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Extended Bipolar Intuitionistic Fuzzy Ideals Framework Through Level Sets and Its Characterization via Regular Ordered Γ-Semigroups

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Abstract. This paper proposes an extended framework for bipolar anti-intuitionistic fuzzy ideals within the context of ordered Γ-semigroups. We introduce and investigate the (δ, τ)-bipolar anti-intuitionistic fuzzy subsemigroups (BPAIFSS), including their associated left ideals, right ideals, ideals, and bi-ideals. These structures generalize existing fuzzy ideal notions by incorporating dual-valued membership and non-membership functions with flexible threshold control. Using level set analysis, we characterize the algebraic properties of these fuzzy ideals and establish their role in determining the regularity of ordered Γ-semigroups. Illustrative examples are provided to validate and demonstrate the applicability of the theoretical results.

1. Introduction

The uncertainties have led to the development of several theories that are uncertain, including fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), and spherical fuzzy sets (SFSs). Since then, a large number of articles on FSs have been published, demonstrating the significance of the idea and its applications to real analysis, measure theory, topology, group theory, logic, and groupoids, among other fields [3–7]. There are several uses for ordered semigroups in computer arithmetic, formal languages, error-correcting codes, and the theory of sequential machines. An FS consists of sets of grades, or MG, ranging from 0 to 1. Despite Atanassov's claims that non-membership grades (NMGs) might be as low as 1, IFS is classified as a membership

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grade (MG). The total of MGs and NMGs may occasionally exceed 1 during the decision-making process. Yager used PFS logic to develop the generalized MG and NMG logics, which attain a maximum value of 1 and are based on the MG and NMG squares. These notions cannot effectively represent the neutral situation, which is neither positive nor negative. Rosenfeld [13] defined fuzzy subgroups and detailed their features in 1971. Kuroki [10] introduced FSS as an extension of traditional semigroups. Mordeson developed a specific fuzzy semigroup categorization [12]. The features of gamma-semigroups were described by Sen et al. [14]. BFSs were first proposed by Zhang [15], who utilized them for modeling and decision analysis. BFSs are FSs whose MG range is expanded from the interval $[0,1]$ to $[-1,1]$. Additionally, research on BFI types has been conducted by researchers like Kang et al. [2], who examined BFSS in semigroups. In semigroups, generalized BFSSs were described by Khamrot et al. [8]. Lekkoksung introduced the idea of Q-FIs in ordered semigroups [11]. Khan et al. [9] were the first to suggest the (δ_1, δ_2)-FBI and the (δ_1, δ_2)-FSS. Jun et al. discussed results on ordered semigroups with $(\sharp, \sharp \vee q)$-FBIs [1].

2. Preliminaries

In this section, we recall some basic definitions and concepts that will be used throughout the paper. These include fundamental operations on subsets of an ordered Γ-semigroup, properties of fuzzy sets, and classical notions of fuzzy ideals. We also establish the necessary notations and conditions for defining various types of bipolar anti-intuitionistic fuzzy subsets and their respective ideal structures. These preliminaries form the foundation for the new framework proposed in later sections.

Definition 2.1. *Let* $\daleth$ *and* $\gimel$ *be subsets of* $\Bbbk$. *Then*

(1) $(\daleth] = \{\mu \in \Bbbk \mid \mu \leq \nu \text{ for some } \nu \in \daleth\}$,
(2) $\daleth\Gamma\gimel = \{A\upsilon B \mid A \in \daleth, B \in \gimel, \upsilon \in \Gamma\}$,
(3) $\daleth_\eta = \{(\zeta, \omega) \in \Bbbk \times \Bbbk \mid \eta \leq \zeta\upsilon\omega\}$.

Definition 2.2. *An FS* τ *of* $\Bbbk$ *is represent an FRI (FLI) of* $\Bbbk$ *if*

(1) $\zeta \leq \omega \Rightarrow \tau(\zeta) \geq \tau(\omega)$,
(2) $\tau(\zeta\gamma\omega) \geq \tau(\zeta)$ *(resp.,* $\tau(\zeta\gamma\omega) \geq \tau(\omega)$*) for all* $\zeta, \omega \in \Bbbk$ *and* $\gamma \in \Gamma$.

Definition 2.3. *An FS* $\flat$ *of* $\Bbbk$ *is represent an FBI of* $\Bbbk$ *if*

(1) $a \leq b \Rightarrow \flat(a) \geq \flat(b)$,
(2) $\flat(xyz) \geq \min\{\flat(x), \flat(z)\}$ *for all* $x, z \in \Bbbk$ *and* $y \in \Gamma$.

Definition 2.4. *Let* C *be an FS, if* $\mathfrak{R}_C$ *is the characteristic function of* C, *then*

$$(\mathfrak{R}_C)^{\iota}_{\gamma}(\tau) := \begin{cases} \iota \ \text{if} \ \tau \in C, \\ \gamma \ \text{otherwise.} \end{cases}$$

Note: $\Bbbk$ is regular if and only if for all RI $\daleth$ and for all LI $\gimel$ of $\Bbbk$, $(\daleth \cap \gimel] = (\daleth \circ \gimel]$.

3. Bipolar anti-intuitionistic fuzzy ideals

Here, $\Bbbk$ refers the ordered Γ-semigroup, $\delta, \tau \in [0,1]$, $0 \geq \underline{\delta} > \underline{\tau} \geq -1$ and $0 \leq \overline{\delta} < \overline{\tau} \leq 1$.

Definition 3.1. *A bipolar anti-intuitionistic fuzzy set (BPIFS)* $\flat = [(\sqsupset, \Delta), (\aleph, \Psi)]$ *of* $\Bbbk$ *is represent a* (δ, τ)*-BPAIFSS of* $\Bbbk$ *if*

(1) $\varrho \leq \natural \Rightarrow \sqsupset(\varrho) \leq \sqsupset(\natural), \aleph(\varrho) \geq \aleph(\natural), \Delta(\varrho) \geq \Delta(\natural), \Psi(\varrho) \leq \Psi(\natural)$,
(2) $\min\{\sqsupset(\varrho\gamma\natural), \overline{\delta}\} \leq \max\{\sqsupset(\varrho), \sqsupset(\natural), \overline{\tau}\}$,
$\max\{\aleph(\varrho\gamma\natural), \overline{\delta}\} \geq \min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\}$,
(3) $\max\{\Delta(\varrho\gamma\natural), \underline{\delta}\} \geq \min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\}$,
$\min\{\Psi(\varrho\gamma\natural), \underline{\delta}\} \leq \max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\}$, *for all* $\varrho, \natural \in \Bbbk$, $\gamma \in \Gamma$.

Definition 3.2. *A BPIFS* $\flat = [(\sqsupset, \Delta), (\aleph, \Psi)]$ *of* $\Bbbk$ *is represent a* (δ, τ)*-BPAIFSLI of* $\Bbbk$ *if*

(1) $\varrho \leq \natural \Rightarrow \sqsupset(\varrho) \leq \sqsupset(\natural), \aleph(\varrho) \geq \aleph(\natural), \Delta(\varrho) \geq \Delta(\natural), \Psi(\varrho) \leq \Psi(\natural)$,
(2) $\min\{\sqsupset(\varrho\gamma_1\natural), \overline{\delta}\} \leq \max\{\sqsupset(\natural), \overline{\tau}\}$,
$\max\{\aleph(\varrho\gamma_1\natural), \overline{\delta}\} \geq \min\{\aleph(\natural), \overline{\tau}\}$,
(3) $\max\{\Delta(\varrho\gamma_1\natural), \underline{\delta}\} \geq \min\{\Delta(\natural), \underline{\tau}\}$,
$\min\{\Psi(\varrho\gamma_1\natural), \underline{\delta}\} \leq \max\{\Psi(\natural), \underline{\tau}\}$, *for* $\varrho, \natural \in \Bbbk, \gamma_1 \in \Gamma$.

Definition 3.3. *A BPIFS* $\flat = [(\sqsupset, \Delta), (\aleph, \Psi)]$ *of* $\Bbbk$ *is represent a* (δ, τ)*-BPAIFSRI of* $\Bbbk$ *if*

(1) $\varrho \leq \natural \Rightarrow \sqsupset(\varrho) \leq \sqsupset(\natural), \aleph(\varrho) \geq \aleph(\natural), \Delta(\varrho) \geq \Delta(\natural), \Psi(\varrho) \leq \Psi(\natural)$,
(2) $\min\{\sqsupset(\varrho\gamma_1\natural), \overline{\delta}\} \leq \max\{\sqsupset(\varrho), \overline{\tau}\}$,
$\max\{\aleph(\varrho\gamma_1\natural), \overline{\delta}\} \geq \min\{\aleph(\varrho), \overline{\tau}\}$,
(3) $\max\{\Delta(\varrho\gamma_1\natural), \underline{\delta}\} \geq \min\{\Delta(\varrho), \underline{\tau}\}$,
$\min\{\Psi(\varrho\gamma_1\natural), \underline{\delta}\} \leq \max\{\Psi(\varrho), \underline{\tau}\}$, *for* $\varrho, \natural \in \Bbbk, \gamma_1 \in \Gamma$.

Definition 3.4. *A BPIFS* $\flat = [(\sqsupset, \Delta), (\aleph, \Psi)]$ *of* $\Bbbk$ *is represent a* (δ, τ)*-BPAIFSBI of* $\Bbbk$ *if*

(1) $\varrho \leq \natural \Rightarrow \sqsupset(\varrho) \leq \sqsupset(\natural), \aleph(\varrho) \geq \aleph(\natural), \Delta(\varrho) \geq \Delta(\natural), \Psi(\varrho) \leq \Psi(\natural)$,
(2) $\min\{\sqsupset(\varrho\gamma_1\natural), \overline{\delta}\} \leq \max\{\sqsupset(\varrho), \sqsupset(\natural), \overline{\tau}\}$,
$\max\{\aleph(\varrho\gamma_1\natural), \overline{\delta}\} \geq \min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\}$,
$\min\{\sqsupset(\varrho\gamma_1\natural\gamma_2\varepsilon), \overline{\delta}\} \leq \max\{\sqsupset(\varrho), \sqsupset(\varepsilon), \overline{\tau}\}$,
$\max\{\aleph(\varrho\gamma_1\natural\gamma_2\varepsilon), \overline{\delta}\} \geq \min\{\aleph(\varrho), \aleph(\varepsilon), \overline{\tau}\}$,
(3) $\max\{\Delta(\varrho\gamma_1\natural), \underline{\delta}\} \geq \min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\}$,
$\min\{\Psi(\varrho\gamma_1\natural), \underline{\delta}\} \leq \max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\}$,
$\max\{\Delta(\varrho\gamma_1\natural\gamma_2\varepsilon), \underline{\delta}\} \geq \min\{\Delta(\varrho), \Delta(\varepsilon), \underline{\tau}\}$,
$\min\{\Psi(\varrho\gamma_1\natural\gamma_2\varepsilon), \underline{\delta}\} \leq \max\{\Psi(\varrho), \Psi(\varepsilon), \underline{\tau}\}$, *for* $\varrho, \natural, \varepsilon \in \Bbbk, \gamma_1, \gamma_2 \in \Gamma$.

Example 3.1. *Let* $\Bbbk = \{\sharp_1, \sharp_2, \sharp_3, \sharp_4\}$ *and* $\Gamma = \{\gamma\}$ *where* γ *is defined on* $\Bbbk$.

γ	$\sharp_1$	$\sharp_2$	$\sharp_3$	$\sharp_4$
$\sharp_1$	$\sharp_1$	$\sharp_1$	$\sharp_1$	$\sharp_1$
$\sharp_2$	$\sharp_1$	$\sharp_2$	$\sharp_3$	$\sharp_4$
$\sharp_3$	$\sharp_1$	$\sharp_3$	$\sharp_3$	$\sharp_3$
$\sharp_4$	$\sharp_1$	$\sharp_3$	$\sharp_3$	$\sharp_3$

The order relation: $\{(\sharp_1,\sharp_1),(\sharp_1,\sharp_2),(\sharp_1,\sharp_3),(\sharp_1,\sharp_4),(\sharp_2,\sharp_2),(\sharp_2,\sharp_3),(\sharp_2,\sharp_4),(\sharp_3,\sharp_3),(\sharp_4,\sharp_3),(\sharp_4,\sharp_4)\}$.
Define a BPIFS $\gimel = [(\beth,\Delta),(\aleph,\Psi)] : \Bbbk \to [-1,0]\times[0,1]$ *as follows:*

$$[(\beth,\Delta),(\aleph,\Psi)](\sharp_1) = (0.31,-0.26),(0.61,-0.56),$$

$$[(\beth,\Delta),(\aleph,\Psi)](\sharp_2) = (0.36,-0.31),(0.41,-0.36),$$

$$[(\beth,\Delta),(\aleph,\Psi)](\sharp_3) = (0.44,-0.44),(0.11,-0.06),$$

$$[(\beth,\Delta),(\aleph,\Psi)](\sharp_4) = (0.51,-0.36),(0.21,-0.16).$$

Hence, $\gimel$ *is a* $(0.51,0.66)$*-BPAIFSS of* $\Bbbk$.

Lemma 3.1. *Let a BPIFS* $\flat_\delta$ *be a* (δ,τ)*-BPAIFSS (BPAIFSLI, BPAIFSRI, BPAIFSBI) of* $\Bbbk$*. Then the lower level set is an SS (LI, RI, BI) of* $\Bbbk$*, where* $\beth_{\overline{\delta}} = \{\varrho\in\Bbbk \mid \beth(\varrho) < \overline{\delta}\}$, $\aleph_{\overline{\delta}} = \{\varrho\in\Bbbk \mid \aleph(\varrho) > \overline{\delta}\}$, $\Delta_{\underline{\delta}} = \{\varrho\in\Bbbk \mid \Delta(\varrho) > \underline{\delta}\}$ *and* $\Psi_{\underline{\delta}} = \{\varrho\in\Bbbk \mid \Psi(\varrho) < \underline{\delta}\}$.

Proof. Suppose that $\flat_{\overline{\delta}}$ is a (δ,τ)-BPAIFSS of $\Bbbk$. Let $\varrho,\natural\in\Bbbk$ and $\gamma\in\Gamma$ be such that $\varrho,\natural\in\beth_{\overline{\delta}}$. Then $\beth(\varrho) < \overline{\delta}, \beth(\natural) < \overline{\delta}$. Hence, $\min\{\beth(\varrho\gamma\natural),\overline{\delta}\} \le \max\{\beth(\varrho),\beth(\natural),\overline{\tau}\} < \max\{\overline{\delta},\overline{\delta},\overline{\tau}\} = \overline{\delta}$. Hence, $\beth(\varrho\gamma\natural) < \overline{\delta}$. It shows that $\varrho\gamma\natural\in\beth_{\overline{\delta}}$. Hence, $\beth_{\overline{\delta}}$ is an SS of $\Bbbk$. Let $\varrho,\natural\in\Bbbk$ and $\gamma\in\Gamma$ be such that $\varrho,\natural\in\aleph_{\overline{\delta}}$. Then $\aleph(\varrho) > \overline{\delta}, \aleph(\natural) > \overline{\delta}$. Hence, $\max\{\aleph(\varrho\gamma\natural),\overline{\delta}\} \ge \min\{\aleph(\varrho),\aleph(\natural),\overline{\tau}\} > \min\{\overline{\delta},\overline{\delta},\overline{\tau}\}$ $= \overline{\tau}$. Hence, $\aleph(\varrho\gamma\natural) > \overline{\delta}$. It shows that $\varrho\gamma\natural\in\aleph_{\overline{\delta}}$. Hence, $\aleph_{\overline{\delta}}$ is an SS of $\Bbbk$. Suppose that $\flat_{\underline{\delta}}$ is a (δ,τ)-BPAIFSS of $\Bbbk$. Let $\varrho,\natural\in\Bbbk$ and $\gamma\in\Gamma$ be such that $\varrho,\natural\in\Delta_{\underline{\delta}}$. Then $\Delta(\varrho) > \underline{\delta}, \Delta(\natural) > \underline{\delta}$. Hence, $\max\{\Delta(\varrho\gamma\natural),\underline{\delta}\} \ge \min\{\Delta(\varrho),\Delta(\natural),\underline{\tau}\} > \min\{\underline{\delta},\underline{\delta},\underline{\tau}\} = \underline{\delta}$. Hence, $\Delta(\varrho\gamma\natural) > \underline{\delta}$. It shows that $\varrho\gamma\natural\in\Delta_{\underline{\delta}}$. Hence, $\Delta_{\underline{\delta}}$ is an SS of $\Bbbk$. Let $\varrho,\natural\in\Bbbk$ and $\gamma\in\Gamma$ be such that $\varrho,\natural\in\Psi_{\underline{\delta}}$. Then $\Psi(\varrho) < \underline{\delta}, \Psi(\natural) < \underline{\delta}$. Hence, $\min\{\Psi(\varrho\gamma\natural),\underline{\delta}\} \le \max\{\Psi(\varrho),\Psi(\natural),\underline{\tau}\} < \max\{\underline{\delta},\underline{\delta},\underline{\tau}\} = \underline{\tau}$. Hence, $\Psi(\varrho\gamma\natural) < \underline{\delta}$ implies that $\varrho\gamma\natural\in\Psi_{\underline{\delta}}$. Hence, $\Psi_{\underline{\delta}}$ is an SS of $\Bbbk$. □

Lemma 3.2. *A subset* $\daleth$ *of* $\Bbbk$ *is an SS (LI, RI, BI) of* $\Bbbk$ *if and only if the BPIFS* $\flat = [(\beth,\Delta),(\aleph,\Psi)]$ *of* $\Bbbk$ *is defined as follows:*

$$\beth(\varrho) = \begin{cases} \le \overline{\tau} \text{ for all } \varrho\in(\daleth] \\ \overline{\delta} \text{ for all } \varrho\notin(\daleth] \end{cases} \qquad \aleph(\varrho) = \begin{cases} \ge \overline{\tau} \text{ for all } \varrho\in(\daleth] \\ \overline{\delta} \text{ for all } \varrho\notin(\daleth] \end{cases}$$

$$\Delta(\varrho) = \begin{cases} \ge \underline{\tau} \text{ for all } \varrho\in(\daleth] \\ \underline{\delta} \text{ for all } \varrho\notin(\daleth] \end{cases} \qquad \Psi(\varrho) = \begin{cases} \le \underline{\tau} \text{ for all } \varrho\in(\daleth] \\ \underline{\delta} \text{ for all } \varrho\notin(\daleth] \end{cases}$$

is a (δ,τ)*-BPAIFSS (BPAIFSLI, BPAIFSRI, BPAIFSBI) of* $\Bbbk$.

Proof. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\daleth]$ then $\varrho\gamma\natural \in (\daleth]$ and $\gamma \in \Gamma$. Hence, $\beth(\varrho\gamma\natural) \leq \overline{\tau}$ and $\aleph(\varrho\gamma\natural) \geq \overline{\tau}$. Thus, $\min\{\beth(\varrho\gamma\natural), \overline{\delta}\} \leq \overline{\tau} = \max\{\beth(\varrho), \beth(\natural), \overline{\tau}\}$ and $\max\{\aleph(\varrho\gamma\natural), \overline{\delta}\} \geq \overline{\tau} = \min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\}$.

If $\varrho \notin (\daleth]$ or $\natural \notin (\daleth]$, then $\max\{\beth(\varrho), \beth(\natural), \overline{\tau}\} = \overline{\delta}$ and $\min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\} = \overline{\tau}$.

That is, $\min\{\beth(\varrho\gamma\natural), \overline{\delta}\} \leq \max\{\beth(\varrho), \beth(\natural), \overline{\tau}\}$ and $\max\{\aleph(\varrho\gamma\natural), \overline{\delta}\} \geq \min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\}$.

Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\daleth]$ then $\varrho\gamma\natural \in (\daleth]$ and $\gamma \in \Gamma$. Hence, $\Delta(\varrho\gamma\natural) \geq \underline{\tau}$ and $\Psi(\varrho\gamma\natural) \leq \underline{\tau}$. Thus, $\max\{\Delta(\varrho\gamma\natural), \underline{\delta}\} \geq \underline{\tau} = \min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\}$ and $\min\{\Psi(\varrho\gamma\natural), \underline{\delta}\} \leq \underline{\tau} = \max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\}$.

If $\varrho \notin (\daleth]$ or $\natural \notin (\daleth]$, then $\min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\} = \underline{\delta}$ and $\max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\} = \underline{\tau}$.

That is, $\max\{\Delta(\varrho\gamma\natural), \underline{\delta}\} \geq \min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\}$ and $\min\{\Psi(\varrho\gamma\natural), \underline{\delta}\} \leq \max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\}$. Hence, $\flat$ is a (δ, τ)-BPAIFSS of $\Bbbk$.

Conversely, assume that $\flat = [\beth, \aleph]$ is a (δ, τ)-BPAIFSS of $\Bbbk$. Let $\varrho, \natural \in (\daleth]$. Then $\beth(\varrho) \leq \overline{\tau}, \beth(\natural) \leq \overline{\tau}$ and $\aleph(\varrho) \geq \overline{\tau}, \aleph(\natural) \geq \overline{\tau}$. Now $\flat = [\beth, \aleph]$ is a (δ, τ)-BPAIFSS of $\Bbbk$. Hence, $\min\{\beth(\varrho\gamma\natural), \overline{\delta}\} \leq \max\{\beth(\varrho), \beth(\natural), \overline{\tau}\} \leq \max\{\overline{\tau}, \overline{\tau}, \overline{\tau}\} = \overline{\tau}$ and $\max\{\aleph(\varrho\gamma\natural), \overline{\delta}\} \geq \min\{\aleph(\varrho), \aleph(\natural), \overline{\tau}\} \geq \min\{\overline{\tau}, \overline{\tau}, \overline{\tau}\} = \overline{\tau}$. It follows that $\varrho\gamma\natural \in (\daleth]$. Let $\varrho, \natural \in (\daleth]$. Then $\Delta(\varrho) \geq \underline{\tau}, \Delta(\natural) \geq \underline{\tau}$, and $\Psi(\varrho) \leq \underline{\tau}, \Psi(\natural) \leq \underline{\tau}$. Now $\flat = [\Delta, \Psi]$ is a (δ, τ)-BPAIFSS of $\Bbbk$. Hence, $\max\{\Delta(\varrho\gamma\natural), \underline{\delta}\} \geq \min\{\Delta(\varrho), \Delta(\natural), \underline{\tau}\} \geq \min\{\underline{\tau}, \underline{\tau}, \underline{\tau}\} = \underline{\tau}$ and $\min\{\Psi(\varrho\gamma\natural), \underline{\delta}\} \leq \max\{\Psi(\varrho), \Psi(\natural), \underline{\tau}\} \leq \underline{\tau}$ implies that $\varrho\gamma\natural \in (\daleth]$. Hence, $\daleth$ is an SS of $\Bbbk$. □

Definition 3.5. *Let* $\flat = [(\beth, \Delta), (\aleph, \Psi)]$ *be a* (δ, τ)*-BPAIFSS of* $\Bbbk$*, and let* $t, s \in (\delta, \tau]$*. Then the* level subset $\flat^{(t,s)}$ *of* $\flat$ *is defined as*

$$\flat^{(t,s)} = \left\{ x \in \Bbbk \;\middle|\; \begin{array}{ll} \beth(x) \leq t, & \aleph(x) \geq t, \\ \Delta(x) \geq s, & \Psi(x) \leq s \end{array} \right\}.$$

Theorem 3.1. *A BPIFS* $\flat = [(\beth, \Delta), (\aleph, \Psi)]$ *is a* (δ, τ)*-BPAIFSS (BPAIFSLI, BPAIFSRI, BPAIFSBI) of* $\Bbbk$ *if and only if each level subset* $\flat^{(t,s)}$ *is an SS (LI, RI, BI) of* $\Bbbk$ *for all* $t \in (\overline{\delta}, \overline{\tau}]$*.*

Proof. Assume that $\flat^{(t,s)}$ is an SS of $\Bbbk$. Let $\varrho_1, \varrho_2 \in \Bbbk$. Let $t = \max\{\beth(\varrho_1), \beth(\varrho_2)\}$. Then $\varrho_1, \varrho_2 \in \beth_t$. Thus, $\min\{\beth(\varrho_1\gamma\varrho_2), \overline{\delta}\} \leq t = \max\{\beth(\varrho_1), \beth(\varrho_2), \overline{\tau}\}$. Let $t = \min\{\aleph(\varrho_1), \aleph(\varrho_2)\}$. Then $\varrho_1, \varrho_2 \in \aleph_t$. Thus, $\max\{\aleph(\varrho_1\gamma\varrho_2), \overline{\delta}\} \geq t = \min\{\aleph(\varrho_1), \aleph(\varrho_2), \overline{\tau}\}$. Let $s = \min\{\Delta(\varrho_1), \Delta(\varrho_2)\}$. Then $\varrho_1, \varrho_2 \in \Delta_s$. Thus, $\max\{\Delta(\varrho_1\gamma\varrho_2), \underline{\delta}\} \geq s = \min\{\Delta(\varrho_1), \Delta(\varrho_2), \underline{\tau}\}$. $s = \max\{\Psi(\varrho_1), \Psi(\varrho_2)\}$. Then $\varrho_1, \varrho_2 \in \Psi_s$. Thus, $\min\{\Psi(\varrho_1\gamma\varrho_2), \underline{\delta}\} \leq s = \max\{\Psi(\varrho_1), \Psi(\varrho_2), \underline{\tau}\}$ implies that $\flat$ is a (δ, τ)-BPAIFSS of $\Bbbk$.

Conversely, assume that $\flat$ is a (δ, τ)-BPAIFSS of $\Bbbk$ and $\varrho_1, \varrho_2 \in \flat^{(t,s)}$. Then $\beth(\varrho_1) \leq t, \beth(\varrho_2) \leq t$. Thus, $\min\{\beth(\varrho_1\gamma\varrho_2), \overline{\delta}\} \leq \max\{\beth(\varrho_1), \beth(\varrho_2), \overline{\tau}\} \leq t$. This implies that $\varrho_1\gamma\varrho_2 \in \flat^{(t,s)}$. Now, $\aleph(\varrho_1) \geq t, \aleph(\varrho_2) \geq t$. Since $\flat$ is a (δ, τ)-BPAIFSS of $\Bbbk$, $\max\{\aleph(\varrho_1\gamma\varrho_2), \overline{\delta}\} \geq \min\{\aleph(\varrho_1), \aleph(\varrho_2), \overline{\tau}\} \geq t$. This implies that $\varrho_1\gamma\varrho_2 \in \aleph_t$. Then $\Delta(\varrho_1) \geq t, \Delta(\varrho_2) \geq s$. Thus, $\max\{\Delta(\varrho_1\gamma\varrho_2), \underline{\delta}\} \geq \min\{\Delta(\varrho_1), \Delta(\varrho_2), \underline{\tau}\} \geq s$. This implies that $\varrho_1\gamma\varrho_2 \in \Delta_t$. Then $\Psi(\varrho_1) \leq s, \Psi(\varrho_2) \leq s$. Since Ψ is an SS of $\Bbbk$, $\min\{\Psi(\varrho_1\gamma\varrho_2), \underline{\delta}\} \leq s$. Hence, $\flat_{(t,s)}$ is an SS of $\Bbbk$. □

Example 3.2. *The BPAIFSS* $\flat$ *of* $\Bbbk$ *is a* (δ, τ)*-BPAIFSS of* $\Bbbk$*, but reverse is not true. From Example 3.1, we define a BPIFS* $\gimel = [(\beth, \Delta), (\aleph, \Psi)] : \Bbbk \to [-1, 0] \times [0, 1]$ *as follows:*

$$[(\beth, \Delta), (\aleph, \Psi)](\sharp_1) = (0.16, -0.15), (0.33, -0.30),$$

$$[(\beth, \Delta), (\aleph, \Psi)](\sharp_2) = (0.23, -0.20), (0.26, -0.23),$$

$$[(\sqsupset, \Delta), (\aleph, \Psi)](\sharp_3) = (0.33, -0.30), (0.16, -0.13),$$

$$[(\sqsupset, \Delta), (\aleph, \Psi)](\sharp_4) = (0.28, -0.25), (0.21, -0.18).$$

Hence, $\flat$ *is a* $(0.24, 0.38)$*-BPAIFSS of* $\Bbbk$ *and not a BPAIFSS.*

Definition 3.6. *The BPIFS* $\mathfrak{R}_{\urcorner}$ *is defined as*

$$\mathfrak{R}^{\overline{\mathfrak{T}}}_{\urcorner}(\varrho) = \begin{cases} \overline{\tau} \text{ if } \varrho \in (\urcorner] \\ \overline{\delta} \text{ if } \varrho \notin (\urcorner] \end{cases} \qquad \mathfrak{R}^{\overline{\mathfrak{F}}}_{\urcorner}(\varrho) = \begin{cases} \overline{\delta} \text{ if } \varrho \in (\urcorner] \\ \overline{\tau} \text{ if } \varrho \notin (\urcorner] \end{cases}$$

$$\mathfrak{R}^{\underline{\mathfrak{T}}}_{\urcorner}(\varrho) = \begin{cases} \underline{\tau} \text{ if } \varrho \in (\urcorner] \\ \underline{\delta} \text{ if } \varrho \notin (\urcorner] \end{cases} \qquad \mathfrak{R}^{\underline{\mathfrak{F}}}_{\urcorner}(\varrho) = \begin{cases} \underline{\delta} \text{ if } \varrho \in (\urcorner] \\ \underline{\tau} \text{ if } \varrho \notin (\urcorner] \end{cases}$$

Theorem 3.2. *A subset* $\urcorner$ *of* $\Bbbk$ *is an SS (LI, RI, BI) of* $\Bbbk$ *if and only if the BPIFS* $\mathfrak{R}_{(\urcorner]}$ *is a* (δ, τ)*-BPAIFSS (BPAIFSLI, BPAIFSRI, BPAIFSBI) of* $\Bbbk$*.*

Proof. Suppose that $\urcorner$ is an SS of $\Bbbk$. Then $\mathfrak{R}_{(\urcorner]}$ is a BPAIFSS of $\Bbbk$ implies $\mathfrak{R}_{(\urcorner]}$ is a (δ, τ)-BPAIFSS of $\Bbbk$.

Conversely, assume that $\mathfrak{R}_{(\urcorner]}$ is a (δ, τ)-BPAIFSS of $\Bbbk$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\urcorner]$. Then $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho) = \overline{\tau} = \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\natural) = \overline{\tau}$. Since $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned} \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho\gamma\natural), \overline{\delta}\} &\leq \max\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\natural), \overline{\tau}\} \\ &= \max\{\overline{\tau}, \overline{\tau}, \overline{\tau}\} \\ &= \overline{\tau} \end{aligned}$$

as $\overline{\delta} > \overline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho\gamma\natural) \leq \overline{\tau}$. Thus, $\varrho\gamma\natural \in (\urcorner]$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\urcorner]$. Then $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\varrho) = \overline{\delta} = \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\natural) = \overline{\delta}$. Since $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned} \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\varrho\gamma\natural), \overline{\delta}\} &\geq \min\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\natural), \overline{\tau}\} \\ &= \min\{\overline{\delta}, \overline{\delta}, \overline{\tau}\} \\ &= \overline{\tau} \end{aligned}$$

as $\overline{\delta} > \overline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\urcorner]}(\varrho\gamma\natural) \geq \overline{\delta}$. Thus, $\varrho\gamma\natural \in (\urcorner]$. Hence, $\urcorner$ is an SS of $\Bbbk$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \notin (\urcorner]$. Then $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho) = \overline{\delta} = \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\natural) = \overline{\delta}$. Since $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned} \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho\gamma\natural), \overline{\delta}\} &\leq \max\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\urcorner]}(\natural), \overline{\tau}\} \\ &= \max\{\overline{\delta}, \overline{\delta}, \overline{\tau}\} \\ &= \overline{\delta} \end{aligned}$$

as $\overline{\delta} > \overline{\tau} \Rightarrow \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]}(\varrho\gamma\natural) \leq \overline{\delta}$. Thus, $\varrho\gamma\natural \notin (\daleth]$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \notin (\daleth]$. Then $\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\varrho) = \overline{\tau} = \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\natural) = \overline{\tau}$. Since $\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned}\max\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural), \overline{\delta}\} &\geq \min\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\varrho), \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\natural), \overline{\tau}\}\\ &= \min\{\overline{\tau}, \overline{\tau}, \overline{\tau}\}\\ &= \overline{\tau}\end{aligned}$$

as $\overline{\delta} > \overline{\tau} \Rightarrow \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural) \geq \overline{\tau}$. Thus, $\varrho\gamma\natural \notin (\daleth]$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\daleth]$. Then $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho) = \underline{\tau} = \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\natural) = \underline{\tau}$. Since $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned}\max\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho\gamma\natural), \underline{\delta}\} &\geq \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\natural), \underline{\tau}\}\\ &= \min\{\underline{\tau}, \underline{\tau}, \underline{\tau}\}\\ &= \underline{\tau}\end{aligned}$$

as $\underline{\delta} < \underline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho\gamma\natural) \geq \underline{\tau}$. Thus, $\varrho\gamma\natural \in (\daleth]$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \in (\daleth]$. Then $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho) = \underline{\delta} = \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\natural) = \underline{\delta}$. Since $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned}\min\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural), \underline{\delta}\} &\leq \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\natural), \underline{\tau}\}\\ &= \max\{\underline{\delta}, \underline{\delta}, \underline{\tau}\}\\ &= \underline{\tau}\end{aligned}$$

as $\underline{\delta} < \underline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural) \leq \underline{\delta}$. Thus, $\varrho\gamma\natural \in (\daleth]$. Hence, $\daleth$ is an SS of $\Bbbk$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \notin (\daleth]$. Then $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho) = \underline{\delta} = \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\natural) = \underline{\delta}$. Since $\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned}\max\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho\gamma\natural), \underline{\delta}\} &\geq \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\natural), \underline{\tau}\}\\ &= \min\{\underline{\delta}, \underline{\delta}, \underline{\tau}\}\\ &= \underline{\delta}\end{aligned}$$

as $\underline{\delta} < \underline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(\varrho\gamma\natural) \geq \underline{\delta}$. Thus, $\varrho\gamma\natural \notin (\daleth]$. Let $\varrho, \natural \in \Bbbk$ be such that $\varrho, \natural \notin (\daleth]$. Then $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho) = \underline{\tau} = \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\natural) = \underline{\tau}$. Since $\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}$ is a (δ, τ)-BPAIFSS, we have

$$\begin{aligned}\min\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural), \underline{\delta}\} &\leq \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\natural), \underline{\tau}\}\\ &= \max\{\underline{\tau}, \underline{\tau}, \underline{\tau}\}\\ &= \underline{\tau}\end{aligned}$$

as $\underline{\delta} < \underline{\tau} \Rightarrow \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(\varrho\gamma\natural) \leq \underline{\tau}$. Thus, $\varrho\gamma\natural \notin (\daleth]$. Hence, $\daleth$ is an SS of $\Bbbk$. □

Definition 3.7. *The BPIFSs and their product $\flat \circ \delta$ is defined as follows:*

$$(\flat^{\underline{\mathfrak{T}}} \circ \delta^{\underline{\mathfrak{T}}})(\varrho) = \begin{cases} \inf\limits_{(s,t)\in\daleth_{\varrho}} \{\flat^{\underline{\mathfrak{T}}}(s) \vee \delta^{\underline{\mathfrak{T}}}(t)\} & \textit{if}\ \daleth_{\varrho} \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$$

$$(\flat^{\underline{\mathfrak{F}}} \circ \delta^{\underline{\mathfrak{F}}})(\varrho) = \begin{cases} \sup\limits_{(s,t)\in \daleth_\varrho} \{\flat^{\underline{\mathfrak{F}}}(s) \wedge \delta^{\underline{\mathfrak{F}}}(t)\} \ if \daleth_\varrho \neq \emptyset \\ 1 \ otherwise \end{cases}$$

$$(\flat^{\overline{\mathfrak{T}}} \circ \delta^{\overline{\mathfrak{T}}})(\varrho) = \begin{cases} \sup\limits_{(s,t)\in \daleth_\varrho} \{\flat^{\overline{\mathfrak{T}}}(s) \wedge \delta^{\overline{\mathfrak{T}}}(t)\} \ if \daleth_\varrho \neq \emptyset \\ 0 \ otherwise \end{cases}$$

$$(\flat^{\overline{\mathfrak{F}}} \circ \delta^{\overline{\mathfrak{F}}})(\varrho) = \begin{cases} \inf\limits_{(s,t)\in \daleth_\varrho} \{\flat^{\overline{\mathfrak{F}}}(s) \vee \delta^{\overline{\mathfrak{F}}}(t)\} \ if \daleth_\varrho \neq \emptyset \\ -1 \ otherwise \end{cases}$$

Definition 3.8. *We define* $(\beth)^{\overline{\tau}}_{\overline{\delta}}(\varrho) = \{\beth(\varrho) \vee \overline{\tau}\} \wedge \overline{\delta}$, $(\aleph)^{\overline{\tau}}_{\overline{\delta}}(\varrho) = \{\aleph(\varrho) \wedge \overline{\tau}\} \vee \overline{\delta}$, $(\Delta)^{\underline{\tau}}_{\underline{\delta}}(\varrho) = \{\Delta(\varrho) \wedge \underline{\tau}\} \vee \underline{\delta}$, $(\Psi)^{\underline{\tau}}_{\underline{\delta}}(\varrho) = \{\Psi(\varrho) \vee \underline{\tau}\} \wedge \underline{\delta}$, *for all* $\varrho \in \Bbbk$.

Lemma 3.3. *Let* $\daleth$ *and* $\gimel$ *be subsets of* $\Bbbk$. *Then*

(1) $\mathfrak{R}_{(\daleth]} \vee^{\tau}_{\delta} \mathfrak{R}_{(\gimel]} = (\mathfrak{R}_{(\daleth \cap \gimel]})^{\tau}_{\delta}$,
(2) $\mathfrak{R}_{(\daleth]} \wedge^{\tau}_{\delta} \mathfrak{R}_{(\gimel]} = (\mathfrak{R}_{(\daleth \cup \gimel]})^{\tau}_{\delta}$,
(3) $\mathfrak{R}_{(\daleth]} \circ^{\tau}_{\delta} \mathfrak{R}_{(\gimel]} = (\mathfrak{R}_{(\daleth \Gamma \gimel]})^{\tau}_{\delta}$.

Proof. (1) and (2) are straightforward.

(3) Let $\varrho \in \Bbbk$. If $\varrho \in (\daleth\Gamma\gimel]$, then $(\mathfrak{R}_{(\daleth\Gamma\gimel]})(\varrho) = \overline{\tau}$. Since $\varrho \leq a\gamma b$ for certain $a \in (\daleth]$, $b \in (\gimel]$, $\gamma \in \Gamma$, we have $(a, b) \in \daleth_\varrho$ and so $\daleth_\varrho \neq \emptyset$. Thus,

$$\begin{aligned}(\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]} \circ \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]})(\varrho) &= \inf_{\varrho = y\gamma z} \max\{\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]}(y), \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]}(z)\} \\ &\leq \max\{\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]}(a), \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]}(b)\} \\ &= \overline{\tau},\end{aligned}$$

$$\begin{aligned}(\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]} \circ \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]})(\varrho) &= \sup_{\varrho = y\gamma z} \min\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(y), \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]}(z)\} \\ &\geq \min\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(a), \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]}(b)\} \\ &= \overline{\delta}.\end{aligned}$$

Hence, $(\mathfrak{R}_{(\daleth]} \circ \mathfrak{R}_{(\gimel]})(\varrho) = (\mathfrak{R}_{(\daleth\Gamma\gimel]})(\varrho)$.

If $\varrho \in (\daleth\Gamma\gimel]$, then $(\mathfrak{R}_{(\daleth\Gamma\gimel]})(\varrho) = \underline{\tau}$. Since $\varrho \leq a\gamma b$ for certain $a \in (\daleth]$, $b \in (\gimel]$, $\gamma \in \Gamma$, we have $(a, b) \in \daleth_\varrho$ and so $\daleth_\varrho \neq \emptyset$. Thus,

$$\begin{aligned}(\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]} \circ \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]})(\varrho) &= \sup_{\varrho = y\gamma z} \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(y), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]}(z)\} \\ &\geq \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(a), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]}(b)\} \\ &= \underline{\tau},\end{aligned}$$

$$
\begin{aligned}
(\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]} \circ \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]})(\varrho) &= \inf_{\varrho = y\gamma z} \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(y), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]}(z)\} \\
&\leq \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(a), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]}(b)\} \\
&= \underline{\delta}.
\end{aligned}
$$

Hence, $(\mathfrak{R}_{(\daleth]} \circ \mathfrak{R}_{(\gimel]})(\varrho) = (\mathfrak{R}_{(\daleth\Gamma\gimel]})(\varrho)$.

If $\varrho \notin (\daleth\Gamma\gimel]$, then $(\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth\Gamma\gimel]})(\varrho) = \overline{\delta}$, $(\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth\Gamma\gimel]})(\varrho) = \overline{\tau}$. Since $\varrho \leq a\gamma b$ for certain $a \notin (\daleth]$, $b \notin (\gimel]$, $\gamma \in \Gamma$. Thus,

$$
\begin{aligned}
(\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]} \circ \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]})(\varrho) &= \inf_{\varrho = y\gamma z} \max\{\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]}(y), \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]}(z)\} \\
&\leq \max\{\mathfrak{R}^{\underline{\mathfrak{T}}}_{(\daleth]}(a), \mathfrak{R}^{\underline{\mathfrak{T}}}_{(\gimel]}(b)\} \\
&= \overline{\delta},
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]} \circ \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]})(\varrho) &= \sup_{\varrho = y\gamma z} \min\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(y), \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]}(z)\} \\
&\geq \min\{\mathfrak{R}^{\underline{\mathfrak{F}}}_{(\daleth]}(a), \mathfrak{R}^{\underline{\mathfrak{F}}}_{(\gimel]}(b)\} \\
&= \overline{\tau}.
\end{aligned}
$$

If $\varrho \notin (\daleth\Gamma\gimel]$, then $(\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth\Gamma\gimel]})(\varrho) = \underline{\delta}$, $(\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth\Gamma\gimel]})(\varrho) = \underline{\tau}$. Since $\varrho \leq a\gamma b$ for certain $a \notin (\daleth]$, $b \notin (\gimel]$, $\gamma \in \Gamma$. Thus,

$$
\begin{aligned}
(\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]} \circ \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]})(\varrho) &= \sup_{\varrho = y\gamma z} \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(y), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]}(z)\} \\
&\geq \min\{\mathfrak{R}^{\overline{\mathfrak{T}}}_{(\daleth]}(a), \mathfrak{R}^{\overline{\mathfrak{T}}}_{(\gimel]}(b)\} \\
&= \underline{\delta},
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]} \circ \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]})(\varrho) &= \inf_{\varrho = y\gamma z} \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(y), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]}(z)\} \\
&\leq \max\{\mathfrak{R}^{\overline{\mathfrak{F}}}_{(\daleth]}(a), \mathfrak{R}^{\overline{\mathfrak{F}}}_{(\gimel]}(b)\} \\
&= \underline{\tau}.
\end{aligned}
$$

Hence, $(\mathfrak{R}_{(\daleth]} \circ \mathfrak{R}_{(\gimel]})(\varrho) = (\mathfrak{R}_{(\daleth\Gamma\gimel]})(\varrho)$. □

Theorem 3.3. *Let $\daleth, \gimel \subseteq \Bbbk$ and $\{\daleth_i \mid i \in I\}$ be a collection of subsets of $\Bbbk$. Then*

(1) $(\daleth] \subseteq (\gimel] \Leftrightarrow (\mathfrak{R}_{(\daleth]})^{\tau}_{\delta} \leq (\mathfrak{R}_{(\gimel]})^{\tau}_{\delta}$,
(2) $(\cap_{i\in I}\mathfrak{R}_{(\daleth_i]})^{\tau}_{\delta} = (\mathfrak{R}_{\cap_{i\in I}(\daleth_i]})^{\tau}_{\delta}$,
(3) $(\cup_{i\in I}\mathfrak{R}_{(\daleth_i]})^{\tau}_{\delta} = (\mathfrak{R}_{\cup_{i\in I}(\daleth_i]})^{\tau}_{\delta}$.

Proof. (1) Assume $(\daleth] \subseteq (\gimel]$. Then for any $x \in \Bbbk$, we have: If $x \in (\daleth]$, then $x \in (\gimel]$, so $\mathfrak{R}_{(\daleth]}(x) = \tau \leq \mathfrak{R}_{(\gimel]}(x) = \tau$. If $x \notin (\daleth]$, then $\mathfrak{R}_{(\daleth]}(x) = \delta \leq \mathfrak{R}_{(\gimel]}(x)$. Hence, $(\mathfrak{R}_{(\daleth]})^{\tau}_{\delta}(x) \leq (\mathfrak{R}_{(\gimel]})^{\tau}_{\delta}(x)$ for all x.

Conversely, assume $(\mathfrak{R}_{(\daleth]})^{\tau}_{\delta} \leq (\mathfrak{R}_{(\gimel]})^{\tau}_{\delta}$. Let $x \in (\daleth]$, then $\mathfrak{R}_{(\daleth]}(x) = \tau$. Thus, we must have $\mathfrak{R}_{(\gimel]}(x) = \tau$, which implies $x \in (\gimel]$. Therefore, $(\daleth] \subseteq (\gimel])$.

(2) Let $x \in \Bbbk$. If $x \in (\bigcap_{i\in I} \daleth_i]$, then $x \in (\daleth_i]$ for all $i \in I$, hence $\mathfrak{R}_{(\daleth_i]}(x) = \tau$ for all i. Thus,

$$\left(\bigcap_{i\in I} \mathfrak{R}_{(\daleth_i]}\right)(x) = \min_{i\in I} \tau = \tau,$$

so the adjusted function gives τ. If $x \notin (\bigcap_{i\in I} \daleth_i]$, then there exists $j \in I$ such that $x \notin (\daleth_j]$, hence $\mathfrak{R}_{(\daleth_j]}(x) = \delta$. Therefore,

$$\left(\bigcap_{i\in I} \mathfrak{R}_{(\daleth_i]}\right)(x) = \min_{i\in I} \mathfrak{R}_{(\daleth_i]}(x) = \delta.$$

Hence, in both cases, we have $(\cap_{i\in I}\mathfrak{R}_{(\daleth_i]})^{\tau}_{\delta}(x) = (\mathfrak{R}_{\cap_{i\in I}(\daleth_i]})^{\tau}_{\delta}(x)$.

(3) Let $x \in \Bbbk$. If $x \in (\bigcup_{i\in I} \daleth_i]$, then there exists $j \in I$ such that $x \in (\daleth_j]$, hence $\mathfrak{R}_{(\daleth_j]}(x) = \tau$. Thus,

$$\left(\bigcup_{i\in I} \mathfrak{R}_{(\daleth_i]}\right)(x) = \max_{i\in I} \mathfrak{R}_{(\daleth_i]}(x) = \tau.$$

If $x \notin (\bigcup_{i\in I} \daleth_i]$, then $x \notin (\daleth_i]$ for all $i \in I$, so all $\mathfrak{R}_{(\daleth_i]}(x) = \delta$, and

$$\left(\bigcup_{i\in I} \mathfrak{R}_{(\daleth_i]}\right)(x) = \max_{i\in I} \delta = \delta.$$

Therefore, $(\cup_{i\in I}\mathfrak{R}_{(\daleth_i]})^{\tau}_{\delta} = (\mathfrak{R}_{\cup_{i\in I}(\daleth_i]})^{\tau}_{\delta}$. □

Definition 3.9. *A BPIFS* $\flat = [(\sqsupset, \Delta), (\aleph, \Psi)]$ *of* $\Bbbk$ *is represent a BPAIFSLI of* $\Bbbk$ *if*

(1) $\varrho \leq \natural \Rightarrow \sqsupset(\varrho) \leq \sqsupset(\natural), \aleph(\varrho) \geq \aleph(\natural), \Delta(\varrho) \geq \Delta(\natural), \Psi(\varrho) \leq \Psi(\natural),$

(2) $\sqsupset(\varrho\gamma_1\natural) \leq \sqsupset(\natural), \aleph(\varrho\gamma_1\natural) \geq \aleph(\natural),$

(3) $\Delta(\varrho\gamma_1\natural) \geq \Delta(\natural), \Psi(\varrho\gamma_1\natural) \leq \Psi(\natural),$ *for* $\varrho, \natural \in \Bbbk, \gamma_1 \in \Gamma.$

The definitions of BPAIFSS and BPAIFSRI can be given analogously by modifying conditions (2) and (3).

Theorem 3.4. *If* $\daleth$ *is a* (δ, τ)*-BPAIFSLI (BPAIFSS, BPAIFSRI) of* $\Bbbk$*, then* $(\daleth)^{\tau}_{\delta}$ *is a BPAIFSLI (BPAIFSS, BPAIFSRI) of* $\Bbbk$.

Proof. Suppose that $\daleth$ is a (δ, τ)-BPAIFSLI of $\Bbbk$. If there exist $\varrho, \natural \in \Bbbk$ and $\gamma \in \Gamma$, then

$$\begin{aligned}
\min\{(\sqsupset)^{\overline{\tau}}_{\overline{\delta}}(\varrho\gamma\natural), \overline{\delta}\} &= \min\{(\{\sqsupset(\varrho\gamma\natural) \vee \overline{\tau}\} \wedge \overline{\delta}), \overline{\delta}\} \\
&= \{\sqsupset(\varrho\gamma\natural) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= \{\sqsupset(\varrho\gamma\natural) \wedge \overline{\delta}\} \vee \{\overline{\tau} \wedge \overline{\delta}\} \\
&= \{(\sqsupset(\varrho\gamma\natural) \wedge \overline{\delta}) \wedge \overline{\delta}\} \vee \overline{\tau} \\
&\leq \{(\sqsupset(\natural) \vee \overline{\tau}) \wedge \overline{\delta}\} \vee \overline{\tau} \\
&\leq (\sqsupset)^{\overline{\tau}}_{\overline{\delta}}(\natural) \vee \overline{\tau},
\end{aligned}$$

$$\begin{aligned}
\max\{(\aleph)_{\overline{\delta}}^{\overline{\tau}}(\varrho\gamma\natural), \overline{\delta}\} &= \max\{(\{\aleph(\varrho\gamma\natural) \wedge \overline{\tau}\} \vee \overline{\delta}), \overline{\delta}\} \\
&= \{\aleph(\varrho\gamma\natural) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= \{(\aleph(\varrho\gamma\natural) \vee \overline{\delta}) \vee \overline{\delta}\} \wedge \overline{\delta} \\
&\geq \{(\aleph(\natural) \wedge \overline{\tau}) \vee \overline{\delta}\} \wedge \overline{\delta} \\
&= \{(\aleph(\natural) \wedge \overline{\tau}) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&\geq (\aleph)_{\overline{\delta}}^{\overline{\tau}}(\natural) \wedge \overline{\tau}.
\end{aligned}$$

and

$$\begin{aligned}
\max\{(\Delta)_{\underline{\delta}}^{\underline{\tau}}(\varrho\gamma\natural), \underline{\delta}\} &= \max\{(\{\Delta(\varrho\gamma\natural) \wedge \underline{\tau}\} \vee \underline{\delta}), \underline{\delta}\} \\
&= \{\Delta(\varrho\gamma\natural) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= \{\Delta(\varrho\gamma\natural) \vee \underline{\delta}\} \wedge \{\underline{\tau} \vee \underline{\delta}\} \\
&= \{(\Delta(\varrho\gamma\natural) \vee \underline{\delta}) \vee \underline{\delta}\} \wedge \underline{\tau} \\
&\geq \{(\Delta(\natural) \wedge \underline{\tau}) \vee \underline{\delta}\} \wedge \underline{\tau} \\
&= \{(\Delta(\natural) \wedge \underline{\tau}) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&\geq (\Delta)_{\underline{\delta}}^{\underline{\tau}}(\natural) \wedge \underline{\tau},
\end{aligned}$$

$$\begin{aligned}
\min\{(\Psi)_{\underline{\delta}}^{\underline{\tau}}(\varrho\gamma\natural), \underline{\delta}\} &= \min\{(\{\Psi(\varrho\gamma\natural) \vee \underline{\tau}\} \wedge \underline{\delta}), \underline{\delta}\} \\
&= \{\Psi(\varrho\gamma\natural) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&= \{\Psi(\varrho\gamma\natural) \wedge \underline{\delta}\} \vee \{\underline{\tau} \wedge \underline{\delta}\} \\
&\leq \{(\Psi(\natural) \vee \underline{\tau}) \wedge \underline{\delta}\} \vee \underline{\delta} \\
&= \{(\Psi(\natural) \vee \underline{\tau}) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&\leq (\Psi)_{\underline{\delta}}^{\underline{\tau}}(\natural) \vee \underline{\tau}.
\end{aligned}$$

Hence, $(\daleth)_{\delta}^{\tau}$ is a BPAIFSLI of $\Bbbk$. ∎

Theorem 3.5. *If* $\daleth$ *is a* (δ,τ)*-BPAIFSRI and* $\beth$ *is a* (δ,τ)*-BPAIFSLI of* $\Bbbk$*, then* $((\daleth \circ_{\Gamma} \beth])_{\delta}^{\tau} \subseteq (\daleth \cap_{\delta}^{\tau} \beth]$.

Proof. Let $\daleth = [(\beth_{\daleth}, \Delta_{\daleth}), (\aleph_{\daleth}, \Psi_{\daleth})]$ be a (δ,τ)-BPAIFSRI and $\beth = [(\beth_{\beth}, \Delta_{\beth}), (\aleph_{\beth}, \Psi_{\beth})]$ be a (δ,τ)-BPAIFSLI of $\Bbbk$. Let $(\varrho, \natural) \in I_{\varepsilon}$. If $I_{\varepsilon} \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\beth_{\daleth}(\varepsilon) \leq \beth_{\daleth}(\varrho\gamma\natural) \leq \beth_{\daleth}(\varrho)$ and $\aleph_{\daleth}(\varepsilon) \geq \aleph_{\daleth}(\varrho\gamma\natural) \geq \aleph_{\daleth}(\varrho)$. Similarly, $\beth_{\beth}(\varepsilon) \leq \beth_{\beth}(\varrho\gamma\natural) \leq \beth_{\beth}(\varrho)$ and $\aleph_{\beth}(\varepsilon) \geq \aleph_{\beth}(\varrho\gamma\natural) \geq \aleph_{\beth}(\varrho)$. Let $(\varrho, \natural) \in I_{\varepsilon}$. If $I_{\varepsilon} \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\Delta_{\daleth}(\varepsilon) \geq \Delta_{\daleth}(\varrho\gamma\natural) \geq \Delta_{\daleth}(\varrho)$ and $\Psi_{\daleth}(\varepsilon) \leq \Psi_{\daleth}(\varrho\gamma\natural) \leq \Psi_{\daleth}(\varrho)$. Similarly, $\Delta_{\beth}(\varepsilon) \geq \Delta_{\beth}(\varrho\gamma\natural) \geq \Delta_{\beth}(\varrho)$ and $\Psi_{\beth}(\varepsilon) \leq \Psi_{\beth}(\varrho\gamma\natural) \leq \Psi_{\beth}(\varrho)$. Thus,

$$\begin{aligned}
(\beth_{(\daleth\circ\beth]})_{\overline{\delta}}^{\overline{\tau}}(\varepsilon) &= (\beth_{(\daleth\circ\beth]}(\varepsilon) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= \Big[[\inf_{\varepsilon\leq\varrho\gamma\natural}\{\beth_{\daleth}(\varrho) \vee \beth_{\beth}(\natural)\} \vee \overline{\tau}]\Big] \wedge \overline{\delta} \\
&= \Big[\inf_{\varepsilon\leq\varrho\gamma\natural}\{\beth_{\daleth}(\varrho) \vee \beth_{\beth}(\natural)\} \vee \overline{\tau} \vee \overline{\tau}\Big] \wedge \overline{\delta}
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\inf_{\varepsilon \leq \varrho\gamma\natural} \{(\beth_{\neg}(\varrho) \vee \overline{\tau}) \vee (\beth_{\jmath}(\natural) \vee \overline{\tau})\} \vee \overline{\tau} \Big] \wedge \overline{\delta} \\
&\geq (\{(\beth_{\neg}(\varepsilon) \wedge \overline{\delta}) \vee (\beth_{\jmath}(\varepsilon) \wedge \overline{\delta})\} \vee \overline{\tau}) \wedge \overline{\delta} \\
&= \{((\beth_{\neg}(\varepsilon) \vee \beth_{\jmath}(\varepsilon)) \wedge \overline{\delta}) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= \{((\beth_{\neg} \vee \beth_{\jmath})(\varepsilon) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= (\beth_{\neg \cap \frac{\overline{\tau}}{\underline{\delta}} \jmath})(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\aleph_{(\neg \circ \jmath]})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) &= (\aleph_{(\neg \circ \jmath]}(\varepsilon) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \Big[[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\aleph_{\neg}(\varrho) \wedge \aleph_{\jmath}(\natural)\} \wedge \overline{\tau}] \Big] \vee \overline{\delta} \\
&= \Big[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\aleph_{\neg}(\varrho) \wedge \aleph_{\jmath}(\natural)\} \wedge \overline{\tau} \wedge \overline{\tau} \Big] \vee \overline{\delta} \\
&= \Big[\sup_{\varepsilon \leq \varrho\gamma\natural} \{(\aleph_{\neg}(\varrho) \wedge \overline{\tau}) \wedge (\aleph_{\jmath}(\natural) \wedge \overline{\tau})\} \wedge \overline{\tau} \Big] \vee \overline{\delta} \\
&\leq (\{(\aleph_{\neg}(\varepsilon) \vee \overline{\delta}) \wedge (\aleph_{\jmath}(\varepsilon) \vee \overline{\delta})\} \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \{((\aleph_{\neg}(\varepsilon) \wedge \aleph_{\jmath}(\varepsilon)) \vee \overline{\delta}) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= \{((\aleph_{\neg} \wedge \aleph_{\jmath})(\varepsilon) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= (\aleph_{\neg \cup \frac{\overline{\tau}}{\delta} \jmath})(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\Delta_{(\neg \circ \jmath]})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Delta_{(\neg \circ \jmath]}(\varepsilon) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \Big[[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\Delta_{\neg}(\varrho) \wedge \Delta_{\jmath}(\natural)\} \wedge \underline{\tau}] \Big] \vee \underline{\delta} \\
&= \Big[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\Delta_{\neg}(\varrho) \wedge \Delta_{\jmath}(\natural)\} \wedge \underline{\tau} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&= \Big[\sup_{\varepsilon \leq \varrho\gamma\natural} \{(\Delta_{\neg}(\varrho) \wedge \underline{\tau}) \wedge (\Delta_{\jmath}(\natural) \wedge \underline{\tau})\} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&\leq (\{(\Delta_{\neg}(\varepsilon) \vee \underline{\delta}) \wedge (\Delta_{\jmath}(\varepsilon) \vee \underline{\delta})\} \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \{((\Delta_{\neg}(\varepsilon) \wedge \Delta_{\jmath}(\varepsilon)) \vee \underline{\delta}) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= \{((\Delta_{\neg} \wedge \Delta_{\jmath})(\varepsilon) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= (\Delta_{\neg \cap \frac{\tau}{\underline{\delta}} \jmath})(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\Psi_{(\neg \circ \jmath]})^{\tau}_{\underline{\delta}}(\varepsilon) &= (\Psi_{(\neg \circ \jmath]}(\varepsilon) \vee \underline{\tau}) \wedge \underline{\delta} \\
&= \Big[[\inf_{\varepsilon \leq \varrho\gamma\natural} \{\Psi_{\neg}(\varrho) \vee \Psi_{\jmath}(\natural)\} \vee \underline{\tau}] \Big] \wedge \underline{\delta} \\
&= \Big[\inf_{\varepsilon \leq \varrho\gamma\natural} \{\Psi_{\neg}(\varrho) \vee \Psi_{\jmath}(\natural)\} \vee \underline{\tau} \vee \underline{\tau} \Big] \wedge \underline{\delta} \\
&= \Big[\inf_{\varepsilon \leq \varrho\gamma\natural} \{(\Psi_{\neg}(\varrho) \vee \underline{\tau}) \vee (\Psi_{\jmath}(\natural) \vee \underline{\tau})\} \vee \underline{\tau} \Big] \wedge \underline{\delta}
\end{aligned}
$$

$$
\begin{aligned}
&\geq (\{(\Psi_{\daleth}(\varepsilon) \wedge \underline{\delta}) \vee (\Psi_{\gimel}(\varepsilon) \wedge \underline{\delta})\} \vee \underline{\tau}) \wedge \underline{\delta} \\
&= \{((\Psi_{\daleth}(\varepsilon) \vee \Psi_{\gimel}(\varepsilon)) \wedge \underline{\delta}) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&= \{((\Psi_{\daleth} \vee \Psi_{\gimel})(\varepsilon) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&= (\Psi_{\daleth \cup^{\tau}_{\underline{\delta}} \gimel})(\varepsilon).
\end{aligned}
$$

Let $\varrho, \natural \notin I_{\varepsilon}$. If $I_{\varepsilon} = \emptyset$, then $(\beth_{\daleth} \circ \beth_{\gimel})(\varepsilon) = 0$ and $(\aleph_{\daleth} \circ \aleph_{\gimel})(\varepsilon) = -1$ and $\gamma \in \Gamma$ implies $\varepsilon \leq \varrho\gamma\natural$. Thus,

$$
\begin{aligned}
(\beth_{(\daleth \circ \gimel]})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) &= (\beth_{(\daleth \circ \gimel]}(\varepsilon) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= 0 \wedge \overline{\delta} \\
&\geq (\beth_{\daleth \cap \gimel}(\varepsilon) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= (\beth_{\daleth \cap \gimel}(\varepsilon) \vee \overline{\tau}),
\end{aligned}
$$

$$
\begin{aligned}
(\aleph_{(\daleth \circ \gimel]})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) &= (\aleph_{(\daleth \circ \gimel]}(\varepsilon) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= -1 \vee \overline{\delta} \\
&= \overline{\delta} \\
&\leq (\aleph_{\daleth \cup \gimel}(\varepsilon) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= (\aleph_{\daleth \cup \gimel}(\varepsilon) \wedge \overline{\tau}).
\end{aligned}
$$

Let $\varrho, \natural \notin I_{\varepsilon}$. If $I_{\varepsilon} = \emptyset$, then $(\Delta_{\daleth} \circ \Delta_{\gimel})(\varepsilon) = 0$ and $(\Psi_{\daleth} \circ \Psi_{\gimel})(\varepsilon) = 1$ and $\gamma \in \Gamma$ implies $\varepsilon \leq \varrho\gamma\natural$. Thus,

$$
\begin{aligned}
(\Delta_{(\daleth \circ \gimel]})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Delta_{(\daleth \circ \gimel]}(\varepsilon) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= 0 \vee \underline{\delta} \\
&\leq (\Delta_{\daleth \cap \gimel}(\varepsilon) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= (\Delta_{\daleth \cap \gimel}(\varepsilon) \wedge \underline{\tau}),
\end{aligned}
$$

$$
\begin{aligned}
(\Psi_{(\daleth \circ \gimel]})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Psi_{(\daleth \circ \gimel]}(\varepsilon) \vee \underline{\tau}) \wedge \underline{\delta} \\
&= 1 \wedge \underline{\delta} \\
&= \underline{\delta} \\
&\geq (\Psi_{\daleth \cup \gimel}(\varepsilon) \vee \underline{\tau}) \wedge \underline{\delta} \\
&= (\Psi_{\daleth \cup \gimel}(\varepsilon) \vee \underline{\tau}).
\end{aligned}
$$

Hence, $((\daleth \circ_{\Gamma} \gimel])^{\tau}_{\delta} \subseteq (\daleth \cap^{\tau}_{\delta} \gimel]$. □

4. Characterization of regular ordered Gamma semigroups via BPAIFIs

This section focuses on establishing necessary and sufficient conditions under which an ordered Γ-semigroup becomes regular in the context of (δ, τ)-bipolar anti-intuitionistic fuzzy ideals. By examining the behavior of fuzzy left and right ideals under Γ-product operations and level set approximations, we provide characterizations that link regularity with ideal-theoretic properties.

Theorem 4.1. *Let* $\daleth$ *be a* (δ,τ)*-BPAIFSRI and* $\gimel$ *be a* (δ,τ)*-BPAIFSLI of* $\Bbbk$*. Then* $\Bbbk$ *is regular if and only if* $((\daleth \circ_\Gamma \gimel])^{\overline{\tau}}_{\overline{\delta}} = (\daleth \cap^{\overline{\tau}}_{\overline{\delta}} \gimel]$.

Proof. Let $\daleth$ be a (δ,τ)-BPAIFSRI and $\gimel$ be an (δ,τ)-BPAIFSLI of $\Bbbk$. Let $(\varrho,\natural) \in I_\varepsilon$. If $I_\varepsilon \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\beth_\daleth(\varepsilon) \leq \beth_\daleth(\varrho\gamma\natural) \leq \beth_\daleth(\varrho)$ and $\aleph_\daleth(\varepsilon) \geq \aleph_\daleth(\varrho\gamma\natural) \geq \aleph_\daleth(\varrho)$. Similarly, $\beth_\gimel(\varepsilon) \leq \beth_\gimel(\varrho\gamma\natural) \leq \beth_\gimel(\varrho)$ and $\aleph_\gimel(\varepsilon) \geq \aleph_\gimel(\varrho\gamma\natural) \geq \aleph_\gimel(\varrho)$. Let $(\varrho,\natural) \in I_\varepsilon$. If $I_\varepsilon \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\Delta_\daleth(\varepsilon) \geq \Delta_\daleth(\varrho\gamma\natural) \geq \Delta_\daleth(\varrho)$ and $\Psi_\daleth(\varepsilon) \leq \Psi_\daleth(\varrho\gamma\natural) \leq \Psi_\daleth(\varrho)$. Similarly, $\Delta_\gimel(\varepsilon) \geq \Delta_\gimel(\varrho\gamma\natural) \geq \Delta_\gimel(\varrho)$ and $\Psi_\gimel(\varepsilon) \leq \Psi_\gimel(\varrho\gamma\natural) \leq \Psi_\gimel(\varrho)$. For $\varepsilon \in \Bbbk$, there exists $x \in \Bbbk$ such that $\varepsilon \leq (\varepsilon\gamma x)\iota\varepsilon$. Then $(\varepsilon\gamma x), \varepsilon \in I_\varepsilon$. Thus,

$$\begin{aligned}
(\beth_{(\daleth\circ\gimel]})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) &= (\beth_{(\daleth\circ\gimel]}(\varepsilon) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= \left[\left[\inf_{\varepsilon \geq \varrho\gamma\natural} \{\beth_\daleth(\varrho) \vee \beth_\gimel(\natural)\} \vee \overline{\tau}\right]\right] \wedge \overline{\delta} \\
&= \left[\left[\inf_{\varepsilon \geq \varrho\gamma\natural} \{\beth_\daleth(\varrho) \vee \beth_\gimel(\natural)\} \vee \overline{\tau} \vee \overline{\tau}\right] \wedge \overline{\delta} \\
&= \left[\left[\inf_{\varepsilon \geq \varrho\gamma\natural} \{(\beth_\daleth(\varrho) \vee \overline{\tau}) \vee (\beth_\gimel(\natural) \vee \overline{\tau})\} \vee \overline{\tau}\right] \wedge \overline{\delta} \\
&\leq (\{(\beth_\daleth(\varepsilon\gamma x) \wedge \overline{\delta}) \vee (\beth_\gimel(\varepsilon) \wedge \overline{\delta})\} \vee \overline{\tau}) \wedge \overline{\delta} \\
&\leq ((\beth_\daleth(\varepsilon) \wedge \overline{\delta}) \vee (\beth_\gimel(\varepsilon) \wedge \overline{\delta}) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= \{((\beth_\daleth(\varepsilon) \vee \beth_\gimel(\varepsilon)) \wedge \overline{\delta}) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= \{((\beth_\daleth \vee \beth_\gimel)(\varepsilon) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= (\beth_{\daleth\cap^{\overline{\tau}}_{\overline{\delta}}\gimel})(\varepsilon),
\end{aligned}$$

$$\begin{aligned}
(\aleph_{(\daleth\circ\gimel]})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) &= (\aleph_{(\daleth\circ\gimel]}(\varepsilon) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \left[\left[\sup_{\varepsilon \geq \varrho\gamma\natural} \{\aleph_\daleth(\varrho) \wedge \aleph_\gimel(\natural)\} \wedge \overline{\tau}\right]\right] \vee \overline{\delta} \\
&= \left[\left[\sup_{\varepsilon \geq \varrho\gamma\natural} \{\aleph_\daleth(\varrho) \wedge \aleph_\gimel(\natural)\} \wedge \overline{\tau} \wedge \overline{\tau}\right] \vee \overline{\delta} \\
&= \left[\left[\sup_{\varepsilon \geq \varrho\gamma\natural} \{(\aleph_\daleth(\varrho) \wedge \overline{\tau}) \wedge (\aleph_\gimel(\natural) \wedge \overline{\tau})\} \wedge \overline{\tau}\right] \vee \overline{\delta} \\
&\geq (\{(\aleph_\daleth(\varepsilon\gamma x) \vee \overline{\delta}) \wedge (\aleph_\gimel(\varepsilon) \vee \overline{\delta})\} \wedge \overline{\tau}) \vee \overline{\delta} \\
&\geq ((\aleph_\daleth(\varepsilon) \vee \overline{\delta}) \wedge (\aleph_\gimel(\varepsilon) \vee \overline{\delta}) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \{((\aleph_\daleth(\varepsilon) \wedge \aleph_\gimel(\varepsilon)) \vee \overline{\delta}) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= \{((\aleph_\daleth \wedge \aleph_\gimel)(\varepsilon) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= (\aleph_{\daleth\cap^{\overline{\tau}}_{\overline{\delta}}\gimel})(\varepsilon),
\end{aligned}$$

$$\begin{aligned}
(\Delta_{(\daleth\circ\gimel]})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Delta_{(\daleth\circ\gimel]}(\varepsilon) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \left[\left[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\Delta_\daleth(\varrho) \wedge \Delta_\gimel(\natural)\} \wedge \underline{\tau}\right]\right] \vee \underline{\delta}
\end{aligned}$$

$$
\begin{aligned}
&= \Big[[\sup_{\varepsilon \leq \varrho\gamma\natural} \{\Delta_{\daleth}(\varrho) \wedge \Delta_{\gimel}(\natural)\} \wedge \underline{\tau} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&= \Big[[\sup_{\varepsilon \leq \varrho\gamma\natural} \{(\Delta_{\daleth}(\varrho) \wedge \underline{\tau}) \wedge (\Delta_{\gimel}(\natural) \wedge \underline{\tau})\} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&\geq (\{(\Delta_{\daleth}(\varepsilon\gamma x) \vee \underline{\delta}) \wedge (\Delta_{\gimel}(\varepsilon) \vee \underline{\delta})\} \wedge \underline{\tau}) \vee \underline{\delta} \\
&\geq ((\Delta_{\daleth}(\varepsilon) \vee \underline{\delta}) \wedge (\Delta_{\gimel}(\varepsilon) \vee \underline{\delta}) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \{((\Delta_{\daleth}(\varepsilon) \wedge \Delta_{\gimel}(\varepsilon)) \vee \underline{\delta}) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= \{((\Delta_{\daleth} \wedge \Delta_{\gimel})(\varepsilon) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= (\Delta_{\daleth \cap^{\underline{\tau}}_{\underline{\delta}} \gimel})(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\Psi_{(\daleth \circ \gimel]})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Psi_{(\daleth \circ \gimel]}(\varepsilon) \vee \underline{\tau}) \wedge \underline{\delta} \\
&= \Big[[\inf_{\varepsilon \leq \varrho\gamma\natural} \{\Psi_{\daleth}(\varrho) \vee \Psi_{\gimel}(\natural)\} \vee \underline{\tau}] \Big] \wedge \underline{\delta} \\
&= \Big[[\inf_{\varepsilon \leq \varrho\gamma\natural} \{\Psi_{\daleth}(\varrho) \vee \Psi_{\gimel}(\natural)\} \vee \underline{\tau} \vee \underline{\tau} \Big] \wedge \underline{\delta} \\
&= \Big[[\inf_{\varepsilon \leq \varrho\gamma\natural} \{(\Psi_{\daleth}(\varrho) \vee \underline{\tau}) \vee (\Psi_{\gimel}(\natural) \vee \underline{\tau})\} \vee \underline{\tau} \Big] \wedge \underline{\delta} \\
&\leq (\{(\Psi_{\daleth}(\varepsilon\gamma x) \wedge \underline{\delta}) \vee (\Psi_{\gimel}(\varepsilon) \wedge \underline{\delta})\} \vee \underline{\tau}) \wedge \underline{\delta} \\
&\leq ((\Psi_{\daleth}(\varepsilon) \wedge \underline{\delta}) \vee (\Psi_{\gimel}(\varepsilon) \wedge \underline{\delta}) \vee \underline{\tau}) \wedge \underline{\delta} \\
&= \{((\Psi_{\daleth}(\varepsilon) \vee \Psi_{\gimel}(\varepsilon)) \wedge \underline{\delta}) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&= \{((\Psi_{\daleth} \vee \Psi_{\gimel})(\varepsilon) \vee \underline{\tau}\} \wedge \underline{\delta} \\
&= (\Psi_{\daleth \cap^{\underline{\tau}}_{\underline{\delta}} \gimel})(\varepsilon).
\end{aligned}
$$

Thus, $((\daleth \circ_{\Gamma} \gimel])^{\tau}_{\delta} \supseteq (\daleth \cap^{\tau}_{\delta} \gimel]$, by Theorem 3.5 and hence, $((\daleth \circ_{\Gamma} \gimel])^{\tau}_{\delta} = (\daleth \cap^{\tau}_{\delta} \gimel]$.

Conversely, assume that $((\daleth \circ_{\Gamma} \gimel])^{\tau}_{\delta} = \daleth \cap^{\tau}_{\delta} \gimel$. Let $\daleth = (\beth_{\daleth}, \aleph_{\daleth})$ be a (δ, τ)-BPAIFSRI and $\gimel = (\beth_{\gimel}, \aleph_{\gimel})$ be a (δ, τ)-BPAIFSLI of $\Bbbk$, by Theorem 3.2, $\mathfrak{R}_{\daleth}$ is a (δ, τ)-BPAIFSRI and $\mathfrak{R}_{\gimel}$ is a (δ, τ)-BPAIFSLI of $\Bbbk$. By Lemma 3.3 and Theorem 3.3, $(\mathfrak{R}_{(\daleth \cap \gimel]})^{\tau}_{\delta} = (\mathfrak{R}_{\daleth} \cap^{\tau}_{\delta} \mathfrak{R}_{\gimel}) = (\mathfrak{R}_{\daleth} \circ \mathfrak{R}_{\gimel})^{\tau}_{\delta} = (\mathfrak{R}_{(\daleth \circ \gimel]})^{\tau}_{\delta}$. This implies that $(\daleth \cap^{\tau}_{\delta} \gimel] = ((\daleth \circ \gimel])^{\tau}_{\delta}$, by $\Bbbk$ is regular. □

Theorem 4.2. *Let $\daleth$ be a (δ, τ)-BPAIFSBI and $\gimel$ be a (δ, τ)-BPAIFSLI of $\Bbbk$. Then $\Bbbk$ is regular if and only if $((\daleth \circ_{\Gamma} \gimel])^{\tau}_{\delta} = (\daleth \cap^{\tau}_{\delta} \gimel]$.*

Proof. Let $\daleth$ be a (δ, τ)-BPAIFSBI and $\gimel$ be a (δ, τ)-BPAIFSLI of $\Bbbk$. Let $(\varrho, \natural) \in I_{\varepsilon}$. If $I_{\varepsilon} \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\beth_{\daleth}(\varepsilon) \leq \beth_{\daleth}(\varrho\gamma\natural) \leq \beth_{\daleth}(\varrho)$ and $\aleph_{\daleth}(\varepsilon) \geq \aleph_{\daleth}(\varrho\gamma\natural) \geq \aleph_{\daleth}(\varrho)$. Similarly, $\beth_{\gimel}(\varepsilon) \leq \beth_{\gimel}(\varrho\gamma\natural) \leq \beth_{\gimel}(\varrho)$ and $\aleph_{\gimel}(\varepsilon) \geq \aleph_{\gimel}(\varrho\gamma\natural) \geq \aleph_{\gimel}(\varrho)$. Let $(\varrho, \natural) \in I_{\varepsilon}$. If $I_{\varepsilon} \neq \emptyset$, then $\varepsilon \leq \varrho\gamma\natural$. Thus, $\Delta_{\daleth}(\varepsilon) \geq \Delta_{\daleth}(\varrho\gamma\natural) \geq \Delta_{\daleth}(\varrho)$ and $\Psi_{\daleth}(\varepsilon) \leq \Psi_{\daleth}(\varrho\gamma\natural) \leq \Psi_{\daleth}(\varrho)$. Similarly, $\Delta_{\gimel}(\varepsilon) \geq \Delta_{\gimel}(\varrho\gamma\natural) \geq \Delta_{\gimel}(\varrho)$ and $\Psi_{\gimel}(\varepsilon) \leq \Psi_{\gimel}(\varrho\gamma\natural) \leq \Psi_{\gimel}(\varrho)$. For $\varepsilon \in \Bbbk$, there exists $x \in \Bbbk$ such that $\varepsilon \leq \varepsilon\gamma_1 x\gamma_2\varepsilon = \varepsilon\gamma_1(x\gamma_2\varepsilon) \leq (\varepsilon\gamma_1 x\gamma_2\varepsilon)\gamma_1(x\gamma_2\varepsilon)$. Then $(\varepsilon\gamma_1 x\gamma_2\varepsilon), (x\gamma_2\varepsilon) \in I_{\varepsilon}$. Thus,

$$(\beth_{\daleth \circ \gimel})^{\overline{\tau}}_{\overline{\delta}}(\varepsilon) = (\beth_{\daleth \circ \gimel}(\varepsilon) \vee \overline{\tau}) \wedge \overline{\delta}$$

$$
\begin{aligned}
&= \Big[[\inf_{\varepsilon \le a_1 \gamma a_2} \{ \beth_{\neg}(a_1) \vee \beth_{\daleth}(a_2)\} \vee \overline{\tau}] \Big] \wedge \overline{\delta} \\
&= \Big[\inf_{\varepsilon \le a_1 \gamma a_2} \{ \beth_{\neg}(a_1) \vee \beth_{\daleth}(a_2)\} \vee \overline{\tau} \vee \overline{\tau} \Big] \wedge \overline{\delta} \\
&= \Big[\inf_{\varepsilon \le a_1 \gamma a_2} \{ (\beth_{\neg}(a_1) \vee \overline{\tau}) \vee (\beth_{\daleth}(a_2) \vee \overline{\tau})\} \vee \overline{\tau} \Big] \wedge \overline{\delta} \\
&\le (\{(\beth_{\neg}(\varepsilon\gamma_1 x \gamma_2 \varepsilon) \wedge \overline{\delta}) \vee (\beth_{\daleth}(x\gamma_2\varepsilon) \wedge \overline{\delta})\} \vee \overline{\tau}) \wedge \overline{\delta} \\
&\le ((\beth_{\neg}(\varepsilon) \wedge \overline{\delta}) \vee (\beth_{\daleth}(\varepsilon) \wedge \overline{\delta}) \vee \overline{\tau}) \wedge \overline{\delta} \\
&= \{((\beth_{\neg}(\varepsilon) \vee \beth_{\daleth}(\varepsilon)) \wedge \overline{\delta}) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= \{((\beth_{\neg} \vee \beth_{\daleth})(\varepsilon) \vee \overline{\tau}\} \wedge \overline{\delta} \\
&= (\beth_{\neg \cap_{\overline{\delta}}^{\overline{\tau}} \daleth})(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\aleph_{\neg \circ \daleth})_{\overline{\delta}}^{\overline{\tau}}(\varepsilon) &= (\aleph_{\neg \circ \daleth}(\varepsilon) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \Big[[\sup_{\varepsilon \le a_1 \gamma a_2} \{ \aleph_{\neg}(a_1) \wedge \aleph_{\daleth}(a_2)\} \wedge \overline{\tau}] \Big] \vee \overline{\delta} \\
&= \Big[\sup_{\varepsilon \le a_1 \gamma a_2} \{ \aleph_{\neg}(a_1) \wedge \aleph_{\daleth}(a_2)\} \wedge \overline{\tau} \wedge \overline{\tau} \Big] \vee \overline{\delta} \\
&= \Big[\sup_{\varepsilon \le a_1 \gamma a_2} \{ (\aleph_{\neg}(a_1) \wedge \overline{\tau}) \wedge (\aleph_{\daleth}(a_2) \wedge \overline{\tau})\} \wedge \overline{\tau} \Big] \vee \overline{\delta} \\
&\ge (\{(\aleph_{\neg}(\varepsilon\gamma_1 x \gamma_2 \varepsilon) \vee \overline{\delta}) \wedge (\aleph_{\daleth}(x\gamma_2\varepsilon) \vee \overline{\delta})\} \wedge \overline{\tau}) \vee \overline{\delta} \\
&\ge ((\aleph_{\neg}(\varepsilon) \vee \overline{\delta}) \wedge (\aleph_{\daleth}(\varepsilon) \vee \overline{\delta}) \wedge \overline{\tau}) \vee \overline{\delta} \\
&= \{((\aleph_{\neg}(\varepsilon) \wedge \aleph_{\daleth}(\varepsilon)) \vee \overline{\delta}) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= \{((\aleph_{\neg} \wedge \aleph_{\daleth})(\varepsilon) \wedge \overline{\tau}\} \vee \overline{\delta} \\
&= (\aleph_{\neg \cap_{\overline{\delta}}^{\overline{\tau}} \daleth})(\varepsilon_3),
\end{aligned}
$$

$$
\begin{aligned}
(\Delta_{\neg \circ \daleth})_{\underline{\delta}}^{\underline{\tau}}(\varepsilon) &= (\Delta_{\neg \circ \daleth}(\varepsilon) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \Big[[\sup_{\varepsilon \le a_1 \gamma a_2} \{ \Delta_{\neg}(a_1) \wedge \Delta_{\daleth}(a_2)\} \wedge \underline{\tau}] \Big] \vee \underline{\delta} \\
&= \Big[\sup_{\varepsilon \le a_1 \gamma a_2} \{ \Delta_{\neg}(a_1) \wedge \Delta_{\daleth}(a_2)\} \wedge \underline{\tau} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&= \Big[\sup_{\varepsilon \le a_1 \gamma a_2} \{ (\Delta_{\neg}(a_1) \wedge \underline{\tau}) \wedge (\Delta_{\daleth}(a_2) \wedge \underline{\tau})\} \wedge \underline{\tau} \Big] \vee \underline{\delta} \\
&\ge (\{(\Delta_{\neg}(\varepsilon\gamma_1 x \gamma_2 \varepsilon) \vee \underline{\delta}) \wedge (\Delta_{\daleth}(x\gamma_2\varepsilon) \vee \underline{\delta})\} \wedge \underline{\tau}) \vee \underline{\delta} \\
&\ge ((\Delta_{\neg}(\varepsilon) \vee \underline{\delta}) \wedge (\Delta_{\daleth}(\varepsilon) \vee \underline{\delta}) \wedge \underline{\tau}) \vee \underline{\delta} \\
&= \{((\Delta_{\neg}(\varepsilon) \wedge \Delta_{\daleth}(\varepsilon)) \vee \underline{\delta}) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= \{((\Delta_{\neg} \wedge \Delta_{\daleth})(\varepsilon) \wedge \underline{\tau}\} \vee \underline{\delta} \\
&= (\Delta_{\neg \cap_{\underline{\delta}}^{\underline{\tau}} \daleth})(\varepsilon),
\end{aligned}
$$

$$\begin{aligned}
(\Psi_{\daleth\circ\beth})^{\underline{\tau}}_{\underline{\delta}}(\varepsilon) &= (\Psi_{\daleth\circ\beth}(\varepsilon)\vee\underline{\tau})\wedge\underline{\delta}\\
&= \Big[[\inf_{\varepsilon\leq a_1\gamma a_2}\{\Psi_{\daleth}(a_1)\vee\Psi_{\beth}(a_2)\}\vee\underline{\tau}]\Big]\wedge\underline{\delta}\\
&= \Big[\inf_{\varepsilon\leq a_1\gamma a_2}\{\Psi_{\daleth}(a_1)\vee\Psi_{\beth}(a_2)\}\vee\underline{\tau}\vee\underline{\tau}\Big]\wedge\underline{\delta}\\
&= \Big[\inf_{\varepsilon\leq a_1\gamma a_2}\{(\Psi_{\daleth}(a_1)\vee\underline{\tau})\vee(\Psi_{\beth}(a_2)\vee\underline{\tau})\}\vee\underline{\tau}\Big]\wedge\underline{\delta}\\
&\leq (\{(\Psi_{\daleth}(\varepsilon\gamma_1x\gamma_2\varepsilon)\wedge\underline{\delta})\vee(\Psi_{\beth}(x\gamma_2\varepsilon)\wedge\underline{\delta})\}\vee\underline{\tau})\wedge\underline{\delta}\\
&\leq ((\Psi_{\daleth}(\varepsilon)\wedge\underline{\delta})\vee(\Psi_{\beth}(\varepsilon)\wedge\underline{\delta})\vee\underline{\tau})\wedge\underline{\delta}\\
&= \{((\Psi_{\daleth}(\varepsilon)\vee\Psi_{\beth}(\varepsilon))\wedge\underline{\delta})\vee\underline{\tau}\}\wedge\underline{\delta}\\
&= \{((\Psi_{\daleth}\vee\Psi_{\beth})(\varepsilon)\vee\underline{\tau}\}\wedge\underline{\delta}\\
&= (\Psi_{\daleth\cap^{\underline{\tau}}_{\underline{\delta}}\beth})(\varepsilon_3).
\end{aligned}$$

Thus, $((\daleth\circ_\Gamma\beth])^\tau_\delta\supseteq(\daleth\cap^\tau_\delta\beth]$ and by Theorem 3.5 and hence, $((\daleth\circ_\Gamma\beth])^\tau_\delta=(\daleth\cap^\tau_\delta\beth]$.

Conversely, assume that $((\daleth\circ_\Gamma\beth])^\tau_\delta\supseteq(\daleth\cap^\tau_\delta\beth]$. Let $\daleth$ be a (δ,τ)-BPAIFSBI and $\beth$ be a (δ,τ)-BPAIFSLI of $\Bbbk$. Since every (δ,τ)-BPAIFSRI of $\Bbbk$ is a (δ,τ)-BPAIFSBI of $\Bbbk$ and by Theorem 4.1, we have $\Bbbk$ is regular. □

5. Conclusion

We have developed an extended framework for (δ,τ)-bipolar anti-intuitionistic fuzzy ideals in ordered Γ-semigroups, covering subsemigroups, left ideals, right ideals, and bi-ideals. By employing level set techniques, we demonstrated how these fuzzy structures can effectively characterize the regularity of the underlying algebraic system. The theoretical results are supported by illustrative examples that illustrate both correctness and applicability. This framework provides a foundation for further studies on generalized fuzzy structures in algebraic systems with uncertainty.

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