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Common Fixed Point Theorems in Orthogonal Sets

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Abstract. In this paper, we introduce new notions of $\perp\ \Lambda$-quasicontraction and $\perp\ \Lambda$-preserving in the setting of orthogonal sets and prove respective common fixed point theorems. Furthermore, we provide an example to clarify our main result. Our results extend, improve, and generalize several known results in the literature.

1. Introduction

Fixed point theory has emerged as a fundamental tool in nonlinear analysis with wide-ranging applications. Z. Kadelburg et al. [10] established pioneering results on common fixed points for quasicontractions in ordered cone metric spaces. Subsequent research has significantly expanded these findings across various mathematical settings. Dhivya and Marudai [4] investigated rational-type contractions in ordered partial metric spaces, while Guan and Li [7] obtained results for weakly contractive mappings in metric-like spaces. Abbas et al. [1] examined coupled fixed points in generalized metric spaces, and Abkar and Eslamian [2] explored fixed point properties in CAT(0) spaces. Further developments by Pragadeeswarar et al. [11–13] addressed common best proximity points in both ordered and fuzzy metric spaces.

The foundational work of Kadelburg et al. [10] includes the following key result for ordered cone metric spaces:

Theorem 1.1 ([10]). *Let (f, g) be a pair of self-maps on a complete ordered cone metric space $(X, \sqsubseteq, d)$ satisfying:*

(1) *$f(X) \subset g(X)$ with existence of $x_0 \in X$ such that $gx_0 \sqsubseteq fx_0$;*

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(2) *f is an ordered g-quasicontraction;*
(3) *$g(X)$ is closed in X;*
(4) *f is g-nondecreasing; and*
(5) *for any nondecreasing sequence $\{g(x_n)\} \subset X$ converging to gz, both $gx_n \sqsubseteq gz$ and $gz \sqsubseteq ggz$ hold.*

Then f and g possess a coincidence point $z \in X$ with $fz = gz$, and if weakly compatible, they admit a common fixed point.

On the other hand, Gordji et al. [5] introduced the innovative concept of orthogonal sets in 2017, creating new research directions in fixed point theory. Important contributions followed, including Gungor and Turkoglu's [8] work using distance functions to establish fixed point results, Baghani et al.'s [3] connection between orthogonality and the axiom of choice, Sawangsup et al.'s [15] extension of F-contractions to orthogonal spaces, and Gordji et al.'s [6] investigation of nonlinear contractions in orthogonal settings.
While orthogonal sets provide a more general framework than partially ordered sets through their weaker conditions, the study of common fixed points in orthogonal spaces remains largely undeveloped. Motivated by these advances in both ordered cone metric spaces [10] and orthogonal set theory [3, 5, 6, 8, 15], this paper establishes novel conditions for the existence of common fixed points in orthogonal spaces, addressing a significant gap in current mathematical literature.

2. Preliminaries

Here we provide some definitions, notations, and concepts needed in the sequel.

Definition 2.1. *[14] Let P be a non-empty set. A common fixed point of a pair of self-mappings $A, B : P \to P$ is a point $z \in P$ for which $Az = Bz = z$.*

Definition 2.2. *[14] Let P be a non-empty set. Consider a pair of self-mappings $A, B : P \to P$. If there exists a point $z \in P$ for which $Az = Bz$, it is known as a coincidence point.*

Definition 2.3. *[5] Let $P \neq \emptyset$ and $\perp \subseteq P \times P$ be a binary relation. If $\perp$ satisfies the following condition:*

$$\exists z_0 : (\forall w, w \perp z_0) \text{ or } (\forall w, z_0 \perp w),$$

then it is called an orthogonal set (briefly O-set). We denote this orthogonal set by $(P, \perp)$.

Example 2.1. *Let $P = [0, \infty]$ and define $m \perp n$ if $mn \in \{m, n\}$. Then, by taking $z_0 = 0$ or $z_0 = 1$, $(P, \perp)$ is an O-set.*

Definition 2.4. *[10] Let $(X, \perp)$ be an O-set. A mapping $\Gamma : X \to X$ is said to be $\perp$-preserving if*

$$\Gamma(x) \perp \Gamma(y) \text{ whenever } x \perp y.$$

Definition 2.5. *[10] Let $(X, \perp)$ be an O-set. A mapping $\Gamma : X \to X$ is said to be weakly $\perp$-preserving if*

$$\Gamma(x) \perp \Gamma(y) \text{ or } \Gamma(y) \perp \Gamma(x) \text{ whenever } x \perp y.$$

Definition 2.6. *[9] Let (Γ, Λ) be a pair of self-maps defined in an orthogonal space such that $\Gamma(X) \subset \Lambda(X)$. The mappings Γ and Λ are said to be weakly compatible if, for each $x \in X$, $\Gamma x = \Lambda x$ implies $\Gamma\Lambda x = \Lambda\Gamma x$.*

Definition 2.7. *[5] Let $(X, \perp)$ be an O-set. A sequence (x_n) is called an orthogonal sequence (briefly, O-sequence) if $(\forall n, x_n \perp x_{n+1})$ or $(\forall n, x_{n+1} \perp x_n)$.*

3. Main Results

First, we define new notions called $\perp$ Λ-quasicontraction and $\perp$-closed which we will use for our main results.

Definition 3.1. *Let $(X, \perp)$ be an O-set with metric d and (Γ, Λ) be a pair of self-maps on X. The mapping Γ is said to be $\perp$ Λ-quasicontraction if there exists $\lambda \in [0, \frac{1}{2})$ such that for each $x, y \in X$ satisfying $\Lambda y \perp \Lambda x$, there exists*

$$u \in M_0^{\Gamma,\Lambda}(x, y) = \Big\{d(\Lambda x, \Lambda y), d(\Lambda x, \Gamma x), d(\Lambda y, \Gamma y), d(\Gamma y, \Lambda x), d(\Lambda y, \Gamma x)\Big\} \tag{3.1}$$

such that $d(\Gamma x, \Gamma y) \leq \lambda \cdot u$ holds.

Definition 3.2. *Let X be an orthogonal set. $A \subset X$ is $\perp$-closed if for every orthogonal sequence (x_n) with $x_n \to x$, the limit x belongs to A.*

Definition 3.3. *Let $(X, \perp)$ be an O-set. Let (Γ, Λ) be a pair of self-maps on X. The mapping Γ is called $\perp$ Λ-preserving if for $x, y \in X$, $\Lambda x \perp \Lambda y$ implies $\Gamma x \perp \Gamma y$.*

Now, we state and prove our main result.

Theorem 3.1. *Let X be an O-set and let d be a metric on X such that $(X, d, \perp)$ is an O-complete set. Let $\Gamma, \Lambda : X \to X$ be two self-mappings such that $\Gamma(X) \subset \Lambda(X)$ and there exists a point $x_0 \in X$ with $\Lambda x_0 \perp \Gamma x_0$. Suppose that:*

1. *Γ is a $\perp$ Λ-quasicontraction;*
2. *$\Lambda(X)$ is $\perp$-closed in X;*
3. *Γ is $\perp$ Λ-preserving;*
4. *If $\{\Lambda(x_n)\} \subset X$ is a Λ-$\perp$ preserving sequence converging to some Λz, then $\Lambda x_n \perp \Lambda z$ and $\Lambda z \perp \Lambda\Lambda z$.*

Then Γ and Λ have a coincidence point, i.e., there exists $z \in X$ such that $\Gamma z = \Lambda z$. Further, if Γ and Λ are weakly compatible, then they have a common fixed point.

Proof. As there exists a point $x_0 \in X$ such that $\Lambda x_0 \perp \Gamma x_0$, we construct a Jungck sequence. Given x_0, choose $x_1 \in X$ such that $\Gamma x_0 = \Lambda x_1$ (since $\Gamma(X) \subset \Lambda(X)$). Now, $\Lambda x_0 \perp \Lambda x_1$ implies that $\Gamma x_0 \perp \Gamma x_1$. Then, there exists $x_2 \in X$ such that $\Gamma x_1 = \Lambda x_2$ and again $\Gamma x_0 \perp \Gamma x_1$ implies that

$\Lambda x_1 \perp \Lambda x_2$ and $\Gamma x_1 \perp \Gamma x_2$. Continuing this procedure, we have:

$$\Gamma x_0 \perp \Gamma x_1 \perp \Gamma x_2 \perp \cdots \perp \Gamma x_n \perp \Gamma x_{n+1} \perp \cdots$$
$$\Lambda x_1 \perp \Lambda x_2 \perp \cdots \perp \Lambda x_{n+1} \perp \Lambda x_{n+2} \perp \cdots$$

We will show that $\{\Lambda x_n\}$ is an O-Cauchy sequence. First, let us prove that:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \frac{\lambda}{1-\lambda} d(\Gamma x_{n-1}, \Gamma x_n) \tag{3.2}$$

for all $n \geq 1$.
Indeed, since $\Lambda x_n \perp \Lambda x_{n+1}$, we apply the condition that Γ is a $\perp$ Λ-quasicontraction to get:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda \cdot u_n \tag{3.3}$$

where

$$\begin{aligned} u_n \in \{&d(\Lambda x_n, \Lambda x_{n+1}), d(\Lambda x_n, \Gamma x_n), d(\Lambda x_{n+1}, \Gamma x_{n+1}), d(\Lambda x_n, \Gamma x_{n+1}), d(\Lambda x_{n+1}, \Gamma x_n)\} \\ = \{&d(\Gamma x_{n-1}, \Gamma x_n), d(\Gamma x_n, \Gamma x_{n+1}), d(\Gamma x_{n-1}, \Gamma x_{n+1}), 0\}. \end{aligned}$$

There are four different cases to consider:

(1) $d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_{n-1}, \Gamma x_n) \leq \frac{\lambda}{1-\lambda} d(\Gamma x_{n-1}, \Gamma x_n)$ since $\lambda \leq \frac{\lambda}{1-\lambda}$.
(2) $d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_n, \Gamma x_{n+1})$; it follows that $d(\Gamma x_n, \Gamma x_{n+1}) = 0$.
(3) $d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_{n-1}, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_{n-1}, \Gamma x_n) + \lambda d(\Gamma x_n, \Gamma x_{n+1})$.
(4) $d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda \cdot 0 = 0$ and $d(\Gamma x_n, \Gamma x_{n+1}) = 0$.

Put $h = \frac{\lambda}{1-\lambda}$ in (3.2), then we get:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq h d(\Gamma x_{n-1}, \Gamma x_n) \leq \cdots \leq h^n d(\Gamma x_0, \Gamma x_1) \tag{3.4}$$

for all $n \geq 1$. Now, for $m, n \in \mathbb{N}, n > m$, we have:

$$\begin{aligned} d(\Gamma x_n, \Gamma x_m) &\leq d(\Gamma x_n, \Gamma x_{n-1}) + d(\Gamma x_{n-1}, \Gamma x_{n-2}) + \cdots + d(\Gamma x_{m+1}, \Gamma x_m) \\ &\leq (h^{n-1} + h^{n-2} + \cdots + h^m) d(\Gamma x_0, \Gamma x_1) \\ &\leq \frac{h^m}{1-h} d(\Gamma x_0, \Gamma x_1) \to 0 \text{ as } m \to \infty. \end{aligned} \tag{3.5}$$

From assumption (4), $\{\Gamma x_n\}$, i.e., $\{\Lambda x_n\}$, is an O-Cauchy sequence. Since X is O-complete and $\Lambda(X)$ is $\perp$-closed, there exists $z \in X$ such that:

$$\Lambda x_n \to \Lambda z \text{ i.e., } \Gamma x_n \to \Lambda z \text{ as } n \to \infty. \tag{3.6}$$

Now, we will prove that $\Gamma z = \Lambda z$. From assumption (4), $\Lambda x_n \perp \Lambda z$. Putting $x = x_n$, $y = z$ in the quasi-contraction condition, we get:

$$d(\Gamma x_n, \Gamma z) \leq \lambda \cdot u_n, \tag{3.7}$$

where $u_n \in \{d(\Lambda x_n, \Lambda z), d(\Lambda x_n, \Gamma x_n), d(\Lambda z, \Gamma z), d(\Lambda z, \Gamma x_n), d(\Lambda x_n, \Gamma z)\}$.

Observe that:

$$d(\Lambda z, \Gamma z) \leq d(\Lambda z, \Gamma x_n) + d(\Gamma x_n, \Gamma z), \tag{3.8}$$

$$d(\Lambda x_n, \Gamma z) \leq d(\Lambda x_n, \Gamma x_n) + d(\Gamma x_n, \Gamma z). \tag{3.9}$$

Now, let $\epsilon > 0$ be given. In all possible cases, there exists $n_0 \in \mathbb{N}$ such that (using (3.7)) one obtains $d(\Gamma x_n, \Gamma z) < \epsilon$:

(1) $d(\Gamma x_n, \Gamma z) \leq \lambda \cdot d(\Lambda x_n, \Lambda z) < \epsilon$ (since $\Lambda x_n \to \Lambda z$ as $n \to \infty$);
(2) $d(\Gamma x_n, \Gamma z) \leq \lambda \cdot d(\Lambda x_n, \Gamma x_n) \leq \lambda[d(\Gamma x_n, \Gamma z) + d(\Lambda z, \Gamma x_n)] < \epsilon$;
(3) $d(\Gamma x_n, \Gamma z) \leq \lambda \cdot d(\Lambda z, \Gamma z) \leq \lambda d(\Lambda z, \Gamma x_n) + \lambda d(\Gamma x_n, \Gamma z)$; it follows that $d(\Gamma x_n, \Gamma z) \leq \frac{\lambda}{1-\lambda} d(\Lambda z, \Gamma x_n) < \epsilon$;
(4) $d(\Gamma x_n, \Gamma z) \leq \lambda \cdot d(\Lambda z, \Gamma x_n) < \epsilon$;
(5) $d(\Gamma x_n, \Gamma z) \leq \lambda \cdot d(\Lambda x_n, \Gamma z) \leq \lambda d(\Lambda x_n, \Gamma x_n) + \lambda d(\Gamma x_n, \Gamma z)$; it follows that $d(\Gamma x_n, \Gamma z) \leq \frac{\lambda}{1-\lambda} d(\Lambda z, \Gamma x_n) < \epsilon$.

It follows that $\Gamma x_n \to \Gamma z$ as $n \to \infty$. The uniqueness of the limit implies $\Gamma z = \Lambda z = t$. Thus, z is a coincidence point of the pair (Γ, Λ) and t is a point of coincidence.
Suppose now that Γ and Λ are weakly compatible. By assumption (4), $\Lambda z \perp \Lambda\Lambda z$, and hence we obtain:

$$\Gamma\Lambda z = \Lambda\Gamma z = \Gamma\Gamma z = \Lambda\Lambda z. \tag{3.10}$$

Suppose $\Gamma z \neq \Gamma\Gamma z$. Then, by the $\perp$ Λ-quasicontraction condition for $x = z$, $y = \Gamma z$, we have:

$$d(\Gamma z, \Gamma\Gamma z) \leq \lambda u, \tag{3.11}$$

where

$$\begin{aligned} u &\in \{d(\Lambda z, \Lambda\Gamma z), d(\Lambda z, \Gamma z), d(\Lambda\Gamma z, \Gamma\Gamma z), d(\Lambda\Gamma z, \Gamma z), d(\Lambda z, \Gamma\Gamma z)\} \\ &= \{d(\Gamma z, \Gamma z), 0, d(\Gamma\Gamma z, \Gamma\Gamma z), d(\Gamma\Gamma z, \Gamma z), d(\Gamma z, \Gamma\Gamma z)\} \\ &= \{0, d(\Gamma z, \Gamma\Gamma z)\}. \end{aligned} \tag{3.12}$$

Thus, we have two possibilities:

(1) $d(\Gamma z, \Gamma\Gamma z) \leq \lambda \cdot 0 = 0 \Rightarrow d(\Gamma z, \Gamma\Gamma z) = 0 \Rightarrow \Gamma z = \Gamma\Gamma z$.
(2) $d(\Gamma z, \Gamma\Gamma z) \leq \lambda \cdot d(\Gamma z, \Gamma\Gamma z) \Rightarrow d(\Gamma z, \Gamma\Gamma z) = 0$ i.e., $\Gamma z = \Gamma\Gamma z$.

In other words, $\Gamma z = \Lambda z$ is a common fixed point of the mappings Γ and Λ. □

Example 3.1. *Let $X = [0, 1)$ be a set in a complete orthogonal space, where $x \perp y$ if $xy \in \{x, y\}$. Define two functions $\Gamma, \Lambda : X \to X$ as follows:*

$$\Gamma(x) = \begin{cases} \frac{x}{8}, & \text{if } x \in \mathbb{Q} \cap [0,1), \\ 0, & \text{if } x \in \mathbb{Q}^C \cap [0,1). \end{cases} \tag{3.13}$$

$$\Lambda(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in \mathbb{Q} \cap [0,1), \\ 0, & \text{if } x \in \mathbb{Q}^C \cap [0,1). \end{cases} \tag{3.14}$$

Let $x \perp y$ and $xy \in \{x, y\}$. We shall prove that Γ is a $\perp$ Λ-quasicontraction. Let $\Lambda x \perp \Lambda y$. With respect to the above defined orthogonal condition, we have $\Lambda x\Lambda y \in \{\Lambda x, \Lambda y\}$, i.e., $\Lambda x = 0$ or $\Lambda y = 0$. Without loss of generality, assume $\Lambda x = 0 \Rightarrow x = 0$ or $x \in \mathbb{Q}^C \cap [0,1)$. Thus,

$$d(\Gamma(x), \Gamma(y)) = |0 - \Gamma(y)| = |-\Gamma(y)| = \Gamma(y) = \begin{cases} \frac{y}{8}, & \text{if } y \in \mathbb{Q} \cap [0,1), \\ 0, & \text{if } y \in \mathbb{Q}^C \cap [0,1). \end{cases} \tag{3.15}$$

$$d(\Lambda(x), \Lambda(y)) = |0 - \Lambda(y)| = |-\Lambda(y)| = \Lambda(y) = \begin{cases} \frac{y}{2}, & \text{if } y \in \mathbb{Q} \cap [0,1), \\ 0, & \text{if } y \in \mathbb{Q}^C \cap [0,1). \end{cases} \tag{3.16}$$

Hence,

$$d(\Gamma(x), \Gamma(y)) = \frac{1}{4} d(\Lambda(x), \Lambda(y)). \tag{3.17}$$

This verifies the Λ-quasicontraction condition.
Now, we shall prove that $\Lambda(X)$ is $\perp$-closed in X. Here, the constant sequence $(0)_n$ is the only orthogonal convergent sequence in X. Hence, it is $\perp$-closed.
Next, we shall prove that Γ is $\perp$ Λ-preserving. Given that $\Lambda x \perp \Lambda y$, by the orthogonal condition definition, we have $\Lambda x\Lambda y \in \{\Lambda x, \Lambda y\}$, i.e., $\Lambda x = 0$ or $\Lambda y = 0$. Without loss of generality, assume $\Lambda x = 0 \Rightarrow x = 0$ or $x \in \mathbb{Q}^C \cap [0,1)$. Similarly, we have $\Gamma x\Gamma y \in \{\Gamma x, \Gamma y\}$, i.e., $\Gamma x = 0$ or $\Gamma y = 0$. Without loss of generality, assume $\Gamma x = 0 \Rightarrow x = 0$ or $x \in \mathbb{Q}^C \cap [0,1)$. Thus, $\Lambda x\Lambda y = 0$ (since $\Lambda x = 0$) and $\Gamma x\Gamma y = 0$ (since $\Gamma x = 0$). Therefore, $\Lambda x \perp \Lambda y \Rightarrow \Gamma x \perp \Gamma y$.
We have already verified that Λ is $\perp$-preserving. Now, let $\{\Lambda(x_n)\} \subset X$ be a Λ $\perp$-preserving sequence. Without loss of generality, assume $\Lambda(x_n) \to 0$. Consider $\Lambda(z) = 0$. Then, $\Lambda(x_n) \perp \Lambda z \Rightarrow \Lambda(x_n)\Lambda z \in \{\Lambda(x_n), \Lambda z\}$, so $\Lambda(x_n) \perp 0 = 0 \in \{\Lambda(x_n), \Lambda z\}$ and $\Lambda z \perp \Lambda\Lambda z \Rightarrow \Lambda\Lambda z\Lambda z \in \{\Lambda z, \Lambda\Lambda z\}$, so $\Lambda\Lambda z \perp 0 = 0 \in \{\Lambda z, \Lambda\Lambda z\}$.
To verify that Λ and Γ are weakly compatible, we need to show that for each $x \in X$, $\Gamma x = \Lambda x$ implies $\Gamma\Lambda x = \Lambda\Gamma x$. Since $X = [0,1)$, it suffices to check for $x = 0$: $\Gamma x = 0 = \Lambda x \Rightarrow \Gamma\Lambda(0) = \Lambda\Gamma(0)$. Hence, it is verified.
Thus, this example satisfies all the hypotheses of the above theorem, and 0 is the common fixed point.

We shall now prove another result in the setting of orthogonal sets.

Theorem 3.2. *Let X be an O-set and let d be a metric on X such that $(X, d, \perp)$ is an O-complete set. Let $\Gamma, \Lambda : X \to X$ be two self-mappings such that $\Gamma(X) \subset \Lambda(X)$ and there exists a point $x_0 \in X$ with $\Lambda x_0 \perp \Gamma x_0$. Suppose that:*

(1) *There exists $\lambda \in [0,1)$ such that for each $x, y \in X$ satisfying $\Lambda y \perp \Lambda x$, there exists*
$$u \in M_1^{\Gamma,\Lambda}(x, y) = \left\{ d(\Lambda x, \Lambda y), d(\Gamma x, \Lambda x), d(\Gamma y, \Lambda y), \frac{d(\Gamma x, \Lambda y) + d(\Gamma y, \Lambda x)}{2} \right\}$$
such that $d(\Gamma x, \Gamma y) \leq \lambda \cdot u$ holds;
(2) *$\Lambda(X)$ is $\perp$-closed in X;*
(3) *Γ is Λ-$\perp$ preserving;*

(4) *If* $\{\Lambda(x_n)\} \subset X$ *is a* $\perp$*-preserving sequence converging to some* Λz*, then* $\Lambda x_n \perp \Lambda z$ *and* $\Lambda z \perp \Lambda\Lambda z$*.*

Then Γ *and* Λ *have a coincidence point. Moreover, if* Γ *and* Λ *are weakly compatible, then they have a common fixed point.*

Proof. As there exists a point $x_0 \in X$ such that $\Lambda x_0 \perp \Gamma x_0$, we construct a Jungck sequence. Given x_0, choose $x_1 \in X$ such that $\Gamma x_0 = \Lambda x_1$ (since $\Gamma(X) \subset \Lambda(X)$). Now, $\Lambda x_0 \perp \Lambda x_1$ implies $\Gamma x_0 \perp \Gamma x_1$. Then, there exists $x_2 \in X$ such that $\Gamma x_1 = \Lambda x_2$ and again $\Gamma x_0 \perp \Gamma x_1$ implies $\Lambda x_1 \perp \Lambda x_2$ and $\Gamma x_1 \perp \Gamma x_2$. Continuing this procedure, we have:

$$\Gamma x_0 \perp \Gamma x_1 \perp \Gamma x_2 \perp \cdots \perp \Gamma x_n \perp \Gamma x_{n+1} \perp \cdots$$
$$\Lambda x_1 \perp \Lambda x_2 \perp \cdots \perp \Lambda x_{n+1} \perp \Lambda x_{n+2} \perp \cdots$$

First, we prove that:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_{n-1}, \Gamma x_n) \text{ for } n \geq 1. \tag{3.18}$$

Since $\Lambda x_n \perp \Lambda x_{n+1}$, we get:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda \cdot u,$$

where

$$u \in \left\{ d(\Lambda x_n, \Lambda x_{n+1}), d(\Gamma x_n, \Lambda x_n), d(\Gamma x_{n+1}, \Lambda x_{n+1}), \frac{d(\Gamma x_n, \Lambda x_{n+1}) + d(\Gamma x_{n+1}, \Lambda x_n)}{2} \right\}$$
$$= \left\{ d(\Gamma x_{n-1}, \Gamma x_n), d(\Gamma x_n, \Gamma x_{n+1}), \frac{d(\Gamma x_{n-1}, \Gamma x_{n+1})}{2} \right\}.$$

We consider the following three cases:

(1) If $u = d(\Gamma x_{n-1}, \Gamma x_n)$, then clearly (3.18) holds.

(2) If $u = d(\Gamma x_n, \Gamma x_{n+1})$, then $d(\Gamma x_n, \Gamma x_{n+1}) = 0$, thus (3.18) holds.

(3) If $u = \frac{d(\Gamma x_{n-1}, \Gamma x_{n+1})}{2}$, then:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda \tfrac{d(\Gamma x_{n-1}, \Gamma x_{n+1})}{2} \leq \tfrac{\lambda}{2} d(\Gamma x_{n-1}, \Gamma x_n) + \tfrac{\lambda}{2} d(\Gamma x_n, \Gamma x_{n+1}).$$

Hence, $d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda d(\Gamma x_{n-1}, \Gamma x_n)$, and we have proved (3.18).

Now, we have:

$$d(\Gamma x_n, \Gamma x_{n+1}) \leq \lambda^n d(\Gamma x_0, \Gamma x_1).$$

We shall show that $\{\Gamma x_n\}$ is an O-Cauchy sequence. For $m, n \in \mathbb{N}, n > m$, we have:

$$d(\Gamma x_n, \Gamma x_m) \leq d(\Gamma x_n, \Gamma x_{n-1}) + d(\Gamma x_{n-1}, \Gamma x_{n-2}) + \cdots + d(\Gamma x_{m+1}, \Gamma x_m),$$

and we obtain:

$$d(\Gamma x_n, \Gamma x_m) \leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) d(\Gamma x_0, \Gamma x_1)$$
$$\leq \frac{\lambda^m}{1-\lambda} d(\Gamma x_0, \Gamma x_1) \to 0 \text{ as } m \to \infty.$$

It follows that for $\epsilon > 0$ and m sufficiently large, $\lambda^m(1-\lambda)^{-1}d(\Gamma x_0, \Gamma x_1) < \epsilon$, and thus $d(\Gamma x_n, \Gamma x_m) < \epsilon$. Hence, $\{\Gamma x_n\}$ is an O-Cauchy sequence.
Since $\Gamma(X) \subset \Lambda(X)$, $\Lambda(X)$ is $\perp$-closed, and X is O-complete, there exists $u \in \Lambda(X)$ such that $\Lambda x_n \to u$ as $n \to \infty$. Consequently, we can find $z \in X$ such that $\Lambda z = u$.
Let us show that $\Gamma z = u$. We have (because $\Lambda x_n \perp \Lambda z$):

$$d(\Gamma z, u) \leq d(\Gamma z, \Gamma x_n) + d(\Gamma x_n, u) \leq \lambda \cdot u_n + d(\Gamma x_n, u),$$

where

$$u_n \in \left\{ d(\Lambda x_n, \Lambda z), d(\Gamma x_n, \Lambda x_n), d(\Gamma z, \Lambda z), \tfrac{d(\Gamma x_n, \Lambda z) + d(\Gamma z, \Lambda x_n)}{2} \right\}.$$

Let $\epsilon > 0$ be given. Since $\Lambda x_n \to \Lambda z$, in each of the following cases, there exists n_0 such that for $n \geq n_0$, we have $d(\Gamma z, u) < \epsilon$:

(1) $d(\Gamma z, u) \leq \lambda \cdot d(\Lambda x_n, \Lambda z) + d(\Gamma x_n, u) < \epsilon$.
(2) $d(\Gamma z, u) \leq \lambda \cdot d(\Gamma x_n, \Lambda x_n) + d(\Gamma x_n, u) \leq \lambda \cdot d(\Gamma x_n, u) + \lambda \cdot d(u, \Lambda x_n) + d(\Gamma x_n, u) = (\lambda+1) \cdot d(\Gamma x_n, u) + \lambda \cdot d(u, \Lambda x_n) < \epsilon$.
(3) $d(\Gamma z, u) \leq \lambda \cdot d(\Gamma z, u) + d(\Gamma x_n, u)$; i.e., $d(\Gamma z, u) < \epsilon$.
(4) $d(\Gamma z, u) \leq \lambda \cdot \frac{d(\Gamma x_n, \Lambda z) + d(\Gamma z, \Lambda x_n)}{2} + d(\Gamma x_n, u) \leq \lambda \cdot \frac{d(\Gamma x_n, \Lambda z) + d(\Lambda x_n, u) + d(u, \Gamma z)}{2} + d(\Gamma x_n, u)$; i.e., $d(\Gamma z, u) \leq \frac{(\lambda+2) d(\Gamma x_n, u) + \lambda d(\Lambda x_n, u)}{2-\lambda} < \epsilon$.

Thus, we conclude $d(\Gamma z, u) = 0$, i.e., $\Gamma z = u$. Hence, we have proved that Γ and Λ have a coincidence point $z \in X$ and a point of coincidence $u \in X$ such that $u = \Gamma(z) = \Lambda(z)$.
If they are weakly compatible, then:

$$\Lambda\Lambda z = \Lambda\Gamma z = \Gamma\Lambda z = \Gamma\Gamma z.$$

We shall prove that $\Gamma z = \Lambda z$ is a common fixed point of the mappings Γ and Λ. Using $\Lambda z \perp \Lambda\Lambda z$, we obtain from our first condition:

$$d(\Gamma z, \Gamma\Gamma z) \leq \lambda \cdot u,$$

where

$$u \in \left\{ d(\Lambda z, \Lambda\Gamma z), d(\Gamma z, \Lambda z), d(\Gamma\Gamma z, \Lambda\Gamma z), \frac{d(\Gamma z, \Gamma\Lambda z) + d(\Gamma\Gamma z, \Lambda z)}{2} \right\}$$
$$= \left\{ d(\Gamma z, \Gamma\Gamma z), 0, \frac{d(\Gamma z, \Gamma\Gamma z) + d(\Gamma\Gamma z, \Lambda z)}{2} \right\} = \{0, d(\Gamma z, \Gamma\Gamma z)\}.$$

Thus, $d(\Gamma z, \Gamma\Gamma z) = 0$, i.e., $\Gamma z = \Gamma\Gamma z$. Similarly, $\Lambda z = \Lambda\Lambda z$, and the theorem is proved. □

Theorem 3.3. *Let X be an O-set and let d be a metric on X such that $(X, d, \perp)$ is an O-complete set. Let $\Gamma, \Lambda : X \to X$ be two self-mappings such that $\Gamma(X) \subset \Lambda(X)$ and there exists a point $x_0 \in X$ with $\Lambda x_0 \perp \Gamma x_0$. Suppose that:*

(1) *There exists $\lambda \in [0, 1)$ such that for each $x, y \in X$ satisfying $\Lambda y \perp \Lambda x$, there exists*

$$u \in M_2^{\Gamma,\Lambda}(x,y) = \left\{ d(\Lambda x, \Lambda y), \frac{d(\Gamma x,\Lambda x)+d(\Gamma y,\Lambda y)}{2}, \frac{d(\Gamma x,\Lambda y)+d(\Gamma y,\Lambda x)}{2} \right\}$$

such that $d(\Gamma x, \Gamma y) \le \lambda \cdot u$ *holds;*

(2) $\Lambda(X)$ *is* $\perp$*-closed in* X*;*

(3) Γ *is* Λ-$\perp$ *preserving;*

(4) *If* $\{\Lambda(x_n)\} \subset X$ *is a* $\perp$*-preserving sequence converging to some* Λz*, then* $\Lambda x_n \perp \Lambda z$ *and* $\Lambda z \perp \Lambda\Lambda z$*.*

Then Γ *and* Λ *have a coincidence point. Moreover, if* Γ *and* Λ *are weakly compatible, then they have a common fixed point.*

Proof. The proof of this theorem is similar to the proofs of Theorems 3.1 and 3.2. □

4. Conclusions

The fixed point results play an important role in ensuring the existence of solutions to various problems in non-linear analysis. In our paper, we have established the existence of common fixed points for multivalued mappings using the concepts of $\perp$ Λ-quasocontraction and $\perp$ Λ-preserving mappings in orthogonal sets. We have also provided an example to support our main result.

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