= |\int_{\tau}^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds - \int_{\tau}^{T-\tau} h(s,x^*(s),I^{1-\gamma}y^*(s))ds$$

$$+ \int_0^t y(s)ds - \int_0^t y^*(s)ds|$$

$$\leq \int_0^T |h(s,x(s),I^{1-\gamma}y(s)) - h(s,x^*(s),I^{1-\gamma}y^*(s))|ds$$

$$+ \int_0^t |y(s) - y^*(s)|ds$$

$$\leq b \int_0^T |x(s) - x^*(s)|ds + b \int_0^T I^{1-\gamma}|y(s) - y^*(s)|d\,s + \int_0^t |y(s) - y^*(s)|ds$$

$$\leq b \int_0^T |x(s) - x^*(s)|ds + b^2\,\lambda \int_0^T I^{1-\gamma}\|y - y^*\|_1 \frac{s^{1-\alpha}}{\Gamma(2-\alpha)}dt + \int_0^t |y(s) - y^*(s)|ds$$

and

$$\|x - x^*\|_C \leq b\,T\|x - x^*\|_c + \lambda\, b^2\, \delta\, \frac{T^{2-\alpha-\gamma}}{\Gamma(2-\gamma)} + \delta = \epsilon$$

which implies that

$$\|x - x^*\|_C \leq \epsilon.$$

(2) Let $\delta > 0$ be given such that $|x_0 - x_0^*| \leq \delta$, x be the solution of (2.2) and x^* be the solution of

$$x^*(t) = x_0^* + \int_{\tau}^{T-\tau} h(s,x^*(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds.$$

Then

$$|x(t) - x^*(t)| = |x_0 + \int_{\tau}^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds + \int_0^t y(s)ds$$

$$- x_0^* - \int_{\tau}^{T-\tau} h(s,x^*(s),I^{1-\gamma}y(s))ds - \int_0^t y(s)ds|$$

$$= |x_0 - x_0^* + \int_{\tau}^{T-\tau} h(s,x(s),I^{1-\gamma}y(s))ds - \int_{\tau}^{T-\tau} h(s,x^*(s),I^{1-\gamma}y(s))ds|$$

$$\leq |x_0 - x_0^*| + \int_0^T |h(s,x(s),I^{1-\gamma}y(s)) - h(s,x^*(s),I^{1-\gamma}y(s))|ds$$

$$\leq \delta + b \int_0^T |x(s) - x^*(s)|ds$$

and

$$\|x - x^*\|_C \leq \delta + b\,T\,\|x - x^*\|_c$$

which implies that

$$\|x - x^*\|_C \leq \frac{\delta}{1 - bT} = \epsilon.$$

Corollary 4.1. *Let the assumptions of Theorem 4.1 be satisfied. Then the solution* $x \in AC(I)$ *of (2.2) depends continuously on the parameter* λ *and the delay function* ϕ*.*

5. Hyers-Ulam stability

Many authors have studied and further developed the definition of Hyers-Ulam stability across various types of problems, see [3, 4, 5, 6]. In light of these definitions and based on the equivalence between the problems (1.1) and (1.2) or (1.1) and (2.2) and the integral equation (2.2), we present the next definition of the Hyers-Ulam stability of the problems (1.1) and (1.2) or (1.1) and (2.2) as follows:

5.1. **Definition.** Let the solution $x \in AC(I)$ of the problem (1.1)- (1.2) be exists, then the problem (1.1)-(1.2) is Hyers-Ulam stable if $\forall\, \epsilon > 0,\ \delta(\epsilon)$ such that for any $\delta-$ approximate solution $x_s \in AC(I)$ satisfies,

$$|\frac{dx_s}{dt} - f(t, \lambda \int_0^{\phi(t)} g(\theta, D^\alpha x_s(\theta))d\theta)| < \delta, \tag{5.1}$$

which implies that $\|x - x_s\|_C < \epsilon$.

Remark 5.1. *Let* $\frac{dx_s}{dt} = y_s(t)$, *in (5.1) we obtain*

$$|y_s(t) - f(t, \lambda \int_0^t g(\theta, I^{1-\alpha} y_s(\theta))d\theta)| < \delta. \tag{5.2}$$

And from (5.1), we get

$$\begin{aligned}
&|\frac{dx_s}{dt} - f(t, \lambda \int_0^{\phi(t)} g(\theta, D^\alpha x_s(\theta))d\theta)| < \delta,\\
-\delta \ <& \frac{dx_s}{dt} - f(t, \lambda \int_0^{\phi(t)} g(\theta, D^\alpha x_s(\theta))d\theta) \ < \ \delta\\
-\delta\, t <& x_s(t) - (x_0 + \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta) + \int_0^t f(t, \lambda \int_0^{\phi(t)} g(\theta, D^\alpha x_s(\theta))d\theta) \ < \delta\, t\\
-\delta\, T <& x_s(t) - (x_0 + \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta) + \int_0^t f(t, \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)\ dt) < \delta\, T
\end{aligned}$$

and

$$|x_s(t) - x_0 - \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta - \int_0^t f(t, \lambda \int_0^{\phi(t)} g(\theta, I^{1-\alpha} y_s(\theta))d\theta)\ dt| < \delta\, T. \tag{5.3}$$

Theorem 5.1. *Let the assumptions of Theorem (3.4) be satisfied, then (2.2) is Hyers-Ulam stable.*

Proof. Firstly we have

$$\begin{aligned}
|y(t) - y_s(t)| =& \ |f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y(\theta))d\theta) - y_s(t)|\\
=& \ |f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y(\theta))d\theta) - f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y_s(\theta))d\theta)|\\
&- y_s(t) + f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y_s(\theta))d\theta)|
\end{aligned}$$

$$\le |f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y(\theta))d\theta) - f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y_s(\theta))d\theta)|$$
$$+ |y_s(t) - f(t,\ \lambda \int_0^t g(\theta, I^{1-\alpha} y_s(\theta))d\theta)|$$
$$\le b_1\ \lambda \int_0^t |g(\theta, I^{1-\alpha} y(\theta)) - g(\theta, I^{1-\alpha} y_s(\theta))| d\theta + \delta$$
$$\le b^2\ \lambda \int_0^t I^{1-\alpha} |y(\theta) - y_s(\theta)| d\theta + \delta$$
$$\le b^2\ \lambda \|y - y_s\|_1\ T^{1-\alpha} + \delta$$

and

$$\|y - y_s\|_1 \le b^2\ \lambda\ \|y - y_s\|_1\ T^2 + \delta\ T$$

which implies that

$$\|y - y_s\|_1 \le \frac{\delta\ T}{1 - b^2\ \lambda\ T^2} = \epsilon_1.$$

Secondly, we have

$$|x(t) - x_s(t)| = |x_0 + \int_\tau^{T-\tau} h(\theta, x(\theta), I^{1-\gamma} y(\theta))d\theta + \int_0^t y(\theta)d\theta - x_s(t)|$$

$$|x(t) - x_s(t)| = |x_0 + \int_\tau^{T-\tau} h(\theta, x(\theta), I^{1-\gamma} y(\theta))d\theta + \int_0^t y(\theta)d\theta - x_s(t)$$
$$+ \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta + \int_0^t y_s(\theta)d\theta$$
$$- \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta - \int_0^t y_s(\theta)d\theta|$$
$$= |x_0 + \int_\tau^{T-\tau} h(\theta, x(\theta), I^{1-\gamma} y(\theta))d\theta - \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta$$
$$+ \int_0^t y(\theta)d\theta - \int_0^t y_s(\theta)d\theta - x_s(t)$$
$$+ \int_\tau^{T-\tau} h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta + \int_0^t y_s(\theta)d\theta|$$
$$\le | \int_0^T h(\theta, x(\theta), I^{1-\gamma} y(\theta))d\theta - \int_0^T h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta$$
$$+ \int_0^t y(\theta)d\theta - \int_0^t y_s(\theta)d\theta$$
$$+ x_0 + \int_0^T h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))d\theta + \int_0^t y_s(\theta)d\theta - x_s(t)|$$
$$\le \int_0^T |h(\theta, x(\theta), I^{1-\gamma} y(\theta)) - h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta))| d\theta + \int_0^t |y(\theta) - y_s(\theta)| d\theta$$
$$+ |x_0 + \int_0^T (h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta)))d\theta + \int_0^t y_s(\theta)d\theta - x_s(t)|$$

$$\leq \int_0^T b_3|x(\theta) - x_s(\theta)|d\theta + \int_0^T b_3 I^{1-\gamma}|y(\theta) - y_s(\theta)|d\theta + \int_0^t |y(\theta) - y_s(\theta)|d\theta$$
$$+ |x_0 + \int_0^T (h(\theta, x_s(\theta), I^{1-\gamma} y_s(\theta)))d\theta + \int_0^t y_s(\theta)d\theta - x_s(t)|$$
$$\leq b \int_0^T |x(\theta) - x_s(\theta)|d\theta + b \int_0^T I^{1-\gamma}|y(\theta) - y_s(\theta)|d\theta + \int_0^t |y(\theta) - y_s(\theta)|d\theta + \delta$$
$$\leq b \int_0^T |x(\theta) - x_s(\theta)|d\theta + b^2 \lambda \, ||y - y_s||_1 \int_0^T (s^{2-\alpha-\gamma} + \delta)ds + \int_0^t |y(\theta) - y_s(\theta)|d\theta + \delta$$

and

$$|x(t) - x_s(t)| \leq b \int_0^T |x(\theta) - x_s(\theta)|d\theta + b^2 \lambda \, ||y - y_s||_1 \, (T^3 + \delta\, T) + \delta$$

$$||x - x_s||_C \leq b\, T||x - x_s||_C + b^2 \lambda \, \epsilon_1 \, (T^3 + \delta\, T) + \epsilon_1 + \delta,$$

then

$$||x - x_s||_C \leq \frac{b^2 \lambda \, \epsilon_1 \, (T^3 + \delta\, T) + \epsilon_1 + \delta}{1 - b\, T} = \epsilon.$$

6. Conclusion

In this paper, we examined a class of nonlocal functional integro-differential equations involving mixed-type derivatives, incorporating both fractional and integer-order operators. Using the Schauder fixed-point theorem, we established suffcient conditions for the existence of solutions. We have demonstrated the uniqueness of the solution under additional constraints, ensuring that the systems behavior is well-posed. Additionally, we have analyzed the continuous dependence of the solution on key parameters, including the initial condition, the parameter, and the delay function, proving the robustness of the model even in the presence of perturbations in system data. Importantly, we have also investigated the Hyers-Ulam stability of the problem, thus conforming that approximate solutions remain close to exact solution within controlled bounds a feature that is especially important for numerical simulations and applications where data may be imprecise. Overall, the results presented in this work contribute to the theoretical understanding of functional integro-differential systems with mixed-type derivatives and provide a rigorous framework for studying more general classes of nonlocal models arising in physical and engineering contexts.

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