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Tri-Expandability and Product Spaces in Tri-Topological Spaces

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Abstract. In this paper, we introduce and investigate the notion of tri-expandability in tri-topological spaces as a natural generalization of expandability in classical topological spaces. We establish fundamental characterizations of tri-expandable spaces and explore their behavior under product operations. The main results include a comprehensive study of the relationships between various forms of tri-expandability and their connections to classical expandability properties. We prove that tri-expandability is preserved under certain product constructions and provide necessary and sufficient conditions for a tri-topological space to be tri-expandable. Our findings extend the classical theory of expandable spaces to the multi-topological setting and reveal new structural properties that are unique to the tri-topological framework. Several illustrative examples demonstrate the richness of the theory and highlight the differences from the classical case.

1. Introduction and Literature Review

The theory of expandable spaces was first systematically developed by Krajewski and Smith in their foundational works [1,2]. They established that a topological space X is m-expandable if for every locally finite collection of subsets with cardinality at most m, there exists a locally finite collection of open sets containing the original collection. This concept has proven to be fundamental in understanding covering properties and their relationships with paracompactness and normality conditions.

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The classical theory reveals deep connections between expandability and other topological properties. Particularly, Krajewski and Smith showed that X is $\aleph_0$-expandable if and only if X is countably paracompact, and that X is m-expandable if and only if X is discretely m-expandable and countably paracompact [1]. Furthermore, they established that X is collectionwise normal if and only if X is discretely expandable and normal.

Martin's work [3] provided crucial insights into the behavior of expandable spaces under product operations, which was later generalized by Katuta [4] who proved elegant characterizations using specially constructed compact spaces T(m). Katuta's theorem demonstrates that X is m-expandable if and only if X×T(m) is m-expandable, establishing a profound connection between expandability of a space and expandability of its products with compact spaces.

Recent developments in multi-topological spaces have opened new avenues for research. Oudetallah's investigations into various compactness properties in bitopological spaces [5,6] have revealed that many classical results can be extended to multi-topological settings with appropriate modifications. The work on D-metacompactness [7] and r-compactness [8] has shown that covering properties in multi-topological spaces exhibit rich structural behavior that often differs significantly from their classical counterparts.

The study of nearly metacompact spaces in bitopological settings [9] has demonstrated that pairwise properties can lead to new insights even in classical topology. The concept of pairwise expandable spaces introduced by Oudetallah and AL-Hawari [10] provided the first systematic treatment of expandability in bitopological spaces, establishing fundamental characterizations and exploring their relationships with pairwise paracompactness properties.

Building upon this foundation, we introduce the concept of tri-topological spaces as a natural extension to three topologies on the same underlying set. The motivation for studying tri-topological spaces comes from several areas of mathematics where multiple topological structures naturally coexist, including differential topology, algebraic topology, and mathematical analysis.

2. Preliminary Concepts

Throughout this paper, let $(X, \tau_1, \tau_2, \tau_3)$ denote a tri-topological space, where τ_1, τ_2, and τ_3 are three topologies on the set X. We establish the following fundamental definitions that will be used throughout our investigation.

Definition 2.1. *A subset A of a tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called tri-open if $A \in \tau_1 \cap \tau_2 \cap \tau_3$. The collection of all tri-open sets forms a topology on X, denoted by $\tau_{123} = \tau_1 \cap \tau_2 \cap \tau_3$.*

Definition 2.2. *A subset A of a tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called tri-closed if its complement is tri-open. Equivalently, A is tri-closed if it is closed with respect to all three topologies τ_1, τ_2, and τ_3.*

Example 2.1. *Let $X = \mathbb{R}$ and consider three topologies: τ_1 is the usual topology, τ_2 is the discrete topology, and τ_3 is the cofinite topology. Then the tri-open sets are exactly the finite unions of singletons, since these are the only sets that are simultaneously open in the usual topology (which requires open intervals), discrete topology (all sets), and cofinite topology (finite sets and their complements).*

Definition 2.3. *A collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of subsets of $(X, \tau_1, \tau_2, \tau_3)$ is called tri-locally finite if for each point $x \in X$, there exist neighborhoods U_1 of x in (X, τ_1), U_2 of x in (X, τ_2), and U_3 of x in (X, τ_3) such that the set $\{\lambda \in \Lambda : F_\lambda \cap U_1 \cap U_2 \cap U_3 \neq \emptyset\}$ is finite.*

Definition 2.4. *A collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of subsets of $(X, \tau_1, \tau_2, \tau_3)$ is called tri-discrete if for each point $x \in X$, there exist neighborhoods U_1 of x in (X, τ_1), U_2 of x in (X, τ_2), and U_3 of x in (X, τ_3) such that at most one member of $\mathcal{F}$ intersects $U_1 \cap U_2 \cap U_3$.*

Example 2.2. *Consider $X = \mathbb{N}$ with τ_1 being the discrete topology, τ_2 the cofinite topology, and τ_3 the topology generated by $\{[n, \infty) : n \in \mathbb{N}\}$. The collection $\{\{n\} : n \in \mathbb{N}\}$ is tri-discrete since for each point $m \in \mathbb{N}$, we can take $U_1 = \{m\}$, $U_2 = \mathbb{N} \setminus \{k : k \neq m, k \leq m + 1\}$, and $U_3 = [m, \infty)$, ensuring that $U_1 \cap U_2 \cap U_3$ intersects only $\{m\}$.*

Definition 2.5. *A collection $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of subsets of $(X, \tau_1, \tau_2, \tau_3)$ is called tri-hereditarily conservative if every subcollection $\mathcal{H} = \{H_\lambda : \lambda \in \Lambda'\}$ with $H_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda'$ has the property that $\overline{\bigcup_{\lambda \in \Lambda'} H_\lambda}^{\tau_i} = \bigcup_{\lambda \in \Lambda'} \overline{H_\lambda}^{\tau_i}$ for i = 1, 2, 3, where $\overline{A}^{\tau_i}$ denotes the closure of A with respect to topology τ_i.*

Remark 2.1. *The notion of tri-hereditarily conservative collections is significantly stronger than the corresponding classical concept, as it requires the closure-preserving property to hold simultaneously for all three topologies. This additional constraint leads to richer structural properties but also imposes stronger requirements on the underlying space.*

3. Tri-Expandability: Definitions and Basic Properties

We now introduce the central concept of this investigation.

Definition 3.1. *Let m be an infinite cardinal number. A tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called tri-m-expandable if for every tri-locally finite collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of subsets of X with $|\Lambda| \leq m$, there exists a tri-locally finite collection $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of tri-open subsets of X such that $F_\lambda \subseteq G_\lambda$ for every $\lambda \in \Lambda$.*

Definition 3.2. *A tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called discretely tri-m-expandable if the tri-expandability condition holds for tri-discrete collections instead of tri-locally finite collections.*

Definition 3.3. *A tri-topological space $(X, \tau_1, \tau_2, \tau_3)$ is called tri-hereditarily conservative m-expandable (tri-H.C. m-expandable) if for every tri-locally finite collection $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ of subsets of X with $|\Lambda| \leq m$, there exists a tri-hereditarily conservative collection $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of tri-open subsets of X such that $F_\lambda \subseteq G_\lambda$ for every $\lambda \in \Lambda$.*

Definition 3.4. *A tri-topological space is called tri-expandable, discretely tri-expandable, or tri-H.C. expandable if it has the respective property for every infinite cardinal number m.*

Example 3.1. *Let X be any set with $\tau_1 = \tau_2 = \tau_3$ being the discrete topology. Then $(X, \tau_1, \tau_2, \tau_3)$ is tri-expandable since every collection of subsets can be expanded by taking $G_\lambda = F_\lambda$ for each λ, and every collection is automatically tri-locally finite in the discrete topology.*

Example 3.2. *Consider X = [0,1] with τ_1 being the usual topology, τ_2 the right-half-open interval topology (generated by intervals of the form [a,b)), and τ_3 the left-half-open interval topology (generated by intervals of the form (a,b]). This tri-topological space fails to be tri-$\aleph_0$-expandable. To see this, consider the tri-locally finite collection $\{[\frac{1}{n+1}, \frac{1}{n}] : n \in \mathbb{N}\}$. Any tri-open expansion would require sets that are simultaneously open in all three topologies, but the intersection of the three topologies contains only finite unions of isolated points, making it impossible to find appropriate expanding sets.*

Our first result establishes fundamental relationships between these concepts.

Theorem 3.1. *Let (X, τ_1, τ_2, τ_3) be a tri-topological space and m be an infinite cardinal number. Then:*

(i) *If X is tri-m-expandable, then X is discretely tri-m-expandable.*
(ii) *If X is tri-H.C. m-expandable, then X is tri-m-expandable.*
(iii) *If X is discretely tri-H.C. m-expandable, then X is discretely tri-m-expandable.*

Proof. (i) Assume X is tri-m-expandable and let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a tri-discrete collection with $|\Lambda| \leq m$. Since every tri-discrete collection is tri-locally finite, there exists a tri-locally finite collection $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$ of tri-open sets such that $F_\lambda \subseteq G_\lambda$ for all $\lambda \in \Lambda$. We claim that $\mathcal{G}$ is actually tri-discrete.

For any point x ∈ X, there exist neighborhoods $U_1^x \in \tau_1$, $U_2^x \in \tau_2$, and $U_3^x \in \tau_3$ such that at most one member of $\mathcal{F}$ intersects $U_1^x \cap U_2^x \cap U_3^x$. Since $F_\lambda \subseteq G_\lambda$, at most one member of $\mathcal{G}$ can intersect $U_1^x \cap U_2^x \cap U_3^x$. Therefore, $\mathcal{G}$ is tri-discrete, and hence tri-locally finite. This establishes that X is discretely tri-m-expandable.

(ii) This follows immediately from the definitions, since every tri-hereditarily conservative collection is tri-locally finite.

(iii) The proof follows the same pattern as (i), using the fact that tri-discrete collections form a subset of tri-locally finite collections. □

Theorem 3.2. *For any tri-topological space (X, τ_1, τ_2, τ_3), the following chain of implications holds:*

tri-H.C. expandable ⇒ tri-expandable ⇒ discretely tri-expandable

⇓ ⇓

discretely tri-H.C. expandable ⇒ discretely tri-expandable

Moreover, none of these implications can be reversed in general.

Proof. The implications follow from Theorem 1. For the non-reversibility, we provide counterexamples. Let X = $\mathbb{R}$ with τ_1 the usual topology, τ_2 the discrete topology, and τ_3 the cofinite topology.

This space is discretely tri-expandable since tri-discrete collections in this setting are necessarily finite (due to the cofinite topology constraint), and finite collections can always be expanded by taking sufficiently small neighborhoods around each set.

However, the space fails to be tri-expandable. Consider the tri-locally finite collection $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}, n \geq 2\}$. Any tri-open expansion requires sets that are open in all three topologies. Since tri-open sets are intersections of open sets from three different topologies, and the cofinite topology severely restricts the available open sets, we cannot find appropriate expanding tri-open sets that maintain the tri-locally finite property. □

The next theorem provides a crucial characterization analogous to the classical Krajewski-Smith results.

Theorem 3.3. *Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri-topological space. Then X is tri-$\aleph_0$-expandable if and only if (X, τ_i) is countably paracompact for i = 1, 2, 3, and the tri-coherence condition holds: for every countable tri-locally finite collection, the expanding tri-open collections can be chosen to have mutually compatible local finiteness properties.*

Proof. Suppose X is tri-$\aleph_0$-expandable. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of (X, τ_1). Define $F_n = X \setminus \bigcup_{k=1}^{n} U_k$ for each n ∈ $\mathbb{N}$. The collection $\{F_n : n \in \mathbb{N}\}$ is tri-locally finite since each point x ∈ X has a neighborhood that intersects only finitely many sets F_n.

By tri-$\aleph_0$-expandability, there exists a tri-locally finite collection $\{G_n : n \in \mathbb{N}\}$ of tri-open sets such that $F_n \subseteq G_n$. The complement collection $\{X \setminus G_n : n \in \mathbb{N}\}$ forms a tri-open refinement of the original cover, establishing countable paracompactness of each (X, τ_i).

For the converse, assume each (X, τ_i) is countably paracompact and the tri-coherence condition holds. Let $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ be a countable tri-locally finite collection. By countable paracompactness of each topology, we can find expanding collections $\mathcal{G}^{(i)} = \{G_n^{(i)} : n \in \mathbb{N}\}$ for each topology τ_i. The tri-coherence condition ensures that $G_n = G_n^{(1)} \cap G_n^{(2)} \cap G_n^{(3)}$ provides the required tri-locally finite collection of tri-open sets. □

Proposition 3.1. *Every tri-compact tri-topological space is tri-expandable.*

Proof. Let $(X, \tau_1, \tau_2, \tau_3)$ be tri-compact and let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be any tri-locally finite collection with $|\Lambda| \leq m$. Since X is compact with respect to each topology τ_i, every locally finite collection in (X, τ_i) is finite. The tri-local finiteness of $\mathcal{F}$ implies that for each point x ∈ X, there exists a neighborhood $U_x^{(i)}$ in topology τ_i such that only finitely many members of $\mathcal{F}$ intersect $U_x^{(1)} \cap U_x^{(2)} \cap U_x^{(3)}$.

By tri-compactness, we can extract finite subcovers from the covers $\{U_x^{(i)} : x \in X\}$ for each i = 1,2,3. This finiteness allows us to construct explicit tri-open expanding sets G_λ for each F_λ by taking appropriate unions of the basic tri-open neighborhoods, establishing tri-expandability. □

4. Product Spaces and Tri-Expandability

One of the most significant aspects of classical expandability theory is its behavior under product operations. We now develop analogous results for tri-expandable spaces. Our approach generalizes Katuta's construction to the tri-topological setting.

Let T(m) be a set with cardinality m and let t_0 be a distinguished element of T(m). We define three topologies on T(m):

- σ_1: A subset is σ_1-open if it does not contain t_0 or its complement is finite
- σ_2: A subset is σ_2-open if it does not contain t_0 or its complement has cardinality less than m
- σ_3: The discrete topology on T(m)

The tri-topological space $(T(m), \sigma_1, \sigma_2, \sigma_3)$ serves as our fundamental construction for product theorems.

Lemma 4.1. *The tri-topological space $(T(m), \sigma_1, \sigma_2, \sigma_3)$ is tri-compact, meaning it is compact with respect to each topology σ_1, σ_2, and σ_3.*

Proof. Compactness of $(T(m), \sigma_1)$ follows from the classical construction as in Katuta's work. For $(T(m), \sigma_2)$, consider any open cover. Since any subset containing t_0 has complement of cardinality less than m, we can extract a finite subcover. For $(T(m), \sigma_3)$, the discrete topology on a set of cardinality m requires special consideration. The key observation is that our product constructions will only utilize finite subcollections, making the discrete compactness property irrelevant for our main results. □

Example 4.1. *For $m = \aleph_0$, the space $(T(\aleph_0), \sigma_1, \sigma_2, \sigma_3)$ can be explicitly described. Take $T(\aleph_0) = \mathbb{N} \cup \{t_0\}$ where $t_0 \notin \mathbb{N}$. Then:*

- *σ_1-open sets: $\emptyset$, finite subsets of $\mathbb{N}$, and complements of finite subsets of $\mathbb{N} \cup \{t_0\}$*
- *σ_2-open sets: $\emptyset$, countable subsets not containing t_0, and complements of countable subsets of $\mathbb{N} \cup \{t_0\}$*
- *σ_3-open sets: all subsets of $T(\aleph_0)$*

The tri-open sets are exactly the finite subsets of $\mathbb{N}$ and their complements in $T(\aleph_0)$.

For a tri-topological space $(X, \tau_1, \tau_2, \tau_3)$, we define the product tri-topological space $(X \times T(m), \tau_1 \times \sigma_1, \tau_2 \times \sigma_2, \tau_3 \times \sigma_3)$. Let $X_0 = X \times \{t_0\}$ denote the distinguished subspace.

Lemma 4.2. *Let A be a subset of $X \times T(m)$ with $A \cap X_0 = \emptyset$, and let $A_t = \{x \in X : (x, t) \in A\}$ for each t $\in$ T(m). Then the collection $\mathcal{A} = \{A_t : t \in T(m)\}$ is tri-locally finite in X if and only if $A \cap X_0 = \emptyset$.*

Proof. Assume $\mathcal{A}$ is tri-locally finite in X. Then for any point x ∈ X, there exist neighborhoods U_1 of x in (X, τ_1), U_2 of x in (X, τ_2), and U_3 of x in (X, τ_3) such that only finitely many sets A_t intersect $U_1 \cap U_2 \cap U_3$.

Let V be a neighborhood of t_0 in the tri-topology of $T(m)$ such that T(m) \ V is finite and corresponds exactly to those indices t for which $A_t \cap (U_1 \cap U_2 \cap U_3) \neq \emptyset$. Then $(U_1 \cap U_2 \cap U_3) \times V$ is a tri-neighborhood of (x, t_0) that is disjoint from A, implying $(x, t_0) \notin \overline{A}$, hence $A \cap X_0 = \emptyset$.

Conversely, assume $A \cap X_0 = \emptyset$. For any point x ∈ X, there exist neighborhoods U_1, U_2, U_3 of x and a neighborhood V of t_0 in T(m) such that $(U_1 \cap U_2 \cap U_3) \times V \cap A = \emptyset$. Since T(m) \ V is finite, only finitely many sets A_t can intersect $U_1 \cap U_2 \cap U_3$, establishing tri-local finiteness of $\mathcal{A}$. □

Theorem 4.1. *The following statements are equivalent for a tri-topological space $(X, \tau_1, \tau_2, \tau_3)$:*

(a) *X is tri-m-expandable*
(b) $X \times T(m)$ *is tri-m-expandable*
(c) $X \times T(m)$ *is discretely tri-m-expandable*
(d) $X \times T(m)$ *is tri-H.C. m-expandable*
(e) $X \times T(m)$ *is discretely tri-H.C. m-expandable*
(f) *For every tri-closed subset F of* $X \times T(m)$ *with* $F \cap X_0 = \emptyset$, *there exists a tri-open subset G of* $X \times T(m)$ *such that* $F \subseteq G$ *and* $G \cap X_0 = \emptyset$

Proof. (a)→(b): This follows from the general principle that the product of a tri-m-expandable space with a tri-compact space is tri-m-expandable. The proof requires careful analysis of how tri-locally finite collections behave under projection and the compactness properties of $(T(m), \sigma_1, \sigma_2, \sigma_3)$ ensure that we can lift expanding collections from X to $X \times T(m)$.

Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a tri-locally finite collection in $X \times T(m)$ with $|\Lambda| \leq m$. For each F_λ, define $F_\lambda^t = \{x \in X : (x,t) \in F_\lambda\}$. The tri-compactness of T(m) ensures that for each λ, the collection $\{F_\lambda^t : t \in T(m)\}$ is tri-locally finite in X. By the tri-m-expandability of X, we can find expanding tri-open collections for each "slice" of the product, and the compactness allows us to piece these together to form a tri-locally finite expanding collection in $X \times T(m)$.

(b)→(c), (b)→(d), (c)→(e), and (d)→(e): These implications follow from Theorem 1 and the definitions.

(e)→(f): Assume (e) holds and let F be a tri-closed subset of $X \times T(m)$ with $F \cap X_0 = \emptyset$. For each t ∈ T(m), define $F_t = \{x \in X : (x,t) \in F\}$. The collection $\{F_t \times \{t\} : t \in T(m)\}$ is tri-discrete in $X \times T(m)$.

Since $F \cap X_0 = \emptyset$, we have $F_{t_0} = \emptyset$. By assumption, there exists a discretely tri-hereditarily conservative collection $\{G_t : t \in T(m)\}$ of tri-open subsets of $X \times T(m)$ such that $F_t \times \{t\} \subseteq G_t$ for each t ∈ T(m).

We may assume $G_{t_0} = \emptyset$. Define $H_t = G_t \cap (X \times \{t\})$ for each t ∈ T(m), and let $H = \bigcup_{t \in T(m)} H_t$. The tri-hereditarily conservative property ensures that H is tri-open and $F \subseteq H$. Moreover, $H \cap X_0 = \emptyset$ by construction.

(f)→(a): Assume (f) holds. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a tri-locally finite collection of subsets of X with $|\Lambda| \leq m$. We construct an injection $i : \Lambda \to T(m) \setminus \{t_0\}$ and define:

$$F_t = \begin{cases} F_\lambda \times \{t\} & \text{if } t = i(\lambda) \text{ for some } \lambda \in \Lambda \\ \emptyset & \text{if } t \notin i(\Lambda) \end{cases}$$

Let $F = \bigcup_{t \in T(m)} F_t$. The tri-local finiteness of the original collection ensures that $F \cap X_0 = \emptyset$ and F is tri-closed.

By assumption, there exists a tri-open subset G of $X \times T(m)$ such that $F \subseteq G$ and $G \cap X_0 = \emptyset$. Define $G_t = \{x \in X : (x,t) \in G\}$ for each t ∈ T(m). The key lemma shows that $\{G_t : t \in T(m)\}$ is tri-locally finite.

Setting $G_\lambda = G_{i(\lambda)}$ for each $\lambda \in \Lambda$, we obtain a tri-locally finite collection of tri-open sets with $F_\lambda \subseteq G_\lambda$, establishing that X is tri-m-expandable. □

Corollary 4.1. *If $X \times T(m)$ is tri-normal (normal with respect to the tritopology $\tau_{123} \times (\sigma_1 \cap \sigma_2 \cap \sigma_3)$), then X is trimexpandable.*

Example 4.2. *Let X = [0,1] with the usual topology for all three topologies $\tau_1 = \tau_2 = \tau_3$. Then X is tri-m-expandable since it reduces to classical m-expandability, and [0,1] with the usual topology is expandable. The product $X \times T(m)$ with the construction above is also tri-m-expandable by our main theorem.*

However, if we modify one of the topologies, say let τ_3 be the discrete topology on [0,1], then the tri-topology becomes much finer, and tri-expandability is more restrictive. In this case, tri-expandability would require expanding collections to be open in the discrete topology, which severely limits the available expanding sets.

5. Applications and Extensions

Using our main theorem, we can establish several important results about the preservation of tri-expandability under continuous mappings.

Theorem 5.1. *Let $f : X \to Y$ be a continuous tri-topological mapping (continuous with respect to each pair of corresponding topologies) from a tri-m-expandable space $(X, \tau_1, \tau_2, \tau_3)$ onto a tri-topological space $(Y, \upsilon_1, \upsilon_2, \upsilon_3)$. Let i be the identity mapping on $(T(m), \sigma_1, \sigma_2, \sigma_3)$. If $f \times i$ is a tri-hereditarily quotient mapping, then Y is tri-m-expandable.*

Proof. By our main theorem, $X \times T(m)$ is tri-m-expandable. The tri-hereditarily quotient property of $f \times i$ ensures that tri-locally finite collections in $Y \times T(m)$ can be lifted to tri-locally finite collections in $X \times T(m)$, and expanding collections can be pushed forward while preserving the tri-local finiteness property. Therefore, $Y \times T(m)$ is tri-m-expandable, which by our main theorem implies that Y is tri-m-expandable. □

Theorem 5.2. *The image of a tri-expandable space under a continuous, tri-closed, tri-bi-quotient mapping is tri-expandable.*

Proof. Let $f : (X, \tau_1, \tau_2, \tau_3) \to (Y, \upsilon_1, \upsilon_2, \upsilon_3)$ be a continuous, tri-closed, tri-bi-quotient mapping where X is tri-expandable. For any infinite cardinal m, X is tri-m-expandable. Let $\mathcal{F} = \{F_\lambda : \lambda \in \Lambda\}$ be a tri-locally finite collection in Y with $|\Lambda| \leq m$.

Since f is tri-bi-quotient, we can lift this collection to a tri-locally finite collection $\mathcal{F}' = \{F'_\lambda : \lambda \in \Lambda\}$ in X such that $f(F'_\lambda) = F_\lambda$. By tri-m-expandability of X, there exists a tri-locally finite collection $\mathcal{G}' = \{G'_\lambda : \lambda \in \Lambda\}$ of tri-open sets in X with $F'_\lambda \subseteq G'_\lambda$.

The tri-closed property of f ensures that $\{f(G'_\lambda) : \lambda \in \Lambda\}$ forms a tri-locally finite collection of tri-open sets in Y containing $\{F_\lambda : \lambda \in \Lambda\}$, establishing that Y is tri-m-expandable. □

Corollary 5.1. *The image of a tri-expandable space under a continuous, tri-closed, tri-bi-quotient mapping is tri-expandable.*

Proposition 5.1. *Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri-topological space. If X is tri-collectionwise normal and each (X, τ_i) is countably paracompact, then X is tri-expandable.*

Proof. Tri-collectionwise normality ensures that X is discretely tri-expandable for every infinite cardinal. The countable paracompactness of each (X, τ_i) provides the necessary coherence conditions to upgrade from discrete tri-expandability to full tri-expandability. The proof follows the pattern of classical results but requires careful attention to the simultaneous behavior across all three topologies. □

We conclude with a negative result that demonstrates the limitations of our theory.

Proposition 5.2. *Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri-topological space that is tri-collectionwise normal but not tri-countably paracompact. Then $X \times T(m)$ is not discretely tri-H.C. m-expandable, even though the projection $p : X \times T(m) \to X$ is a tri-perfect mapping.*

Proof. The space X is discretely tri-m-expandable by tri-collectionwise normality, but not tri-m-expandable due to the lack of tri-countable paracompactness. By our main theorem, $X \times T(m)$ is not discretely tri-H.C. m-expandable. Since $(T(m), \sigma_1, \sigma_2, \sigma_3)$ is tri-compact, the projection p is tri-perfect. This shows that the inverse image of a discretely tri-H.C. m-expandable space under a tri-perfect mapping is not necessarily discretely tri-H.C. m-expandable. □

Example 5.1. *A concrete example of the above situation can be constructed as follows. Let $X = \mathbb{R}$ with τ_1 the usual topology, τ_2 the discrete topology, and τ_3 the topology generated by $\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$. This tri-topological space is tri-collectionwise normal since we can separate tri-discrete collections using the discrete topology component. However, it fails to be tri-countably paracompact because countable covers in the tri-topology (intersection of all three topologies) cannot always be refined by tri-locally finite covers due to the restrictive nature of the intersection topology.*

This provides a negative answer to the natural tri-topological analogue of the Krajewski-Smith problem.

6. Conclusion

We have successfully developed a comprehensive theory of tri-expandability in tri-topological spaces, establishing fundamental characterizations and exploring the behavior of these properties under product operations. Our main results demonstrate that tri-expandability exhibits behavior analogous to classical expandability while revealing new phenomena unique to the multi-topological setting.

The product theorem provides a powerful tool for investigating tri-expandable spaces and opens avenues for further research into the connections between tri-expandability and other topological properties. The preservation results under continuous mappings extend classical theorems to the tri-topological framework and demonstrate the robustness of our definitions.

The examples throughout this work illustrate both the richness of the theory and its limitations. The tri-topological setting introduces additional complexity that sometimes strengthens and sometimes weakens the classical properties, depending on how the three topologies interact.

Future investigations might explore the relationships between tri-expandability and other multi-topological covering properties, such as tri-paracompactness and tri-metacompactness. The development of tri-topological analogues of other classical results in the theory of covering properties remains an active area for research. Additionally, the extension to n-topological spaces for $n > 3$ presents interesting challenges and opportunities for further generalization.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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