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Generalized Interval Valued Fuzzy Ideals Which Coincide in Ordered Semigroups

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Abstract. In this article, we give a definition of a generalized interval valued bipolar fuzzy ideal. We can find necessary and sufficient conditions for types of generalized interval valued bipolar fuzzy ideals in ordered semigroups.

1. Introduction

The world of fuzzy sets started with the work of the renowned scientist by L. A. Zadeh in 1965 [14]. The theory of fuzzy semigroups contained by Kuroki in 1979 [10]. Later the theory of interval valued fuzzy sets was introduced by L. A. Zadeh in 1975 [15] as a generalization of the notion of fuzzy sets. Interval valued fuzzy sets have various applications in several areas like medical science [2], image processing [1], decision making [17], etc. In 2006, Narayanan and Manikantan [12] developed the theory of interval valued fuzzy subsemigroup and studied types interval valued fuzzy ideals in semigroups. In 1994 Zhang [16] introduced the notion of bipolar fuzzy sets with the extension of fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$, and used them for modeling and decision analysis. In 2000, Lee [11] used the term bipolar valued fuzzy sets and applied it to algebraic structures. In 2016, Mumtaz Ali et al. extended concept of interval valued fuzzy set and bipolar fuzzy set to interval valued bipolar fuzzy set. In 2019, K. Arulmozhi et al. studied interval valued bipolar fuzzy set in algebra structure. Jun et al [9] studied generalized fuzzy bi-ideal in ordered semigroup and characterization of regular ordered semigroups in terms of $(\varepsilon, \varepsilon, \vee_q)$-fuzzy bi-ideals. In 2021, S. Lekkoksung [13] developed interval valued bipolar fuzzy ideal in ordered semigroup and characterized regular ordered semigroup in terms generalized interval valued bipolar fuzzy ideal and bi-ideal.

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In this paper, we establish the concept of an generalized interval valued bipolar fuzzy ideal. We prove necessary and sufficient conditions for types of generalized interval valued bipolar fuzzy ideals in ordered semigroups.

2. Preliminaries

In this section, we give some definition and theory helpful in later sections.

An ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. A non-empty subset $\mathcal{L}$ of an ordered semigroup $\mathcal{G}$ is called

(1) a *subsemigroup* of $\mathcal{G}$ if $\mathcal{L}^2 \subseteq \mathcal{L}$,
(2) a *left* (right) ideal of $\mathcal{G}$ if $(\mathcal{G}\mathcal{L}] \subseteq \mathcal{L}$ $((\mathcal{L}\mathcal{G}] \subseteq \mathcal{L})$,
(3) a *bi-ideal* of $\mathcal{G}$ if $\mathcal{L}$ is a subsemigorup and $(\mathcal{L}\mathcal{G}\mathcal{L}] \subseteq \mathcal{L}$,
(4) an *interior ideal* of $\mathcal{G}$ if $\mathcal{L}$ is a subsemigorup and $(\mathcal{G}\mathcal{L}\mathcal{G}] \subseteq \mathcal{L}$.

An ordered semigroup $\mathcal{G}$ is called a *regular* if for each $\mathfrak{u} \in \mathcal{G}$, there exists $x \in \mathcal{G}$ such that $u \leq uxu$. An ordered semigroup semigroup $\mathcal{G}$ is called a *left* (right) regular if for each $\mathfrak{u} \in \mathcal{G}$, there exists $\mathfrak{a} \in \mathcal{G}$ such that $\mathfrak{u} \leq \mathfrak{a}\mathfrak{u}^2$ (resp. $\mathfrak{u} \leq \mathfrak{u}^2\mathfrak{a}$). An ordered semigroup semigroup $\mathcal{G}$ called an *intra-regular* if for each $\mathfrak{u} \in \mathcal{G}$, there exist $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}$ such that $\mathfrak{u} \leq au^2b$. An ordered semigroup semigroup $\mathcal{G}$ is called a *semisimple* if for every $\mathfrak{u} \in \mathcal{G}$, that is there exist $\mathfrak{w}, \mathfrak{y}, \mathfrak{z} \in \mathcal{G}$ such that $\mathfrak{u} \leq \mathfrak{w}\mathfrak{u}\mathfrak{y}\mathfrak{u}\mathfrak{z}$. [7].

For any $\mathfrak{p}_i \in [0,1]$, where $i \in \mathcal{A}$, define

$$\bigvee_{i\in\mathcal{A}} \mathfrak{k}_i := \sup_{i\in\mathcal{A}}\{\mathfrak{k}_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{A}} \mathfrak{k}_i := \inf_{i\in\mathcal{A}}\{\mathfrak{p}_i\}.$$

We see that for any $\mathfrak{k}, \mathfrak{o} \in [0,1]$, we have

$$\mathfrak{k} \vee \mathfrak{o} = \max\{\mathfrak{k}, \mathfrak{o}\} \quad \text{and} \quad \mathfrak{k} \wedge \mathfrak{o} = \min\{\mathfrak{k}, \mathfrak{o}\}.$$

A *fuzzy set* of a non-empty set $\mathcal{L}$ is a function $\omega : \mathcal{L} \to [0,1]$.

Let $\Omega[0,1]$ be the set of all closed subintervals of $[0,1]$, i.e.,

$$\Omega[0,1] = \{\overline{\omega} = [\omega^-, \omega^+] \mid 0 \leq \omega^- \leq \omega^+ \leq 1\}.$$

We note that $[\omega, \omega] = \{\omega\}$ for all $\omega \in [0,1]$. For $\omega = 0$ or 1 we shall denote $[0,0]$ by $\overline{0}$ and $[1,1]$ by $\overline{1}$.

Let $\overline{\omega} = [\omega^-, \omega^+]$ and $\overline{\varpi} = [\varpi^-, \varpi^+] \in \Omega[0,1]$. Define the operations $\preceq, =, \curlywedge$ and $\curlyvee$ as follows:

(1) $\overline{\omega} \preceq \overline{\varpi}$ if and only if $\omega^- \leq \varpi^-$ and $\omega^+ \leq \varpi^+$
(2) $\overline{\omega} = \overline{\varpi}$ if and only if $\omega^- = \varpi^-$ and $\omega^+ = \varpi^+$
(3) $\overline{\omega} \curlywedge \overline{\varpi} = [(\omega^- \wedge \varpi^-), (\omega^+ \wedge \varpi^+)]$
(4) $\overline{\omega} \curlyvee \overline{\varpi} = [(\omega^- \vee \varpi^-), (\omega^+ \vee \varpi^+)]$.

If $\overline{\omega} \succeq \overline{\varpi}$, we mean $\overline{\varpi} \preceq \overline{\omega}$.

For each interval $\overline{\omega}_i = [\omega_i^-, \omega_i^+] \in \Omega[0,1]$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\curlywedge_{i\in\mathcal{A}} \overline{\omega}_i = [\bigwedge_{i\in\mathcal{A}} \omega_i^-, \bigwedge_{i\in\mathcal{A}} \omega_i^+] \quad \text{and} \quad \curlyvee_{i\in\mathcal{A}} \overline{\omega}_i = [\bigvee_{i\in\mathcal{A}} \omega_i^-, \bigvee_{i\in\mathcal{A}} \omega_i^+].$$

Definition 2.1. *[12] Let $\mathcal{L}$ be a non-empty set. Then the function $\overline{\omega} : \mathcal{L} \to \Omega[0,1]$ is called* interval valued fuzzy set *(shortly, IVF set) of $\mathcal{L}$.*

Definition 2.2. *[12] Let $\mathcal{L}$ be a subset of a non-empty set $\mathcal{G}$. An* interval valued characteristic function *of $\mathcal{L}$ is defined to be a function $\overline{\chi}_{\mathcal{L}} : \mathcal{G} \to \Omega[0,1]$ by*

$$\overline{\chi}_{\mathcal{L}}(\mathfrak{e}) = \begin{cases} \overline{1} & \text{if} \quad \mathfrak{e} \in \mathcal{L}, \\ \overline{0} & \text{if} \quad \mathfrak{e} \notin \mathcal{L} \end{cases}$$

for all $\mathfrak{e} \in \mathcal{G}$.

Now, we reivew definition of bipolar valued fuzzy set and basic properties used in next section.

Definition 2.3. *[11] Let $\mathcal{L}$ be a non-empty set. A bipolar fuzzy set (BF set) ω on $\mathcal{L}$ is an object having the form*

$$\omega := \{(\mathfrak{e}, \omega^p(\mathfrak{e}), \omega^n(\mathfrak{e})) \mid \mathfrak{e} \in \mathcal{L}\},$$

where $\omega^p : \mathcal{L} \to [0,1]$ and $\omega^n : \mathcal{L} \to [-1,0]$.

Remark 2.1. *For the sake of simplicity we shall use the symbol $\omega = (\mathcal{L}; \omega^p, \omega^n)$ for the BF set $\omega = \{(\mathfrak{e}, \omega^p(\mathfrak{e}), \omega^n(\mathfrak{e})) \mid \mathfrak{e} \in \mathcal{L}\}$.*

The following example of a BF set.

Example 2.1. *Let $\mathcal{T} = \{21, 22, 23...\}$. Define $\omega^p : \mathcal{T} \to [0,1]$ is a function*

$$\omega^p(u) = \begin{cases} 0 & \text{if } \mathfrak{u} \text{ is old number} \\ 1 & \text{if } \mathfrak{u} \text{ is even number} \end{cases}$$

and $\omega^n : \mathcal{T} \to [-1,0]$ is a function

$$\omega^n(u) = \begin{cases} -1 & \text{if } \mathfrak{u} \text{ is old number} \\ 0 & \text{if } \mathfrak{u} \text{ is even number.} \end{cases}$$

Then $\omega = (\mathcal{T}; \omega^p, \omega^n)$ is a BF set.

For $\mathfrak{k} \in \mathcal{T}$, define $F_{\mathfrak{k}} = \{(\mathfrak{y}, \mathfrak{z}) \in \mathcal{T} \times \mathcal{T} \mid \mathfrak{k} = \mathfrak{y}\mathfrak{z}\}$.

Definition 2.4. *[6] Let $\mathcal{I}$ be a non-empty set of a semigroup $\mathcal{L}$. A positive characteristic function and a negative characteristic function are respectively defined by*

$$\chi^p_{\mathcal{I}} : \mathcal{L} \to [0,1], \mathfrak{k} \mapsto \lambda^p_{\mathcal{I}}(\mathfrak{k}) := \begin{cases} 1 & \mathfrak{k} \in \mathcal{I}, \\ 0 & \mathfrak{k} \notin \mathcal{I}, \end{cases}$$

and

$$\chi^n_{\mathcal{I}} : \mathcal{L} \to [-1,0], k \mapsto \lambda^n_{\mathcal{I}}(\mathfrak{k}) := \begin{cases} -1 & \mathfrak{k} \in \mathcal{I}, \\ 0 & \mathfrak{k} \notin \mathcal{I}. \end{cases}$$

Remark 2.2. *For the sake of simplicity we shall use the symbol $\chi_{\mathcal{I}} = (\mathcal{L}; \chi^p_{\mathcal{I}}, \chi^n_{\mathcal{I}})$ for the BF set $\chi_{\mathcal{I}} := \{(\mathfrak{k}, \chi^p_{\mathcal{I}}(\mathfrak{k}), \chi^n_{\mathcal{I}}(\mathfrak{k})) \mid \mathfrak{k} \in \mathcal{I}\}$.*

Now, we review definition of an interval valued bipolar fuzzy set and basic properties used in next section.

Definition 2.5. *[13] An* interval valued bipolar fuzzy set *(shortly, IVBF subset)* $\overline{\mathcal{T}}$ *on an ordered semigroup* $\mathcal{G}$ *is form*

$$\overline{\mathcal{T}} := \{\mathfrak{k}, \overline{\omega}^{p}(\mathfrak{k}), \overline{\omega}^{n}(\mathfrak{k}) \mid \mathfrak{k} \in G\},$$

where $\overline{\omega}^{p} : G \to \Omega[0,1]$ *and* $\overline{\omega}^{n} : G \to \Omega[-1,0]$.

In this page we shall use the symbol $\overline{\mathcal{T}} = (\overline{\omega}^{p}, \overline{\omega}^{n})$ instead of the IVBF set $\overline{\mathcal{T}} := \{\mathfrak{k}, \overline{\omega}^{p}(\mathfrak{k}), \overline{\omega}^{n}(\mathfrak{k}) \mid \mathfrak{k} \in G\}$.

For two IVBF sets $\overline{\mathcal{T}}_1 = (\overline{\omega}^{p}, \overline{\omega}^{n})$ and $\overline{\mathcal{T}}_2 = (\overline{\omega}^{p}, \overline{\omega}^{n})$ of an ordered semigroup $\mathcal{G}$, define

(1) $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ if and only if $\overline{\omega}^{p}(\mathfrak{k}) \leq \overline{\omega}^{p}(\mathfrak{k})$ and $\overline{\omega}^{n}(e) \leq \overline{\omega}^{n}(\mathfrak{k})$ for all $\mathfrak{k} \in \mathcal{G}$,

(2) $\overline{\mathcal{T}}_1 = \overline{\mathcal{T}}_1$ if and only if $\overline{\mathcal{T}}_1 \sqsubseteq \overline{\mathcal{T}}_2$ and $\overline{\mathcal{T}}_2 \sqsubseteq \overline{\mathcal{T}}_1$,

(3) $\overline{\mathcal{T}}_1 \overline{\sqcup} \overline{\mathcal{T}}_2$ if and only if $\overline{\omega} \cup \overline{\omega}$ where $(\overline{\omega}^{p} \cup \overline{\omega}^{p})(\mathfrak{k}) = \overline{\omega}^{p}(\mathfrak{k}) \vee \overline{\omega}^{p}(\mathfrak{k})$ and $(\overline{\omega}^{n} \cup \overline{\omega}^{n})(\mathfrak{k}) = \overline{\omega}^{n}(\mathfrak{k}) \wedge \overline{\omega}^{n}(\mathfrak{k})$ for all $\mathfrak{k} \in \mathcal{G}$,

(4) $\overline{\mathcal{T}}_1 \overline{\sqcap} \overline{\mathcal{T}}_2$ if and only if $\overline{\omega} \cap \overline{\omega}$ where $(\overline{\omega}^{p} \cap \overline{\omega}^{p})(\mathfrak{k}) = \overline{\omega}^{p}(\mathfrak{k}) \wedge \overline{\omega}^{p}(\mathfrak{k})$ and $(\overline{\omega}^{n} \cap \overline{\omega}^{n})(\mathfrak{k}) = \overline{\omega}^{n}(\mathfrak{k}) \vee \overline{\omega}^{n}(\mathfrak{k})$ for all $\mathfrak{k} \in \mathcal{G}$,

(5) $\overline{\mathcal{T}}_1 \overline{\circ} \overline{\mathcal{T}}_2$ if and on if $\overline{\omega} \circ \overline{\omega}$ where

$$(\overline{\omega}^{p} \circ \overline{\omega}^{p})(\mathfrak{k}) = \begin{cases} \bigvee_{(\mathfrak{t},\mathfrak{h}) \in F_{\mathfrak{k}}} \{\overline{\omega}^{p}(\mathfrak{t}) \wedge \overline{\omega}^{p}(\mathfrak{h})\} & \text{if } F_{\mathfrak{k}} \neq \emptyset, \\ \overline{0} & \text{if } F_{\mathfrak{k}} = \emptyset, \end{cases}$$

and

$$(\overline{\omega}^{n} \circ \overline{\omega}^{n})(\mathfrak{k}) = \begin{cases} \bigwedge_{(\mathfrak{t},\mathfrak{h}) \in F_{\mathfrak{k}}} \{\overline{\omega}^{n}(\mathfrak{t}) \vee \overline{\omega}^{n}(\mathfrak{h})\} & \text{if } F_{\mathfrak{k}} \neq \emptyset, \\ \overline{0} & \text{if } F_{\mathfrak{k}} = \emptyset, \end{cases}$$

where $F_{\mathfrak{k}} := \{(\mathfrak{t}, \mathfrak{h}) \in \mathcal{G} \times \mathcal{G} \mid \mathfrak{k} \leq \mathfrak{t}\mathfrak{h}\}$ for all $\mathfrak{k} \in \mathcal{G}$.

Definition 2.6. *[13] Let* $\mathcal{I}$ *be a non-empty set of an ordered semigroup* $\mathcal{G}$*. An interval valued bipolar characteristic function are respectively defined by*

$$\overline{\chi}^{p}_{\mathcal{I}} : G \to \Omega[0,1], \mathfrak{k} \mapsto \overline{\chi}^{p}_{\mathcal{I}}(\mathfrak{k}) := \begin{cases} \overline{1} & \mathfrak{k} \in \mathcal{I}, \\ \overline{0} & \mathfrak{k} \notin \mathcal{I}, \end{cases}$$

and

$$\overline{\chi}^{n}_{\mathcal{I}} : \mathcal{G} \to \Omega[-1,0], \mathfrak{k} \mapsto \overline{\chi}^{n}_{\mathcal{I}}(\mathfrak{k}) := \begin{cases} -\overline{1} & \mathfrak{k} \in \mathcal{I}, \\ \overline{0} & \mathfrak{k} \notin \mathcal{I}. \end{cases}$$

Remark 2.3. *For the sake of simplicity we shall use the symbol* $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^{p}_{\mathcal{I}}, \overline{\chi}^{n}_{\mathcal{I}})$ *for the BF set* $\overline{\chi}_{\mathcal{I}} := \{(\mathfrak{k}, \overline{\chi}^{p}_{\mathcal{I}}(\mathfrak{k}), \overline{\chi}^{n}_{\mathcal{I}}(\mathfrak{k})) \mid \mathfrak{k} \in \mathcal{I}\}$.

Now, we let $\overline{\lambda}^{p}, \overline{\delta}^{p} \in \Omega[0,1]$ be such that $\overline{0} \leq \overline{\lambda}^{p} < \overline{\delta}^{p} \leq \overline{1}$ and $\overline{\lambda}^{n}, \overline{\delta}^{n} \in \Omega[-1,0]$ be such that $-\overline{1} \leq \overline{\delta}^{n} < \overline{\lambda}^{n} \leq \overline{1}$. Both $\overline{\lambda}, \overline{\delta}$ are arbitrary but fixed.

3. Main Results

In this section, we give the concept of types generalized interval valued bipolar fuzzy ideals and investigate necessary and sufficient conditions for types of generalized interval valued bipolar fuzzy ideals in ordered semigroups.

Definition 3.1. *[13] An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$ if*

(1) $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$.

(2) $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\omega}^n(\mathfrak{k}_2) \wedge \overline{\delta}^n$.

(3) *If* $\mathfrak{k}_1 \geq \mathfrak{k}_2$, *then* $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ *and* $\overline{\omega}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$,

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$.

Definition 3.2. *[13] An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathcal{G}$ if*

(1) $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$.

(2) $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$.

(3) *If* $\mathfrak{k}_1 \geq \mathfrak{k}_2$, *then* $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ *and* $\overline{\omega}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$.

Definition 3.3. *[13] An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathcal{G}$ if*

(1) $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\delta}^p$.

(2) $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \geq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\delta}^n$.

(3) *If* $\mathfrak{k}_1 \geq \mathfrak{k}_2$, *then* $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ *and* $\overline{\omega}^n(e_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$.

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$.

An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$ if it is both $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal and $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathcal{G}$.

Definition 3.4. *An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathcal{G}$ if*

(1) $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *is an* $(\overline{\lambda}, \overline{\delta})$*-IVBF subsemigroup of* $\mathcal{G}$

(2) $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$

(3) $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$.

The following example is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of a semigroup.

Example 3.1. *Let us consider an ordered semigroup $(\mathcal{G},\cdot)$ defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	c
d	a	c	c	c

An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ in $\mathcal{G}$ as follows: $\overline{\mu}^p = \{((a),[0.7,0.8]),((b),[0.4,0.6]),((c),[0.6,0.7]),$ $((d),[0.3,0.5])\}$ and $\overline{\mu}^n = \{((a),[-0.6,-0.7]),((b),[-0.5,-0.6]),((c),[-0.5,-0.6]),$ $((d),[-0.3,-0.5])\}$ and define a partial order ration $\leq$ on $\mathcal{G}$ as follows:
$\leq$: $\{(a,b),(a,c),(a,d),(b,c),(b,d),(d,c)\} \cup \triangle_G$, where $\triangle_G$ is an equality relation on $\mathcal{G}$. By routine calculation, $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $([0.3,0.3],[0.5,0.5])$-IVBF interior ideal of $\mathcal{G}$.

Theorem 3.1. *Every $(\overline{\lambda},\overline{\delta})$-IVBF ideal of an ordered semigroup $\mathcal{G}$ is an $(\overline{\lambda},\overline{\delta})$-IVBF interior ideal of $\mathcal{G}$.*

Proof. Let $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ be a $(\overline{\lambda},\overline{\delta})$-IVBF ideal of $\mathcal{G}$ and let $\mathfrak{k}_1,\mathfrak{k}_2 \in \mathcal{G}$ with $\mathfrak{k}_1 \geq \mathfrak{k}_2$. Then $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Let $\mathfrak{k}_1,\mathfrak{k}_2 \in \mathcal{G}$. Then
$\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF left ideal and an $(\overline{\lambda},\overline{\delta})$-IVBF right ideal of $\mathcal{G}$. Thus,
$\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Hence,
$\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$.

This show that $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$. Let $\mathfrak{k}_1,\mathfrak{k}_2,\mathfrak{k}_3 \in \mathcal{G}$. Then,

$$\begin{aligned}
\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p &= (\overline{\omega}^p(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{k}_3)) \vee \overline{\lambda}^p) \vee \overline{\lambda}^p \geq (\overline{\omega}^p(\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= ((\overline{\omega}^p(\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \geq ((\overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\
&= (\overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p.
\end{aligned}$$

and

$$\begin{aligned}
\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n &= (\overline{\omega}^p(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{k}_3)) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n \leq (\overline{\omega}^n(\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\delta}^n) \wedge \overline{\lambda}^p \\
&= ((\overline{\omega}^n(\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n \leq ((\overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n \\
&= (\overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n.
\end{aligned}$$

Thus, $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF interior ideal of $\mathcal{G}$. □

Remark 3.1. *In example 3.1 we can show that the converse of the above theorem is not true in general. Consider $\overline{\omega}^p(bd) \vee \overline{\lambda}^p = [0.3,0.5] \ngeq [0.4,0.5] = \overline{\omega}^p(b) \wedge \overline{\delta}^p$. Thus $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda},\overline{\delta})$-IVBF ideal of $\mathcal{G}$.*

The following theorem show that the $(\overline{\lambda},\overline{\delta})$-IVBF interior ideals and $(\overline{\lambda},\overline{\delta})$-IVBF ideals coincide for some types of ordered semigroups.

Theorem 3.2. *In regular, left (right) regular, intra-regular and semisimple oredered semigroup, the $(\overline{\lambda},\overline{\delta})$-IVBF interior ideals and $(\overline{\lambda},\overline{\delta})$-IVBF ideals coincide*

Proof. Let $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ be an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of a regular orered semigroup and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$ with $\mathfrak{k}_1 \geq \mathfrak{k}_2$. Then $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Since $\mathcal{G}$ is regular, we have there exists $\mathfrak{r} \in \mathcal{G}$ such that $\mathfrak{k}_1 \leq \mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1$. Thus,

$$\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq (\overline{\omega}^p((\mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1)\mathfrak{k}_2) \vee \overline{\lambda}^p = \overline{\omega}^p((\mathfrak{k}_1\mathfrak{r})\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\delta}^p$$

and

$$\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n((\mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1)\mathfrak{k}_2) \wedge \overline{\lambda}^n = \overline{\omega}^n((\mathfrak{k}_1\mathfrak{r})\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\delta}^n.$$

Hence $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathcal{G}$. Similarly, we can prove that $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathcal{G}$. Thus $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$.

Similarly, we can prove the other cases also. □

Definition 3.5. *An IVBF set $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ of an ordered semigroup $\mathcal{G}$ is called an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$ if*

(1) $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ *is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$*
(2) $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(\mathfrak{k}_3) \wedge \overline{\delta}^p$
(3) $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\omega}^n(k_3) \vee \overline{\delta}^n$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$.

Lemma 3.1. *Every $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of an ordered semigroup $\mathcal{G}$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$.*

Proof. Suppose that $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$ and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$. Since $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$, we have that $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathcal{G}$. Thus, $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^P \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\delta}^P$ and $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\delta}^n$ and so $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(\mathfrak{k}_3) \wedge \overline{\delta}^p$, and $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\omega}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Hence, $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, k_3 \in \mathcal{G}$. Since $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$, we have that $\overline{\mathcal{T}} = (\overline{\omega}^p, \overline{\omega}^n)$ is a $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathcal{G}$. Thus,

$$\begin{aligned} \overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p &= (\overline{\omega}^p((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \vee \overline{\lambda}^p) \vee \overline{\lambda}^p \\ &\geq (\overline{\omega}^p(\mathfrak{k}_3) \wedge \overline{\delta}^p) \vee \overline{\lambda}^p \\ &\geq (\overline{\omega}^p(\mathfrak{k}_3) \wedge \overline{\delta}^p) \wedge \overline{\delta}^p \end{aligned}$$

and

$$\begin{aligned} \overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n &= (\overline{\omega}^p(\mathfrak{k}_1(\mathfrak{k}_2\mathfrak{k}_3)) \wedge \overline{\lambda}^n) \wedge \overline{\lambda}^n \\ &\leq (\overline{\omega}^n((\mathfrak{k}_1\mathfrak{k}_2)\mathfrak{k}_3) \vee \overline{\delta}^n) \wedge \overline{\lambda}^p \\ &\leq \overline{\omega}^n(\mathfrak{k}_3) \vee \overline{\delta}^n. \end{aligned}$$

and so $\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(k_3) \wedge \overline{\delta}^p$ and $\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1) \vee \overline{\omega}^n(k_3) \vee \overline{\delta}^n$. Hence $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$. □

In order to consider the converse of Lemma 3.1, we need to strengthen the condition of a semigroup $\mathcal{S}$.

Theorem 3.3. *In regular and left (right) regular the $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideals and $(\overline{\lambda}, \overline{\delta})$-IVBF ideals coincide*

Proof. Suppose that $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$ and let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$. Since $\mathcal{G}$ is regular, we have $\mathfrak{k}_1\mathfrak{k}_2 \in (\mathfrak{k}_1\mathcal{G}\mathfrak{k}_1)\mathcal{G} \subseteq \mathfrak{k}_1\mathcal{G}$ which implies that $\mathfrak{k}_1\mathfrak{k}_2 \leq \mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1$ for some $r \in \mathcal{G}$. Thus,

$$\overline{\omega}^p(\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p = \overline{\omega}^p(\mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p = \overline{\omega}^p((\mathfrak{k}_1\mathfrak{r})\mathfrak{k}_1\mathfrak{k}_2) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_1) \wedge \overline{\delta}^p$$

and

$$\overline{\omega}^n(\mathfrak{k}_1\mathfrak{k}_2) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1\mathfrak{r}\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\omega}^n(\mathfrak{k}_1)\mathfrak{k}_1 \vee \overline{\omega}^n(\mathfrak{k}_1)\mathfrak{k}_1 \vee \overline{\delta}^n.$$

Hence, $(\overline{\lambda}, \overline{\delta})$-IVBF right ideal of $\mathcal{G}$. Similarly, we can show that $(\overline{\lambda}, \overline{\delta})$-IVBF left ideal of $\mathcal{S}$. Thus, $(\overline{\lambda}, \overline{\delta})$-IVBF ideal of $\mathcal{G}$. Similarly, we can prove the other cases also. □

In the following theorem, we give a relationship between a subsemigroup and the intervval valued bipolar characteristic function, which proved easily.

Theorem 3.4. *Let $\mathcal{I}$ be a non-empty subset of an ordered semigroup $\mathcal{G}$. Then I is a subsemigroup (left ideal, right ideal, ideal) of $\mathcal{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$ if and only if $\overline{\chi}_I = (\mathcal{G}; \overline{\chi}^p_I, \overline{\chi}^n_I)$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup (left ideal, right ideal, ideal) of $\mathcal{G}$.*

Theorem 3.5. *Let $\mathcal{I}$ be a non-empty subset of an ordered semigroup $\mathcal{G}$. Then $\mathcal{I}$ is an interior ideal of $\mathcal{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$ if and only if $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathcal{G}$.*

Proof. Suppose that $\mathcal{I}$ is an interior ideal of G. Then $\mathcal{I}$ is a subsemigroup of $\mathcal{G}$. Thus by Theorem 3.4, $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigrup of $\mathcal{G}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$.

If $\mathfrak{k}_1 \geq \mathfrak{k}_2$, then $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$.

If $\mathfrak{k}_2 \in \mathcal{I}$, then $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathcal{I}$. Thus, $\overline{1} = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_2) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$ and $-\overline{1} = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_2) = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$. Hence, $\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_2) \vee \overline{\delta}^n$.

If $\mathfrak{k}_2 \notin \mathcal{I}$, then $\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_2) \vee \overline{\delta}^n$.

Let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$ with $\mathfrak{k}_1 \geq \mathfrak{k}_2$. Then $\overline{\chi}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\chi}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\chi}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\chi}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Thus $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathcal{G}$.

Conversely, suppose that $\overline{\chi}_{\mathcal{I}} = (G; \overline{\chi}^p_I, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF interior ideal of $\mathcal{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$. Then $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$. Thus by Theorem 3.4, $\mathcal{I}$ subsemigroup of $\mathcal{G}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$ and $\mathfrak{k}_2 \in \mathcal{I}$. Then $\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_2) = \overline{1}$ and $\overline{\chi}^n_I(\mathfrak{k}_2) = -\overline{1}$. By assumtion,

$$\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_2) \wedge \overline{\delta}^p \text{ and } \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_2) \vee \overline{\delta}^n. \tag{3.1}$$

If $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \notin \mathcal{I}$, then by (3.2) $\overline{\lambda}^p \geq \overline{\delta}^p$and $\overline{\lambda}^n \leq \overline{\delta}^n$. It is a contradiction. Hence $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathcal{I}$. Therefore $\mathcal{I}$ is an interior ideal of $\mathcal{G}$. □

Theorem 3.6. *Let $\mathcal{I}$ be a non-empty subset of an ordered semigroup $\mathcal{G}$. Then $\mathcal{I}$ is a bi-ideal of $\mathcal{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$ if and only if $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$.*

Proof. Suppose that $\mathcal{I}$ is an interior ideal of G. Then $\mathcal{I}$ is a subsemigroup of $\mathcal{G}$. Thus by Theorem 3.4, $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigrup of $\mathcal{G}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$.

If $\mathfrak{k}_1 \geq \mathfrak{k}_2$, then $\overline{\omega}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\omega}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$.

If $\mathfrak{k}_1, \mathfrak{k}_3 \in \mathcal{I}$, then $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathcal{I}$. Thus, $\overline{1} = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_3) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$ and $-\overline{1} = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1) = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_3) = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$. Hence, $\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1) \wedge \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_3) \wedge \overline{\delta}^p$ and $\overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1) \vee \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_3) \vee \overline{\delta}^n$

If $\mathfrak{k}_1, \mathfrak{k}_3 \notin \mathcal{I}$, then $\overline{1} = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_3) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$ and $-\overline{1} = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1) = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_3) = \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3)$.

Let $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{G}$ with $\mathfrak{k}_1 \geq \mathfrak{k}_2$. Then $\overline{\chi}^p(\mathfrak{k}_1) \vee \overline{\lambda}^p \geq \overline{\chi}^p(\mathfrak{k}_2) \wedge \overline{\delta}^p$ and $\overline{\chi}^n(\mathfrak{k}_1) \wedge \overline{\lambda}^n \leq \overline{\chi}^n(\mathfrak{k}_2) \vee \overline{\delta}^n$. Thus $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$.

Conversely, suppose that $\overline{\chi}_{\mathcal{I}} = (G; \overline{\chi}^p_I, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF bi-ideal of $\mathcal{G}$ with $\overline{\lambda}^p < \overline{\delta}^p$ and $\overline{\lambda}^n > \overline{\delta}^n$. Then $\overline{\chi}_{\mathcal{I}} = (\mathcal{G}; \overline{\chi}^p_{\mathcal{I}}, \overline{\chi}^n_{\mathcal{I}})$ is an $(\overline{\lambda}, \overline{\delta})$-IVBF subsemigroup of $\mathcal{G}$. Thus by Theorem 3.4, $\mathcal{I}$ subsemigroup of $\mathcal{G}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathcal{G}$ and $\mathfrak{k}_1, \mathfrak{k}_3, \in \mathcal{I}$. Then $\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1) = \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_3) = \overline{1}$ and $\overline{\chi}^n_I(\mathfrak{k}_1) = \overline{\chi}^n_I(\mathfrak{k}_3) = -\overline{1}$. By assumtion,

$$\overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \vee \overline{\lambda}^p \geq \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_1) \wedge \overline{\chi}^p_{\mathcal{I}}(\mathfrak{k}_3) \wedge \overline{\delta}^p \text{ and } \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3) \wedge \overline{\lambda}^n \leq \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_1) \vee \overline{\chi}^n_{\mathcal{I}}(\mathfrak{k}_3) \vee \overline{\delta}^n. \tag{3.2}$$

If $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \notin \mathcal{I}$, then by (3.2) $\overline{\lambda}^p \geq \overline{\delta}^p$and $\overline{\lambda}^n \leq \overline{\delta}^n$. It is a contradiction. Hence $\mathfrak{k}_1\mathfrak{k}_2\mathfrak{k}_3 \in \mathcal{I}$. Therefore, $\mathcal{I}$ is a bi-ideal of $\mathcal{G}$. □

4. Conclusion

The aim of the paper is to give the concept of generalized interval valued bipolar fuzzy bi-ideals and interior ideals. We prove properties of generalized interval valued bipolar fuzzy bi-ideals and interior ideals. In future work, we can study other generalized interval valued bipolar fuzzy quasi-ideals and their fuzzifications in an ordered ternary semigroup.

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