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The Doubly Generalized Weibull Power Series Frailty Distribution

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Abstract. This paper introduces the doubly generalized Weibull power series frailty (DGWPSF) model, an extension of the Weibull–k-truncated power series family incorporating a gamma-frailty term. The model enhances flexibility for lifetime data by accommodating unobserved heterogeneity and latent risk factors in survival and reliability studies. We derive its fundamental properties, including the probability density, distribution, survival, and hazard functions, and highlight notable special cases, including the binomial, Poisson, geometric, and logarithmic models. To address the challenges of parameter estimation, we develop the expectation-maximization algorithm and the Bayesian inference procedures. The DGWPSF framework offers a flexible structure for lifetime data analysis, capturing diverse frailty patterns while improving model interpretability and robustness.

1. Introduction

The Weibull distribution is a fundamental tool in reliability and survival analysis, as its shape parameter allows it to capture a wide range of hazard rate behaviors [1]. Despite its versatility, it has limitations in modeling complex system dynamics, such as varying numbers of components, which has motivated the development of numerous extensions [2, 3]. One notable advancement is the generalized Weibull-power series distribution with left k-truncation (GWPS), proposed by Rahmouni [4]. This model combines the k-th order statistic of the Weibull distribution with a left k-truncated power series, providing enhanced flexibility for modeling ordered failure times in multi-component systems. However, the GWPS model assumes homogeneity across units, limiting its applicability to heterogeneous populations. Many real-world scenarios require models that account for evolving truncation thresholds and unobserved heterogeneity. To address these limitations, we introduce a family of generalized Weibull power series frailty models, which extend the GWPS framework by incorporating frailty terms to capture unobserved variability and correlations among lifetimes.

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The inclusion of frailty in these models is motivated by the need to address unobserved heterogeneity and dependence structures in complex systems, which are often inadequately captured by standard distributional assumptions. Frailty terms, typically modeled via a random effect such as a gamma-distributed variable, account for unobserved factors—such as latent patient characteristics in medical trials, varying material properties in engineering components, or systemic risks in financial portfolios—that influence failure times [5]. By incorporating frailty, these models capture correlated lifetimes within clusters or groups, enhance the flexibility of hazard rate modeling, and improve prediction accuracy in heterogeneous populations [6].

This study builds upon the theoretical foundations of the compound class of Weibull-power series distributions [4,7,8] and the extensive literature on frailty modeling [5,6], and proposes a flexible framework for analyzing complex lifetime data. The remainder of the paper is organized as follows. Section 2 presents the proposed family of models, derives their key distributional functions, and outlines several important special cases, including the doubly generalized Weibull-geometric frailty (DGWGF), Poisson frailty (DGWPF), logarithmic frailty (DGWLF), and binomial frailty (DGWBF) models. Section 3 examines the mathematical properties of the family, such as hazard rate behavior and moments. Estimation procedures are developed in Section 4, while Section 5 summarizes the main findings and discusses potential avenues for future research.

2. The distribution

The doubly generalized Weibull power series frailty (DGWPSF) distribution models the k-th order statistic $Y = X_{(k)}$ drawn from a sample of random size N, incorporating a latent frailty variable Z. The k-th order statistic $X_{(k)}$ is derived from a Weibull-distributed sample, the sample size N follows a k-truncated power series distribution, and Z is a gamma-distributed frailty that captures unobserved heterogeneity. The truncation point k may vary with context, allowing the model to represent dynamic or condition-dependent sampling mechanisms.

Theorem 2.1 (PDF and CDF of the DGWPSF distribution)**.** *Let $Y = X_{(k)}$ denote the k-th order statistic from a sample of random size $N \geq k$, where each observation is subject to a shared gamma frailty $Z \sim Gamma(\lambda, \lambda)$ and follows a Weibull distribution conditional on Z. If N follows a k-truncated power series distribution defined by coefficients $a_n \geq 0$ and parameter $\eta > 0$ (with $\eta \in (0,1)$ for the geometric and binomial cases), then:*

- *The PDF of Y is:*

$$
\begin{aligned}
f_Y(y;\beta,\theta,\eta,k,\lambda) = \frac{k\beta\theta^{\beta}y^{\beta-1}\lambda^{\lambda+1}}{C_k(\eta)} \sum_{n=k}^{\infty} a_n\eta^n \binom{n}{k} \\
\times \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left([n-k+j+1](\theta y)^{\beta}+\lambda\right)^{-(\lambda+1)}, \quad y>0.
\end{aligned}
\tag{2.1}
$$

- *The CDF of Y is:*

$$F_Y(y;\beta,\theta,\eta,k,\lambda) = \lambda^{\lambda} \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \sum_{j=k}^{n} \binom{n}{j} \sum_{m=0}^{j} \binom{j}{m} (-1)^m \left([m+n-j](\theta y)^{\beta} + \lambda\right)^{-\lambda}, \tag{2.2}$$

where $\beta, \theta, \lambda > 0$ are the shape, scale, and frailty parameters, respectively; $k \in \mathbb{N}$ is the truncation point; and $C_k(\eta) = \sum_{n=k}^{\infty} a_n \eta^n$ is the normalizing constant.

Proof. Assume $X \mid Z = z \sim \text{Weibull}(\beta, \theta z^{-1/\beta})$, so that:

$$F_X(y \mid Z = z) = 1 - e^{-z(\theta y)^{\beta}}, \quad f_X(y \mid Z = z) = \beta \theta^{\beta} y^{\beta-1} z e^{-z(\theta y)^{\beta}}.$$

The conditional PDF of $X_{(k)}$ for $N = n$ and $Z = z$ is:

$$f_{X_{(k)}|Z=z}(y) = k \binom{n}{k} \left(1 - e^{-z(\theta y)^{\beta}}\right)^{k-1} e^{-(n-k)z(\theta y)^{\beta}} z \beta \theta^{\beta} y^{\beta-1}.$$

The marginal PDF is obtained by integrating over $Z \sim \text{Gamma}(\lambda, \lambda)$ and summing over N:

$$f_Y(y) = \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \int_0^{\infty} f_{X_{(k)}|Z=z}(y) \frac{\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z}}{\Gamma(\lambda)} dz,$$

where

$$g(z;\lambda) = \frac{\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z}}{\Gamma(\lambda)}, \quad z > 0,\ \lambda > 0.$$

Using the binomial expansion for $(1 - e^{-z(\theta y)^{\beta}})^{k-1}$ and evaluating the gamma integral using standard identities (e.g., $\int_0^{\infty} z^a e^{-bz} dz = \Gamma(a+1)/b^{a+1}$), we obtain the PDF. The CDF is derived similarly (see Appendix A1 for more details).

Remark 2.1 (Doubly generalized exponential power series frailty distribution (DGEPSF))**.** *When the shape parameter $\beta = 1$, the Weibull distribution reduces to an exponential distribution with rate parameter θ, yielding the DGEPSF model. This special case is particularly suited for applications requiring a constant failure rate, such as reliability analysis of systems with exponential lifetimes or survival studies with homogeneous hazard rates. The model retains the flexibility of the DGWPSF framework through the gamma frailty $Z \sim \text{Gamma}(\lambda, \lambda)$ and the random sample size N governed by a k-truncated power series distribution with coefficients $a_n \geq 0$ and parameter $\eta > 0$. The PDF and CDF of the DGEPSF model are given by:*

- *The PDF:*

$$f_Y(y;\theta,\eta,k,\lambda) = \theta \lambda^{\lambda+1} k \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \left([j+n-k+1]\theta y + \lambda\right)^{-(\lambda+1)}, \tag{2.3}$$

- *The CDF:*

$$F_Y(y;\theta,\eta,k,\lambda) = \lambda^{\lambda} \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \sum_{j=k}^{n} \binom{n}{j} \sum_{m=0}^{j} \binom{j}{m} (-1)^m \left([m+n-j]\theta y + \lambda\right)^{-\lambda}, \tag{2.4}$$

2.1. **Special cases.** The DGWPSF distribution admits a number of tractable special cases, determined by the choice of the underlying power series distribution that governs the random sample size N. Each case is characterized by a specific sequence of coefficients a_n, a corresponding normalizing constant $C_k(\eta) = \sum_{n=k}^{\infty} a_n \eta^n$, and a weight function $\frac{a_n \eta^n}{C_k(\eta)}$ representing the probability mass of $N = n$ under k-truncation. Table 1 summarizes some notable special cases, including the doubly generalized Weibull-geometric frailty (DGWGF), Poisson frailty (DGWPF), logarithmic frailty (DGWLF), and binomial frailty (DGWBF) models. These variations illustrate the flexibility of the DGWPSF family of distributions and highlight how different discrete distributions for N affect the resulting frailty-adjusted order statistic distribution.

Table 1. Special cases of the DGWPSF distribution

Model	a_n	$C_k(\eta)$	Weight: $\frac{a_n\eta^n}{C_k(\eta)}$	η
DGWGF (Geometric)	1	$\frac{\eta^k}{1-\eta}$	$(1-\eta)\eta^{n-k}$	$\eta \in (0,1)$
DGWPF (Poisson)	$\frac{1}{n!}$	$e^{\eta} - \sum_{j=0}^{k-1} \frac{\eta^j}{j!}$	$\frac{\eta^n}{n!\left(e^{\eta}-\sum_{j=0}^{k-1}\frac{\eta^j}{j!}\right)}$	$\eta \in (0,\infty)$
DGWLF (Logarithmic)	$\frac{1}{n}$	$-\log(1-\eta) - \varphi(k)\sum_{j=1}^{k-1}\frac{\eta^j}{j}$	$\frac{\eta^n}{n\left(-\log(1-\eta)-\varphi(k)\sum_{j=1}^{k-1}\frac{\eta^j}{j}\right)}$	$\eta \in (0,1)$
DGWBF (Binomial)	$\binom{N}{n}$	$\sum_{n=k}^{N}\binom{N}{n}\eta^n$	$\frac{\binom{N}{n}\eta^n}{\sum_{m=k}^{N}\binom{N}{m}\eta^m}$	$\eta \in (0,1)$

Note: For DGWLF, define $\varphi(k) = 1$ if $k \geq 2$, and 0 otherwise. For DGWBF, $N \geq k$ denotes the maximum (fixed) sample size.

In the following subsections, we examine these models in more detail, beginning with the doubly generalized Weibull-geometric frailty (DGWGF) distribution.

2.1.1. *Doubly generalized Weibull-geometric-frailty (DGWGF).* For the geometric case where $a_n = 1$ and $\eta \in (0,1)$, the normalizing constant becomes $C_k(\eta) = \frac{\eta^k}{1-\eta}$, and the corresponding weight is $(1-\eta)\eta^{n-k}$. This setting induces a geometric frailty mechanism, introducing additional heterogeneity into the survival model through the latent counting structure of N.

The PDF of the DGWGF distribution is given by:

$$f_Y(y) = k(1-\eta)\beta\theta^{\beta}y^{\beta-1}\lambda^{\lambda+1}\sum_{n=k}^{\infty}\binom{n}{k}\eta^{n-k}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left([n-k+j+1](\theta y)^{\beta}+\lambda\right)^{-(\lambda+1)}.$$

The geometric-frailty specialization of the DGWPSF distribution with $\beta = 1$, $k = 1$, and a latent sample size N following a geometric distribution, $\Pr(N = n) = (1-\eta)\eta^{n-1}$ for $n = 1, 2, \ldots$ transforms the model into a two-layer mixture. Conditional on $N = n$ and Z, the minimum of n i.i.d. $\mathrm{Exp}(\theta Z)$ lifetimes is $\mathrm{Exp}(n\theta Z)$. Integrating out the gamma frailty Z yields a Pareto–Lomax kernel of the form $\frac{n\theta\lambda^{\lambda+1}}{(\lambda+n\theta y)^{\lambda+1}}$, and mixing over N with geometric weights $(1-\eta)\eta^{n-1}$ produces closed-form series expressions for the PDF and CDF. This transformation simplifies the normalizing constant and provides a clear probabilistic interpretation: Y represents the minimum lifetime of a geometrically distributed number of exponentially frail components. The resulting series forms are tractable for applications such as reliability and survival analysis and may be approximated or summed in closed form for specific parameter regimes.

$$f_Y(y) = (1-\eta)\theta\lambda^{\lambda+1}\sum_{n=1}^{\infty} n\eta^{n-1}\Big(\lambda+n\theta y\Big)^{-(\lambda+1)}, \quad y>0. \tag{2.5}$$

$$F_Y(y) = (1-\eta)\sum_{n=1}^{\infty}\eta^{n-1}\left[1-\left(\frac{\lambda}{\lambda+n\,\theta\,y}\right)^{\lambda}\right], \quad y>0. \tag{2.6}$$

Equivalently,

$$F_Y(y) = 1-(1-\eta)\lambda^{\lambda}\sum_{n=1}^{\infty}\eta^{n-1}(\lambda+n\theta y)^{-\lambda}$$

These expressions highlight the influence of the geometric structure in the power series formulation, where the parameter η plays a key role in regulating the tail behavior and overall dispersion of the distribution. Figure 1 displays the PDF of the DGWGF distribution for various combinations of the shape parameter β and the frailty parameter λ, while keeping $\theta=1, k=4$, and $\eta=0.5$ fixed. The plots demonstrate that smaller values of β lead to heavy-tailed, monotonically decreasing densities, whereas larger values of β yield unimodal, right-skewed shapes with increasing sharpness. Likewise, increasing λ results in more peaked and concentrated distributions, reflecting reduced unobserved heterogeneity due to frailty. These dynamics confirm the DGWGF model's flexibility in modeling diverse failure time behaviors, capturing variations in both the baseline hazard and latent frailty components.

2.1.2. *Doubly generalized Weibull-Poisson-frailty (DGWPF).* In this special case, the underlying power series distribution is Poisson, with coefficients $a_n=\frac{1}{n!}$ and normalizing constant

$$C_k(\eta) = e^{\eta}-\sum_{j=0}^{k-1}\frac{\eta^j}{j!}, \quad \eta>0.$$

This yields weights of the form:

$$\frac{\eta^n}{n!\left(e^{\eta}-\sum_{j=0}^{k-1}\frac{\eta^j}{j!}\right)}.$$

The PDF of the DGWPF distribution is given by:

$$f_Y(y) = k\beta\theta^{\beta}y^{\beta-1}\lambda^{\lambda+1}\sum_{n=k}^{\infty}\binom{n}{k}\frac{\eta^n}{n!\left(e^{\eta}-\sum_{r=0}^{k-1}\frac{\eta^r}{r!}\right)}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left([j+n-k+1]\theta y+\lambda\right)^{-(\lambda+1)}.$$

For the special case with $k=1$ and $\beta=1$, the PDF simplifies to:

$$f_Y(y) = \frac{\theta\lambda^{\lambda+1}}{e^{\eta}-1}\sum_{n=1}^{\infty}\frac{\eta^n}{(n-1)!}\cdot(n\theta y+\lambda)^{-(\lambda+1)}, \quad y>0,$$

and the CDF becomes:

$$F_Y(y) = \frac{\lambda^{\lambda}}{e^{\eta}-1}\sum_{n=1}^{\infty}\frac{\eta^n}{n!}\sum_{j=1}^{n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}(-1)^m\left([m+n-j]\theta y+\lambda\right)^{-\lambda}, \quad y>0$$

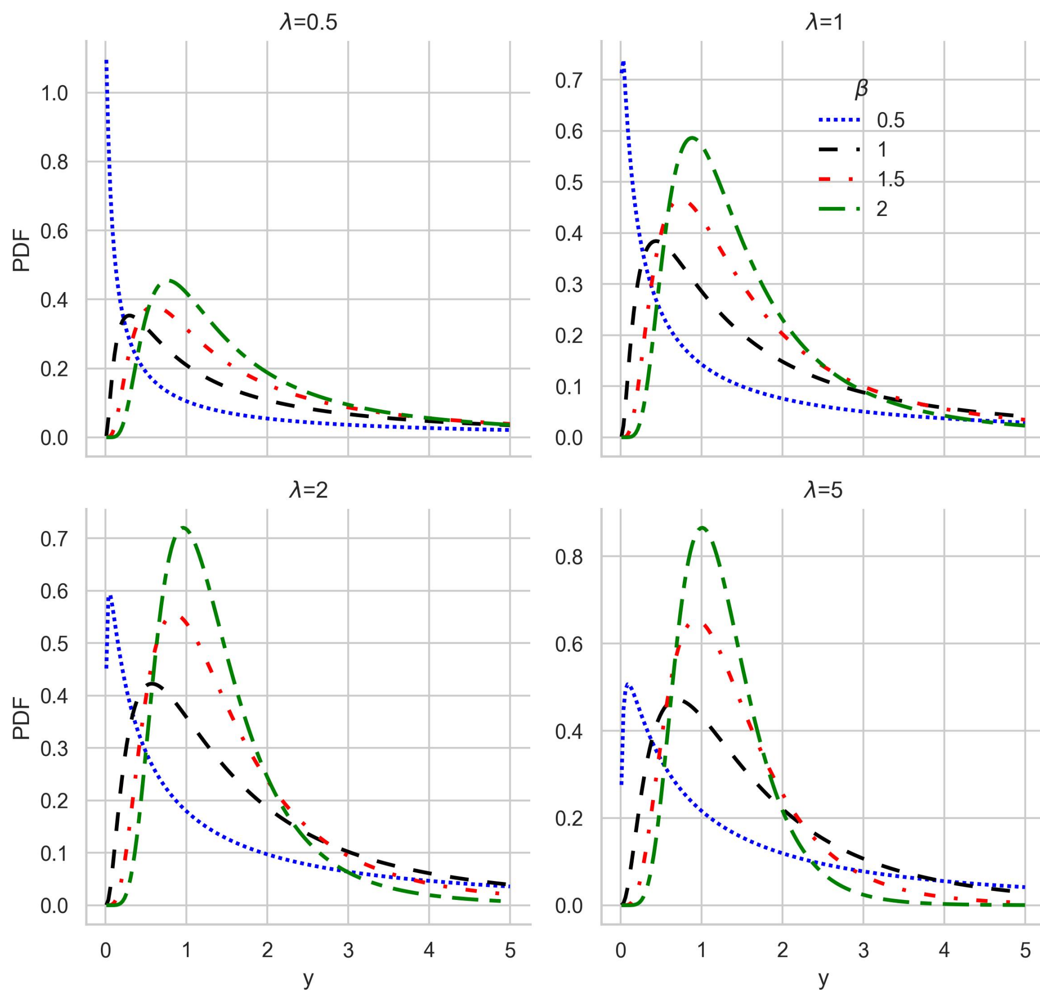

FIGURE 1. PDF of the DGWGF distribution for $k = 4$, varying shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$, with $\theta = 1$ and $\eta = 0.5$.

The integrated marginal PDF form simplifies the CDF to:

$$F_Y(y) = \frac{1}{e^{\eta} - 1} \sum_{n=1}^{\infty} \frac{\eta^n}{n!} \left(1 - \left(\frac{\lambda}{n\theta y + \lambda} \right)^{\lambda} \right).$$

This model introduces Poisson-based frailty into the survival structure, where the stochastic number of latent risks follows a truncated Poisson distribution, thereby enriching the model's flexibility in capturing real-world heterogeneity.

Figure 2 illustrates the PDF of the DGWPF distribution under various combinations of the shape parameter β and frailty parameter λ, with fixed values $\theta = 1$, $k = 4$, and $\eta = 0.5$. The plot reveals how the density shape is modulated by these parameters. As λ increases, the distribution becomes more concentrated around its mode, indicating reduced heterogeneity due to frailty. Lower values of β produce heavier tails, while higher values shift the mode to the right and result in sharper peaks. This behavior highlights the flexibility of the DGWPF model in capturing a wide range of

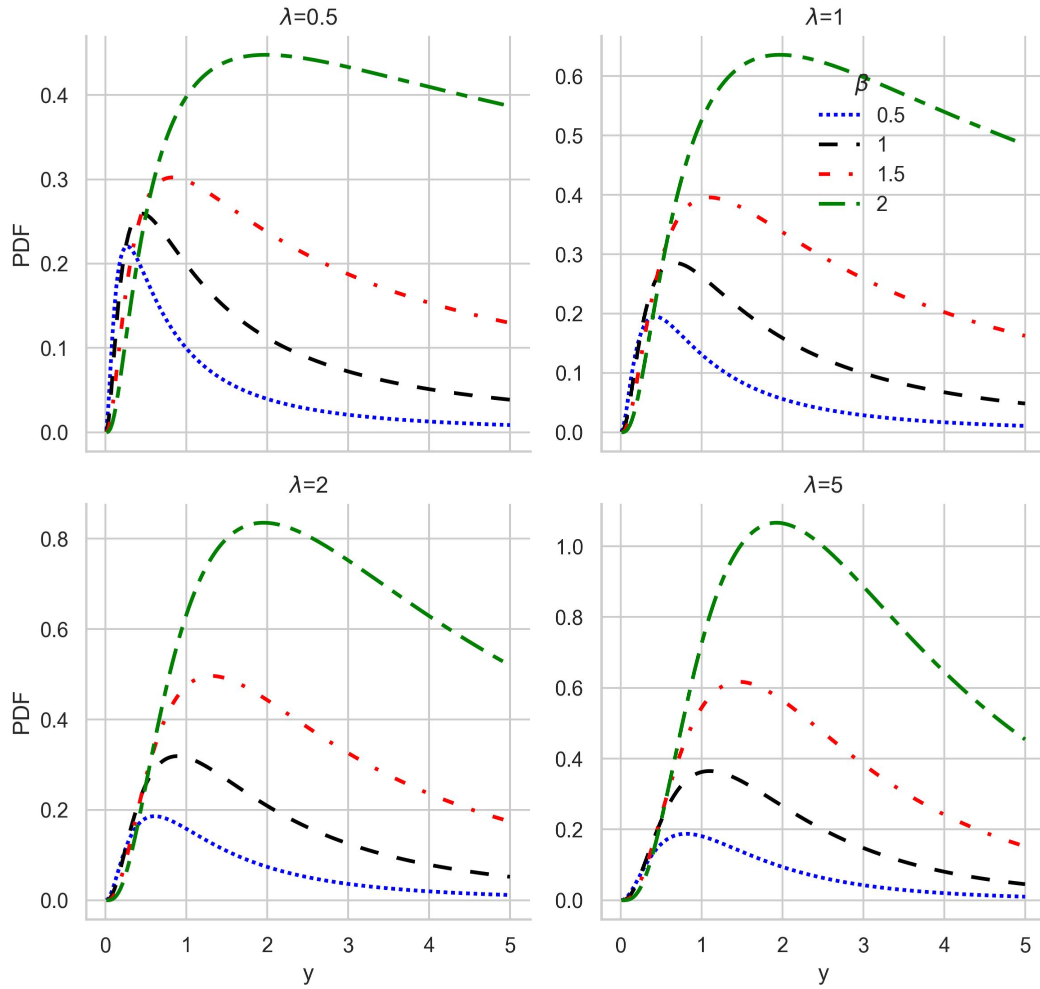

FIGURE 2. PDF of the DGWPF distribution for $k = 4$, with varying shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$, fixing $\theta = 1$ and $\eta = 0.5$.

failure time behaviors, especially in overdispersed or heterogeneous survival data, due to its latent Poisson frailty component.

2.1.3. *Doubly generalized Weibull-logarithmic-frailty (DGWLF).* This special case corresponds to compounding with a truncated logarithmic distribution, characterized by coefficients $a_n = \frac{1}{n}$, with support $n \geq k$, and normalization constant:

$$C_k(\eta) = -\log(1-\eta) - \varphi(k) \sum_{j=1}^{k-1} \frac{\eta^j}{j}, \quad \eta \in (0,1),$$

where $\varphi(k) = 1$ for $k \geq 2$, and $\varphi(k) = 0$ otherwise, to properly account for the truncation of lower-order terms. The associated weights are:

$$\frac{\eta^n}{n\left(-\log(1-\eta) - \varphi(k) \sum_{j=1}^{k-1} \frac{\eta^j}{j}\right)}.$$

The PDF of the DGWLF model is:

$$f_Y(y) = \frac{k\beta\theta^\beta y^{\beta-1}\lambda^{\lambda+1}}{-\log(1-\eta) - \varphi(k)\sum_{r=1}^{k-1}\frac{\eta^r}{r}} \sum_{n=k}^{\infty} \frac{\eta^n}{n}\binom{n}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left([n-k+j+1](\theta y)^\beta + \lambda\right)^{-(\lambda+1)}, \quad y > 0.$$

For the special case where $k = 1$ and $\beta = 1$, the PDF simplifies to:

$$f_Y(y) = \frac{\theta\lambda^{\lambda+1}}{-\log(1-\eta)} \sum_{n=1}^{\infty} \eta^n \left(n\theta y + \lambda\right)^{-(\lambda+1)}, \quad y > 0,$$

and the CDF to:

$$F_Y(y) = \frac{\lambda^\lambda}{-\log(1-\eta)} \sum_{n=1}^{\infty} \frac{\eta^n}{n} \sum_{j=1}^{n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}(-1)^m \left([m+n-j]\theta y + \lambda\right)^{-\lambda}, \quad y > 0.$$

This expression exactly matches the binomial expansion form of the CDF and is equivalent to the simpler version:

$$F_Y(y) = \frac{1}{-\log(1-\eta)} \sum_{n=1}^{\infty} \frac{\eta^n}{n}\left(1 - \left(\frac{\lambda}{n\theta y + \lambda}\right)^\lambda\right), \quad y > 0$$

Figure 3 illustrates the PDF of the DGWLF distribution for various combinations of the shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$, with fixed values $\theta = 1$, $k = 4$, and $\eta = 0.5$. The figure demonstrates that increasing β leads to a transition from heavy-tailed, monotonically decreasing densities to unimodal, right-skewed forms with sharper peaks, indicating more concentrated failure times. On the other hand, the frailty parameter λ influences the dispersion of the distribution: smaller values of λ yield flatter, more dispersed shapes due to higher unobserved heterogeneity, while larger λ values result in more peaked and concentrated densities. These patterns emphasize the flexibility of the DGWLF model in capturing diverse failure time behaviors driven by both shape and frailty effects.

2.1.4. *Doubly generalized Weibull-binomial-frailty (DGWBF).* The DGWBF model arises when the compounding distribution is a truncated binomial distribution with fixed upper bound $N \geq k$. The corresponding power series coefficients are $a_n = \binom{N}{n}$, and the normalizing constant is:

$$C_k(\eta) = \sum_{n=k}^{N}\binom{N}{n}\eta^n, \quad \eta > 0.$$

The weights for each component are given by:

$$\frac{\binom{N}{n}\eta^n}{\sum_{m=k}^{N}\binom{N}{m}\eta^m}.$$

The resulting PDF of the DGWBF model is:

$$f_Y(y) = \frac{k\beta\theta^\beta y^{\beta-1}\lambda^{\lambda+1}}{\sum\limits_{m=k}^{N}\binom{N}{m}\eta^m} \sum_{n=k}^{N}\binom{N}{n}\eta^n\binom{n}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left([n-k+j+1](\theta y)^\beta + \lambda\right)^{-(\lambda+1)}, \quad y > 0.$$

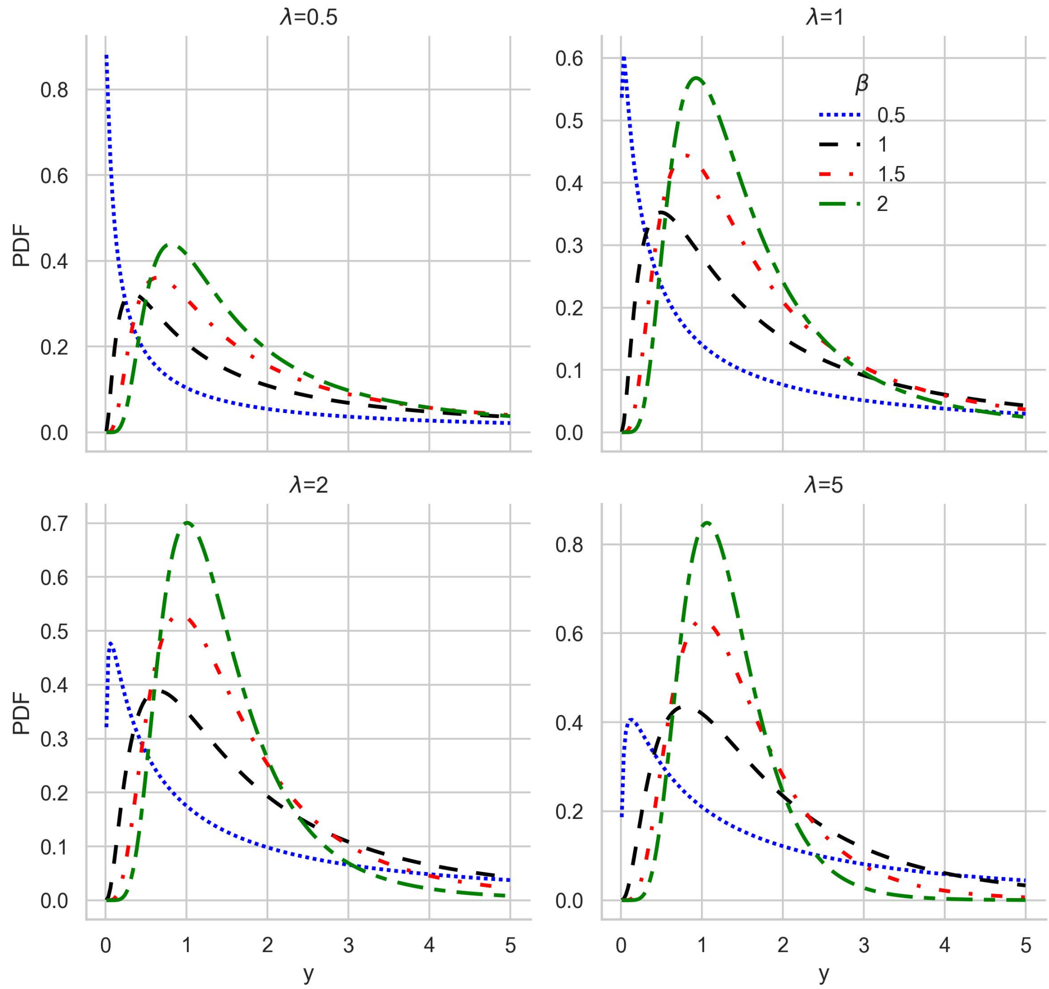

Figure 3. PDF of the DGWLF distribution for $k = 4$, varying shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$ with $\theta = 1$ and $\eta = 0.5$.

where $y > 0$, and β, θ, λ are positive parameters. The binomial compounding leads to a finite mixture, making the model particularly useful for bounded or discrete population settings.

For the simplified scenario where $k = 1$ and $\beta = 1$, the PDF reduces to:

$$f_Y(y) = \frac{\theta\lambda^{\lambda+1}}{\sum\limits_{m=1}^{N} \binom{N}{m}\eta^m} \sum_{n=1}^{N} n\eta^n \binom{N}{n} (n\theta y + \lambda)^{-(\lambda+1)},$$

and the CDF of the DGWBF distribution simplifies to:

$$F_Y(y) = \frac{1}{\sum\limits_{n=1}^{N} \binom{N}{n}\eta^n} \sum_{n=1}^{N} \binom{N}{n}\eta^n \left(1 - \left(\frac{\lambda}{n\theta y + \lambda}\right)^{\lambda}\right).$$

This case illustrates the influence of discrete binomial frailty on survival behavior, introducing non-monotonicity and potentially multimodal behavior depending on N and η.

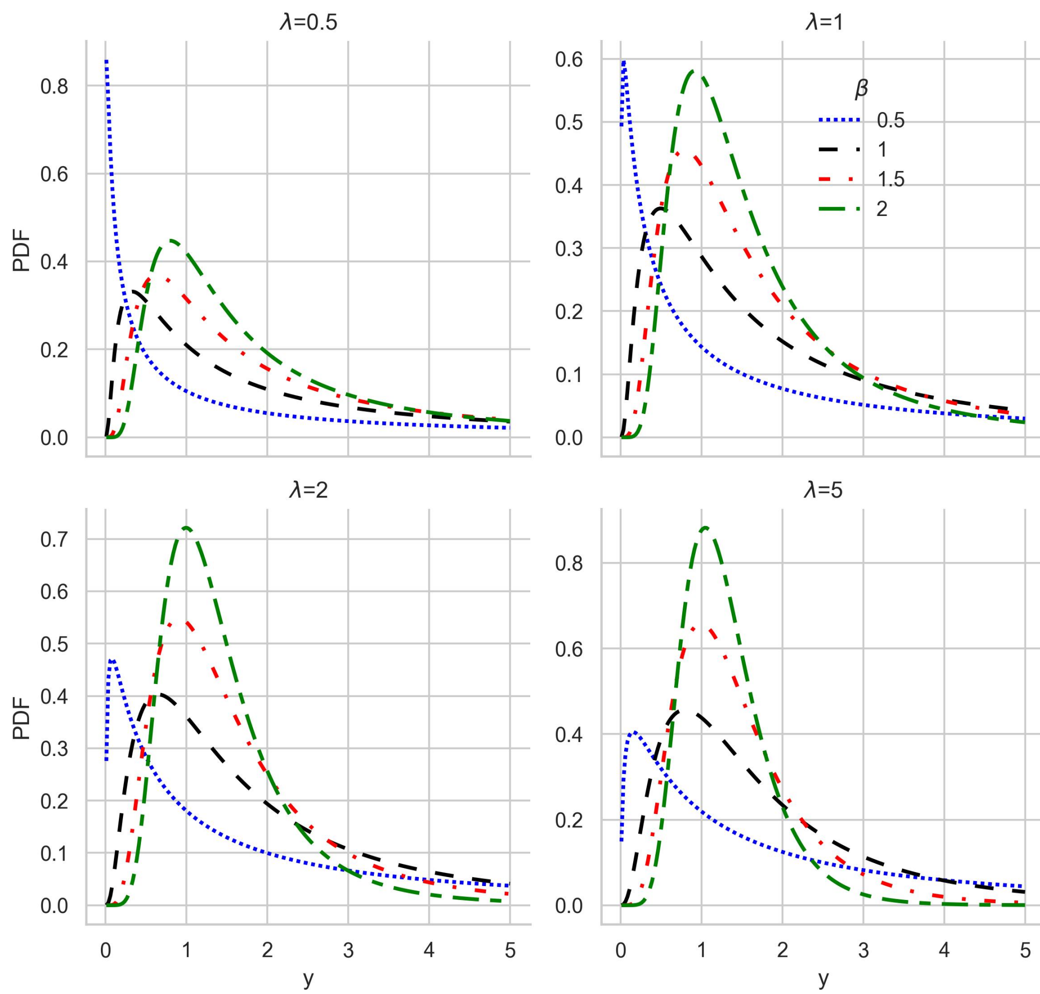

FIGURE 4. PDF of the DGWBF distribution for $k = 4$, $N = 10$, varying shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$ with $\theta = 1$ and $\eta = 0.5$.

Figure 4 presents PDF of the DGWBF distribution for varying values of the shape parameter $\beta \in \{0.5, 1, 1.5, 2\}$ and frailty parameter $\lambda \in \{0.5, 1, 2, 5\}$, with fixed parameters $\theta = 1$, $k = 4$, and $\eta = 0.5$. The plots reveal that the shape parameter β significantly affects the modality and skewness of the distribution: lower values of β result in highly right-skewed, heavy-tailed distributions, while higher values produce more pronounced peaks and increased concentration around the mode. Additionally, as the frailty parameter λ increases, the densities become more peaked and less dispersed, indicating reduced heterogeneity in the population. These trends underscore the DGWBF model's capacity to flexibly model a range of lifetime data characteristics by adjusting both baseline shape and latent frailty influences.

3. Properties

This section derives the moments, quantiles, hazard rate function (HRF), survival function, and moment-generating function (MGF), providing insights into the model's behavior and computational considerations.

3.1. **Moments.** The r-th moment of $Y = X_{(k)}$, denoted $\mathbb{E}[Y^r]$, is obtained by integrating the conditional expectation over the frailty Z and summing over the sample size N:

$$\mathbb{E}[Y^r] = \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \int_0^{\infty} \mathbb{E}[Y^r \mid N = n, Z = z] \cdot \frac{\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z}}{\Gamma(\lambda)} \, \mathrm{d}z.$$

The conditional moment $\mathbb{E}[Y^r \mid N = n, Z = z]$ is derived for Y, the k-th order statistic from n i.i.d. Weibull variables with frailty Z. The PDF of the k-th order statistic is:

$$f_{X_{(k)}}(y \mid N = n, Z = z) = k\binom{n}{k} z\beta\theta^{\beta} y^{\beta-1} \left(1 - e^{-z(\theta y)^{\beta}}\right)^{k-1} e^{-z(n-k+1)(\theta y)^{\beta}}.$$

Thus, the conditional moment is:

$$\mathbb{E}[Y^r \mid N = n, Z = z] = k\binom{n}{k} z\beta\theta^{\beta} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \int_0^{\infty} y^{r+\beta-1} e^{-z(n-k+j+1)(\theta y)^{\beta}} \, \mathrm{d}y.$$

Substitute $u = z(\theta y)^{\beta}$, so $y = \left(\frac{u}{z\theta^{\beta}}\right)^{1/\beta}$, $\mathrm{d}y = \frac{1}{\beta}\left(\frac{u}{z\theta^{\beta}}\right)^{1/\beta-1} \frac{\mathrm{d}u}{z\theta^{\beta}}$, and let $\tau = n - k + j + 1$. Thus

$$\int_0^{\infty} y^{r+\beta-1} e^{-z\tau(\theta y)^{\beta}} \, \mathrm{d}y = \frac{1}{\beta} (z\theta^{\beta})^{-r/\beta-1} \tau^{-r/\beta-1} \Gamma\left(\frac{r}{\beta} + 1\right),$$

and

$$\mathbb{E}[Y^r \mid N = n, Z = z] = k\binom{n}{k} \theta^{-r} z^{-r/\beta} \Gamma\left(\frac{r}{\beta} + 1\right) \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \tau^{-r/\beta-1}.$$

Integrating over the frailty Z:

$$\int_0^{\infty} z^{-r/\beta} \cdot \frac{\lambda^{\lambda} z^{\lambda-1} e^{-\lambda z}}{\Gamma(\lambda)} \, \mathrm{d}z = \lambda^{r/\beta} \frac{\Gamma(\lambda - r/\beta)}{\Gamma(\lambda)}.$$

We obtain:

$$\mathbb{E}[Y^r \mid N = n] = k\binom{n}{k} \theta^{-r} \lambda^{r/\beta} \Gamma\left(\frac{r}{\beta} + 1\right) \frac{\Gamma(\lambda - r/\beta)}{\Gamma(\lambda)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (n - k + j + 1)^{-r/\beta-1}.$$

The r-th moment is:

$$\mathbb{E}[Y^r] = k\theta^{-r} \lambda^{r/\beta} \Gamma\left(\frac{r}{\beta} + 1\right) \frac{\Gamma(\lambda - r/\beta)}{\Gamma(\lambda)} \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (n - k + j + 1)^{-r/\beta-1}.$$

The moment exists for $r < \beta\lambda$, ensuring $\Gamma(\lambda - r/\beta)$ is defined [9,10]. The convergence of the infinite sum depends on the weights a_n and the parameter η. The first moment with $r = 1$, $k = 1$, and $\beta = 1$ is:

$$\mathbb{E}[Y] = \theta^{-1} \frac{\lambda}{\lambda - 1} \sum_{n=1}^{\infty} \frac{a_n \eta^n \binom{n}{1}}{C_1(\eta)} \cdot \frac{1}{n^2} = \frac{1}{\theta} \frac{\lambda}{\lambda - 1} \frac{1}{C_1(\eta)} \sum_{n=1}^{\infty} \frac{a_n \eta^n}{n}, \quad \lambda > 1.$$

3.2. **Quantile function.** The quantile function $Q(p)$ of the random variable Y satisfies:

$$F_Y(Q(p); \beta, \theta, \eta, k, \lambda) = p,$$

Due to the complexity of the CDF $F_Y(y)$, which involves infinite summations and nested binomial expansions, the quantile function $Q(p) = F_Y^{-1}(p)$ generally lacks a closed-form expression and should be computed numerically. Quantile estimation for the DGWPSF requires solving the nonlinear equation $F_Y(y) = p$ for a given cumulative probability level $p \in (0, 1)$. Because $F_Y(y)$ comprises infinite or truncated series without analytic inverses, root-finding algorithms such as Newton-Raphson, bisection, or Brent's method are typically employed to obtain accurate approximations of $Q(p)$.

While the special-case models share a common conditional structure for the kth order statistic given frailty and sample size, they differ in the specification of the random variable N. This affects the shape and tail behavior of the marginal distribution of Y. For instance, increasing η in the Poisson case can shift more mass toward larger N, causing the small-p quantiles of DGWPF to initially dominate those of DGWGF and later fall below them as the frailty effect stabilizes. This is illustrated in Figure 5, where DGWGF, DGWPF, DGWLF, and DGWBF are compared under varying power series parameters. The results highlight the importance of the power series component in shaping early failure behavior, a key concern in survival and reliability contexts.

To gain further theoretical insight, small-p quantile behavior can be studied via asymptotic approximations. In the special case $k = 1$ (i.e., the time to first failure), the CDF admits the approximation (see Appendix A2 for more details):

$$F_Y(y) \approx C \cdot y^{\beta}, \quad \text{as } y \to 0,$$

where the constant C depends on the parameters $(\theta, \beta, \eta, k, \lambda)$ and the form of the power series governing N. Inverting this expression yields an approximate formula for the small-p quantiles:

$$y_p \approx \left(\frac{p}{C}\right)^{1/\beta}, \quad \text{for small } p.$$

This approximation is specific to $k = 1$; for $k \geq 2$, the leading-order behavior near zero is proportional to $y^{k\beta}$.

3.3. **Moment-generating function.** The moment-generating function (MGF) of the DGWPSF random variable Y is defined by

$$M_Y(t) \;=\; \mathbb{E}\Big[e^{tY}\Big] \;=\; \int_0^{\infty} e^{ty} f_Y(y)\, \mathrm{d}y,$$

where $f_Y(y)$ is the PDF given in Equation (2.1). Substituting the series representation of $f_Y(y)$ and interchanging sum and integral (under suitable convergence conditions) yields

$$M_Y(t) = \sum_{n=k}^{\infty} \frac{a_n\, \eta^n}{C_k(\eta)} \int_0^{\infty} e^{ty} \left[\int_0^{\infty} f_{X_{(k)}|Z}(y; n, \beta, \theta, z)\, g(z; \lambda)\, \mathrm{d}z \right] \mathrm{d}y.$$

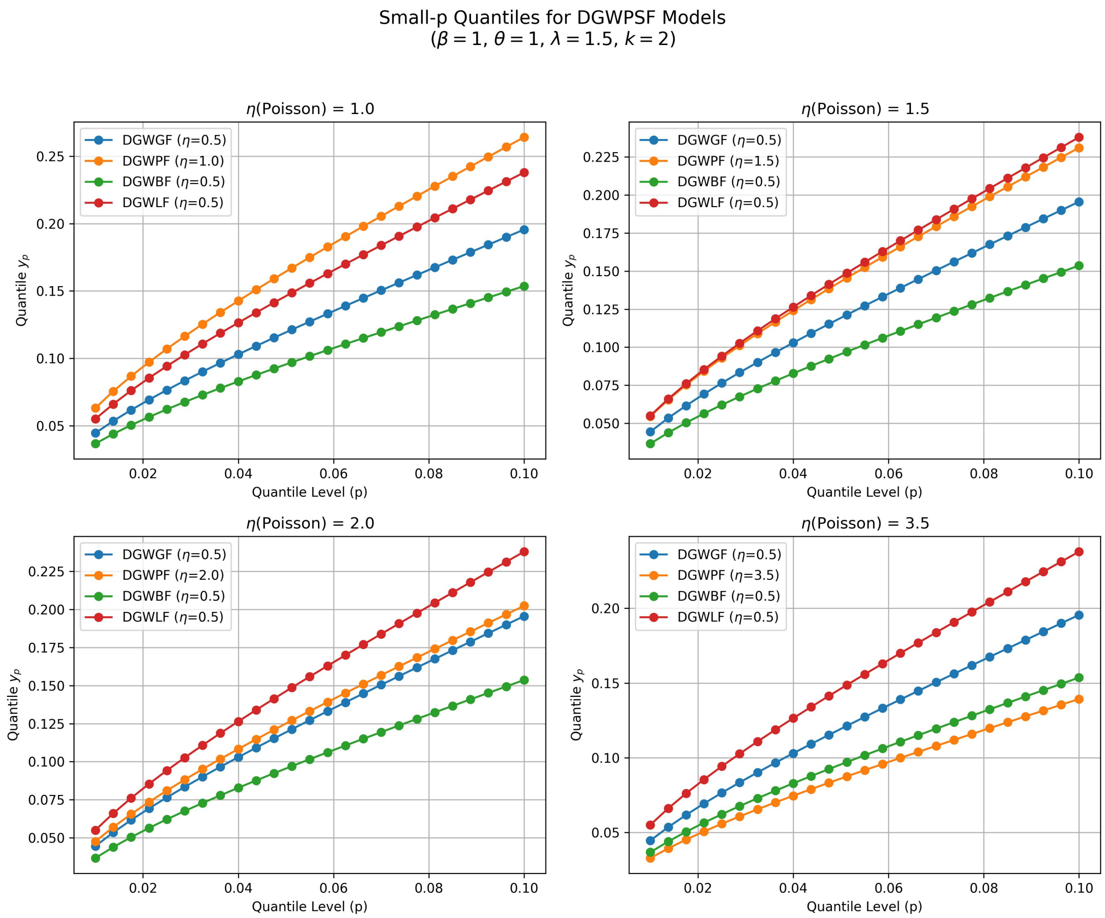

Figure 5. Quantile curves y_p for DGWGF, DGWPF, DGWLF, and DGWBF models at small $p \in [0.01, 0.1]$, with increasing values of η in the Poisson case: (a) $\eta = 1.0$, (b) $\eta = 1.5$, (c) $\eta = 2.0$, and (d) $\eta = 3.5$. Fixed parameters: $\beta = 1$, $\theta = 1$, $\lambda = 1.5$, $k = 2$.

Exchanging integrals gives

$$M_Y(t) = \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \int_0^{\infty} \left[\int_0^{\infty} e^{ty} f_{X_{(k)}|Z}(y; n, \alpha, \beta, z) \, \mathrm{d}y \right] g(z; \lambda) \, \mathrm{d}z.$$

Since $X_{(k)} \mid Z = z$ is the kth order statistic from n independent Weibull$(\beta, \theta z^{-1/\beta})$ draws, its conditional MGF can be written in closed form in terms of incomplete gamma functions. Denoting

$$M_{X_{(k)}|Z}(t; n, z) = \int_0^{\infty} e^{ty} f_{X_{(k)}|Z}(y; n, \beta, \theta, z) \, \mathrm{d}y,$$

we obtain the final representation

$$M_Y(t) = \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \int_0^{\infty} M_{X_{(k)}|Z}(t; n, z) \, g(z; \lambda) \, \mathrm{d}z.$$

In practice, both the outer sum and inner integral are evaluated numerically, and asymptotic or Monte Carlo approximations may be employed when closed-form expressions are unwieldy.

3.4. **Hazard rate and survival functions.** In reliability and survival analysis, the hazard rate and the survival function are fundamental tools for understanding how risk evolves over time. For the DGWPSF family, these quantities reflect both the baseline Weibull behavior and the additional variation induced by the frailty and power-series compounding.

The survival function of Y, denoted $S_Y(y)$, is defined as

$$S_Y(y) \;=\; 1 - F_Y(y) \;=\; \sum_{n=k}^{\infty} \frac{a_n\,\eta^n}{C_k(\eta)} \int_0^{\infty} \Big[1 - F_{X_{(k)}|Z}(y;\eta,\beta,\theta,z)\Big]\, g(z;\lambda)\,\mathrm{d}z,$$

where $F_Y(y)$ is the marginal CDF (see Equation (2.2)); $F_{X_{(k)}|Z}(y;n,\beta,\theta,z)$ is the conditional order-statistic CDF; $g(z;\lambda)$ is the Gamma(λ,λ) frailty density; a_n and $C_k(\eta)$ encode the k-truncated power-series weights.

The hazard rate function $h_Y(y)$ is

$$h_Y(y) \;=\; \frac{f_Y(y)}{S_Y(y)}\,,$$

where $f_Y(y)$ is the DGWPSF PDF given in Equation (2.1). Explicitly:

$$h_Y(y) = \frac{\dfrac{k\beta\theta^{\beta}y^{\beta-1}\lambda^{\lambda+1}}{C_k(\eta)} \displaystyle\sum_{n=k}^{\infty} a_n\eta^n \binom{n}{k} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \Big([n-k+j+1](\theta y)^{\beta} + \lambda\Big)^{-(\lambda+1)}}{1 - F_Y(y)}\,.$$

Figures 6 and 7 show the estimated hazard rate and survival functions for the four DGWPSF special cases (geometric, Poisson, logarithmic, and binomial), with parameters $\beta \in \{0.5, 1, 1.5, 2\}$, $\theta = 1$, $\eta = 0.5$, $\lambda \in \{0.5, 1, 2.5\}$, $k = 4$, and for the binomial case $N = 10$. These survival and hazard plots illustrate three key phenomena. First, introducing gamma frailty (λ) into the Weibull baseline smooths and flattens the hazard curve, dispersing risk more gradually rather than peaking sharply. Second, the choice of power-series compounding (geometric vs. Poisson vs. logarithmic vs. binomial) shifts the balance between early-time and late-time failures: compounding distributions that place more mass on small sample sizes N elevate the initial hazard, while those allowing or enforcing larger N defer failures to later periods. Finally, the binomial-frailty case (DGWBF) exhibits a bathtub-shaped hazard-high risk at both early "infant mortality" and late "wear-out" phases with a comparatively safer mid-life interval–due to its finite upper bound on N.

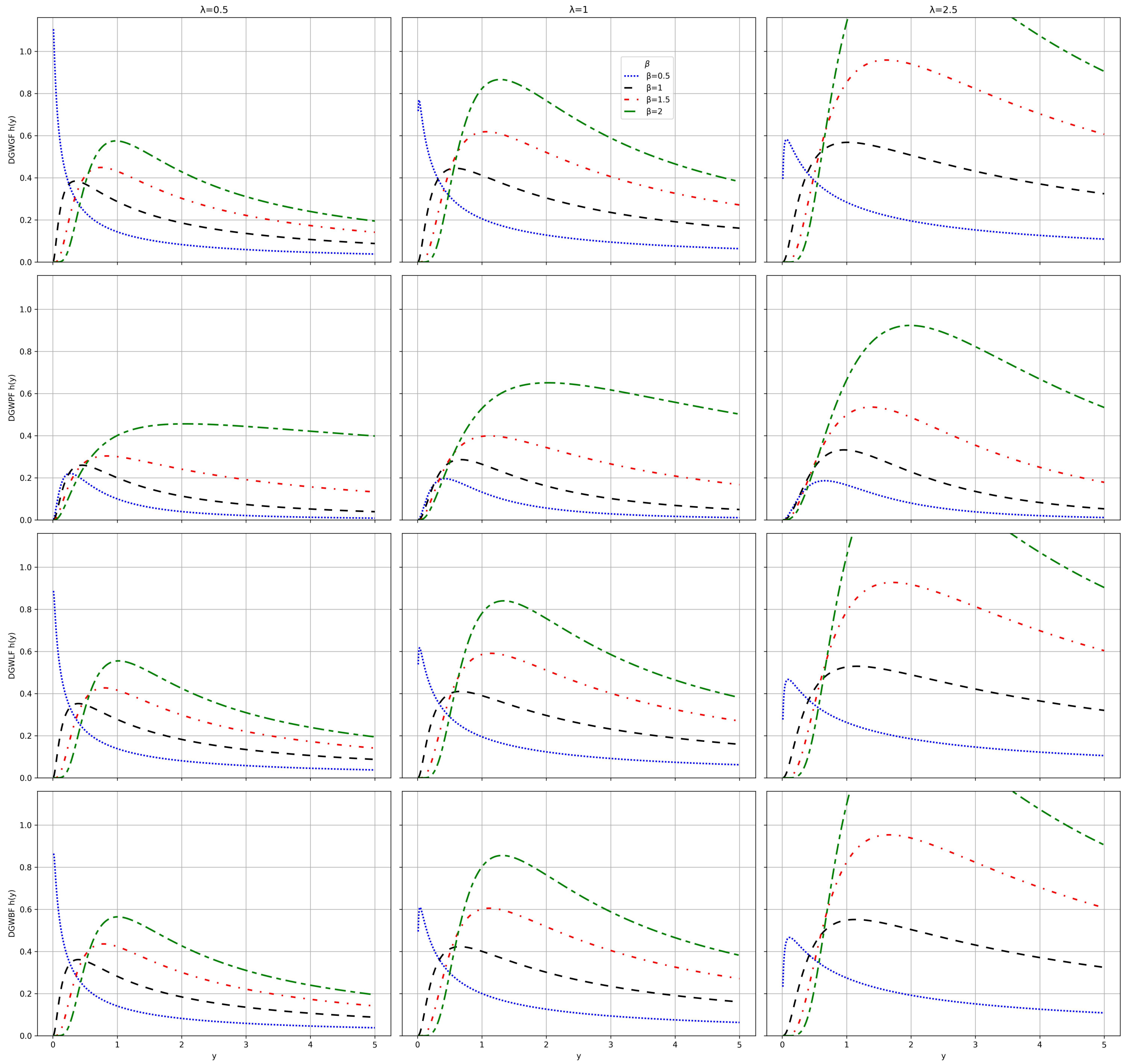

Figure 6. Hazard rate functions $h_Y(y)$ for the DGWPSF special cases (DGWGF, DGWPF, DGWLF, DGWBF). Parameters: $\beta \in \{0.5, 1, 1.5, 2\}$, $\theta = 1$, $\eta = 0.5$, $\lambda \in \{0.5, 1, 2.5\}$, $k = 4$.

4. Parameter estimation

4.1. **Maximum Likelihood Estimation (MLE).** MLE estimates $\boldsymbol{\varphi} = (\beta, \theta, \eta, k, \lambda)$ by maximizing the likelihood for observed failure times $\mathbf{y} = (y_1, \dots, y_m)$. The likelihood function is:

$$L(\boldsymbol{\varphi}) = \prod_{i=1}^{m} f_Y(y_i; \beta, \theta, \eta, k, \lambda) = \prod_{i=1}^{m} \beta \theta^{\beta} y_i^{\beta-1} \lambda^{\lambda+1} k$$
$$\times \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{([j+n-k+1](\theta y_i)^{\beta} + \lambda)^{\lambda+1}},$$

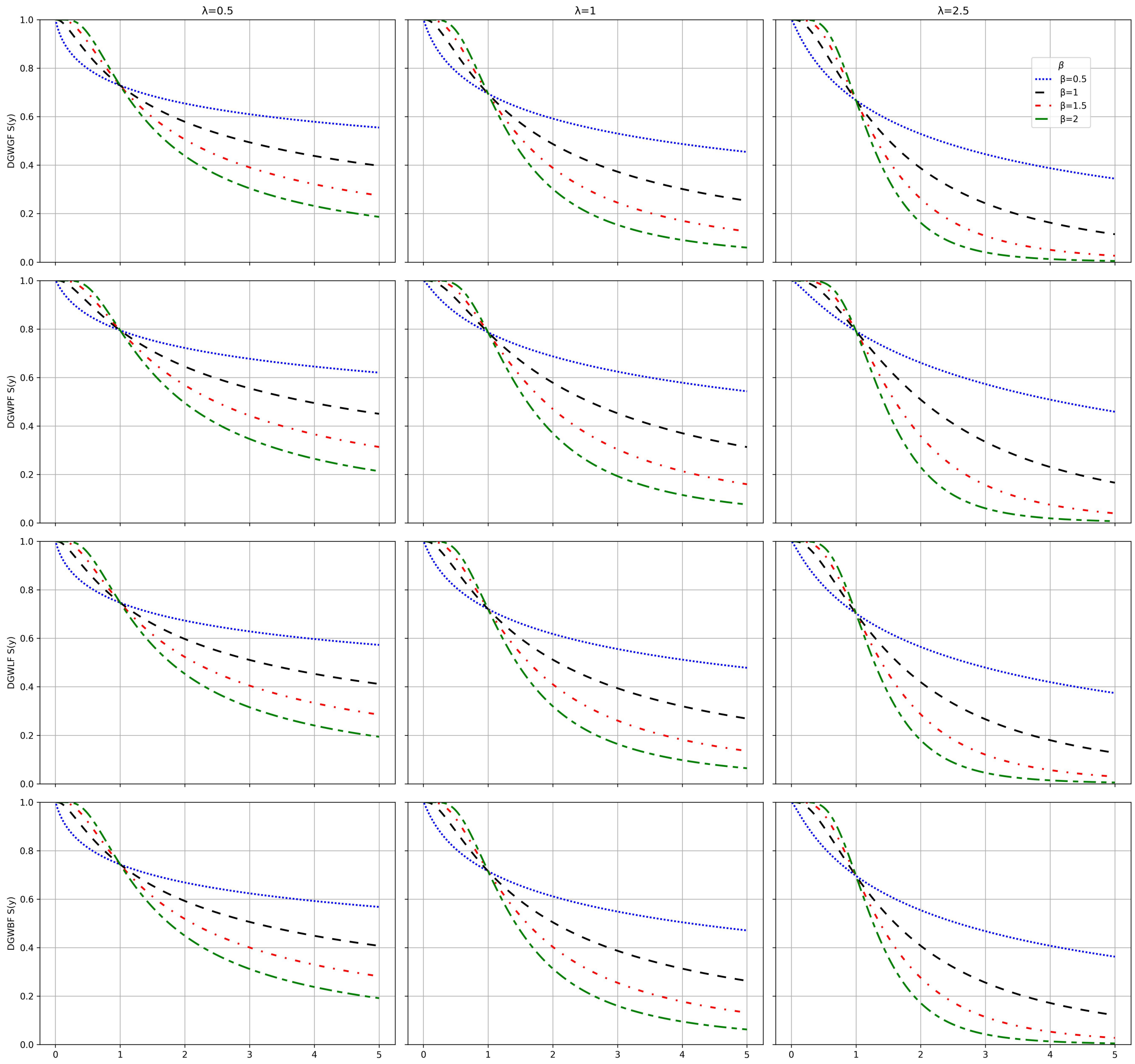

FIGURE 7. Survival functions $S_Y(y)$ for the DGWPSF special cases (DGWGF, DGWPF, DGWLF, DGWBF). Parameters: $\beta \in \{0.5, 1, 1.5, 2\}$, $\theta = 1$, $\eta = 0.5$, $\lambda \in \{0.5, 1, 2.5\}$, $k = 4$.

where $f_Y(y_i; \beta, \theta, \eta, k, \lambda)$ is the PDF of Y_i given in Equation 2.1. a_n, $C_k(\eta)$ are defined in Table 1.

The log-likelihood is:

$$\ell(\boldsymbol{\varphi}) = \sum_{i=1}^{m} \left[\log\beta + \beta\log\theta + (\beta-1)\log y_i + (\lambda+1)\log\lambda + \log k + \log S_i(\boldsymbol{\varphi})\right],$$

where:

$$S_i(\boldsymbol{\varphi}) = \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+1}}.$$

The partial derivatives are:

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{m} \left[\frac{1}{\beta} + \log \theta + \log y_i + \frac{1}{S_i(\boldsymbol{\varphi})} \frac{\partial S_i(\boldsymbol{\varphi})}{\partial \beta} \right],$$

$$\frac{\partial S_i(\boldsymbol{\varphi})}{\partial \beta} = -(\lambda+1) \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{[j+n-k+1](\theta y_i)^\beta \log(\theta y_i)}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+2}},$$

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{m} \left[\frac{\beta}{\theta} + \frac{1}{S_i(\boldsymbol{\varphi})} \frac{\partial S_i(\boldsymbol{\varphi})}{\partial \theta} \right],$$

$$\frac{\partial S_i(\boldsymbol{\varphi})}{\partial \theta} = -(\lambda+1)\beta\theta^{\beta-1} \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{[j+n-k+1] y_i^\beta}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+2}},$$

$$\frac{\partial \ell}{\partial \eta} = \sum_{i=1}^{m} \frac{1}{S_i(\boldsymbol{\varphi})} \cdot \frac{\partial S_i(\boldsymbol{\varphi})}{\partial \eta}$$

$$\begin{aligned}\frac{\partial S_i(\boldsymbol{\varphi})}{\partial \eta} = &\sum_{n=k}^{\infty} \left[\frac{a_n \binom{n}{k} n \eta^{n-1}}{C_k(\eta)} - \frac{a_n \eta^n \binom{n}{k} C_k'(\eta)}{C_k(\eta)^2} \right] \\ &\times \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+1}},\end{aligned}$$

$$\frac{\partial S_i(\boldsymbol{\varphi})}{\partial \eta} = \sum_{n=k}^{\infty} \left[\frac{a_n \binom{n}{k} n \eta^{n-1}}{C_k(\eta)} - \frac{a_n \eta^n \binom{n}{k} C_k'(\eta)}{C_k(\eta)^2} \right] \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+1}},$$

where

$$C_k'(\eta) = \sum_{n=k}^{\infty} n a_n \eta^{n-1}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{m} \left[\frac{\lambda+1}{\lambda} + \log \lambda + \frac{1}{S_i(\boldsymbol{\varphi})} \frac{\partial S_i(\boldsymbol{\varphi})}{\partial \lambda} \right],$$

$$\frac{\partial S_i(\boldsymbol{\varphi})}{\partial \lambda} = \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{-(\lambda+1)\log\Big([j+n-k+1](\theta y_i)^\beta + \lambda\Big) - 1}{\left([j+n-k+1](\theta y_i)^\beta + \lambda\right)^{\lambda+1}}.$$

Due to infinite sums and numerical instability, direct optimization is challenging, motivating the EM algorithm.

4.2. **Expectation-maximization (EM) algorithm.** The EM algorithm estimates $\boldsymbol{\varphi}$ by treating $Z_i \sim \Gamma(\lambda, \lambda)$ and $N_i \sim \mathrm{PS}(\eta, k)$ as latent variables. The complete-data likelihood for observation (y_i, Z_i, N_i) is:

$$\begin{aligned} L_c(\boldsymbol{\varphi}; y_i, z_i, n_i) &= P(N_i = n_i \mid \eta, k) f_{X_{(k)}|Z}(y_i; n_i, \beta, \theta, z_i) g(z_i; \lambda) \\ &= \frac{a_{n_i}\eta^{n_i}}{C_k(\eta)} \left[k\binom{n_i}{k} \left(1 - e^{-z_i(\theta y_i)^\beta}\right)^{k-1} e^{-(n_i-k+1)z_i(\theta y_i)^\beta} \right. \\ &\quad \left. \times z_i \beta \theta^\beta y_i^{\beta-1} \right] \cdot \frac{\lambda^\lambda z_i^{\lambda-1} e^{-\lambda z_i}}{\Gamma(\lambda)}. \end{aligned}$$

The complete-data log-likelihood for a single observation is:

$$\begin{aligned} \log L_c(\boldsymbol{\varphi}; y_i, z_i, n_i) &= \log a_{n_i} + n_i \log \eta - \log C_k(\eta) + \log k + \log \binom{n_i}{k} \\ &\quad + (k-1)\log\left(1 - e^{-z_i(\theta y_i)^\beta}\right) - (n_i - k + 1) z_i (\theta y_i)^\beta \\ &\quad + \log \beta + \beta \log \theta + (\beta - 1)\log y_i + \lambda \log z_i \\ &\quad + \lambda \log \lambda - \lambda z_i - \log \Gamma(\lambda) \end{aligned}$$

E-step: Compute:

$$Q(\boldsymbol{\varphi} \mid \boldsymbol{\varphi}^{(r)}) = \mathbb{E}_{Z_i, N_i | y_i, \boldsymbol{\varphi}^{(r)}}[\ell_c(\boldsymbol{\varphi}) \mid \mathbf{y}, \boldsymbol{\varphi}^{(r)}].$$

This requires computing:

$$\begin{aligned} P(N_i = n \mid y_i, \boldsymbol{\varphi}^{(r)}) &= \frac{f_Y(y_i \mid N_i = n, \boldsymbol{\varphi}^{(r)}) P(N_i = n \mid \eta^{(r)}, k)}{f_Y(y_i; \boldsymbol{\varphi}^{(r)})}, \\ f_Y(y_i \mid N_i = n, \boldsymbol{\varphi}^{(r)}) &= \int_0^\infty f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz. \end{aligned}$$

Thus,

$$P(N_i = n \mid y_i, \boldsymbol{\varphi}^{(r)}) = \frac{\frac{a_n (\eta^{(r)})^n}{C_k(\eta^{(r)})} \int_0^\infty f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}{\sum_{m=k}^\infty \frac{a_m (\eta^{(r)})^m}{C_k(\eta^{(r)})} \int_0^\infty f_{X_{(k)}|Z}(y_i; m, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}.$$

The conditional expectations:

$$\begin{aligned} \mathbb{E}[Z_i \mid y_i, N_i = n, \boldsymbol{\varphi}^{(r)}] &= \frac{\int_0^\infty z f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}{\int_0^\infty f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}, \\ \mathbb{E}[\log Z_i \mid y_i, N_i = n, \boldsymbol{\varphi}^{(r)}] &= \frac{\int_0^\infty \log z f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}{\int_0^\infty f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})\, dz}. \end{aligned}$$

M-step: Maximize:

$$\boldsymbol{\varphi}^{(r+1)} = \arg\max_{\boldsymbol{\varphi}} Q(\boldsymbol{\varphi} \mid \boldsymbol{\varphi}^{(r)}).$$

The expected log-likelihood is:

$$Q(\boldsymbol{\varphi} \mid \boldsymbol{\varphi}^{(r)}) = \sum_{i=1}^m \sum_{n=k}^\infty P(N_i = n \mid y_i, \boldsymbol{\varphi}^{(r)}) \int_0^\infty \log L_c(\boldsymbol{\varphi}; y_i, z, n) P(Z_i = z \mid y_i, N_i = n, \boldsymbol{\varphi}^{(r)})\, dz,$$

where $P(Z_i = z \mid y_i, N_i = n, \boldsymbol{\varphi}^{(r)}) \propto f_{X_{(k)}|Z}(y_i; n, \beta^{(r)}, \theta^{(r)}, z) g(z; \lambda^{(r)})$.

4.3. **Bayesian estimation.** Bayesian inference provides a flexible alternative to likelihood-based estimation by incorporating prior beliefs about the parameters and updating them with observed data. The prior distributions on the parameter vector $\boldsymbol{\varphi} = (\beta, \theta, \eta, \lambda)$ are:

$$\beta, \theta, \lambda \sim \text{Gamma}(a, b), \quad \eta \sim \text{Beta}(c, d),$$

where the hyperparameters (a, b, c, d) are chosen to reflect either prior knowledge or noninformative priors.

The posterior distribution is proportional to the product of the likelihood and the priors:

$$\pi(\boldsymbol{\varphi} \mid \mathbf{y}) \propto L(\boldsymbol{\varphi}; \mathbf{y}) \cdot \pi(\boldsymbol{\varphi}).$$

Given the complexity of the DGWPSF likelihood, direct sampling is infeasible. Instead, one can use Markov Chain Monte Carlo (MCMC) methods such as Metropolis-Hastings or Hamiltonian Monte Carlo (HMC) to draw samples from the posterior distribution. These samples can be used to compute point estimates (e.g., posterior means or medians) and credible intervals for the parameters. While more computationally intensive, Bayesian methods can offer a robust alternative, especially under model uncertainty or limited information.

5. Conclusion

The doubly generalized Weibull power series frailty (DGWPSF) model significantly enhances the generalized Weibull-left k-truncated power series model [4] by incorporating a gamma-distributed frailty term. These features address limitations in modeling dynamic and heterogeneous systems. The gamma frailty term models unobserved heterogeneity and correlated lifetimes in clustered data, such as patient groups in medical studies or financial assets in portfolios [5,6]. By accounting for shared latent factors (e.g., treatment protocols in hospitals or systemic risks in finance), the DGWPSF model improves predictive accuracy over homogeneous models like the generalized Weibull-power series distribution, particularly in survival analysis with clustered data [6].

Future work may explore alternative frailty distributions (e.g., inverse-Gaussian or log-normal), the incorporation of time-varying covariates for dynamic risk modeling, and scalable estimation methods for high-dimensional cluster structures. These extensions will further broaden the applicability of power-series frailty models across economics, engineering, and financial domains.

Appendix

A1. Proof of PDF and CDF of the DGWPSF Distribution. Let $X \mid Z = z \sim \text{Weibull}(\beta, \theta z^{-1/\beta})$. Then, the conditional cumulative distribution function (CDF) and probability density function (PDF) of X given $Z = z$ are:

$$F_X(y \mid Z = z) = 1 - e^{-z(\theta y)^\beta}, \quad f_X(y \mid Z = z) = \beta\theta^\beta y^{\beta-1} z e^{-z(\theta y)^\beta}.$$

Let $X_{(k)}$ denote the k-th order statistic from $N = n$ i.i.d. samples of $X \mid Z = z$. The conditional PDF of $X_{(k)} \mid Z = z, N = n$ is:

$$f_{X_{(k)}|Z=z,N=n}(y) = k\binom{n}{k} [F_X(y \mid z)]^{k-1} [1 - F_X(y \mid z)]^{n-k} f_X(y \mid z).$$

Substituting the expressions for F_X and f_X, we obtain:

$$f_{X_{(k)}|Z=z,N=n}(y) = k\binom{n}{k}\left(1 - e^{-z(\theta y)^\beta}\right)^{k-1} e^{-(n-k+1)z(\theta y)^\beta} \beta\theta^\beta y^{\beta-1} z.$$

Now integrate over the frailty variable $Z \sim \text{Gamma}(\lambda, \lambda)$, with PDF:

$$g(z) = \frac{\lambda^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\lambda z}.$$

The marginal PDF of $Y = X_{(k)}$ conditional on $N = n$ is:

$$f_Y(y \mid N = n) = \int_0^\infty f_{X_{(k)}|Z=z,N=n}(y) g(z)\, dz.$$

To integrate, apply the binomial expansion:

$$\left(1 - e^{-z(\theta y)^\beta}\right)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-jz(\theta y)^\beta}.$$

Thus,

$$\begin{aligned} f_Y(y \mid N = n) &= k\binom{n}{k}\beta\theta^\beta y^{\beta-1} \frac{\lambda^\lambda}{\Gamma(\lambda)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \int_0^\infty z^\lambda e^{-([n-k+j+1](\theta y)^\beta + \lambda) z}\, dz \\ &= \beta\theta^\beta y^{\beta-1} \lambda^{\lambda+1} k\binom{n}{k} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{([n-k+j+1](\theta y)^\beta + \lambda)^{\lambda+1}}. \end{aligned}$$

Now consider $N \sim \text{PS}_k(a_n, \eta)$, a k-truncated power series distribution with PMF:

$$P(N = n) = \frac{a_n \eta^n}{C_k(\eta)}, \quad C_k(\eta) = \sum_{n=k}^\infty a_n \eta^n.$$

The marginal PDF of Y is:

$$\begin{aligned} f_Y(y) &= \sum_{n=k}^{\infty} P(N=n)\cdot f_Y(y \mid N=n) \\ &= \sum_{n=k}^{\infty} \frac{a_n \eta^n}{C_k(\eta)} \cdot \left[\beta\theta^\beta y^{\beta-1}\lambda^{\lambda+1} k \binom{n}{k} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{([n-k+j+1](\theta y)^\beta+\lambda)^{\lambda+1}}\right] \\ &= \beta\theta^\beta y^{\beta-1}\lambda^{\lambda+1} k \sum_{n=k}^{\infty} \frac{a_n \eta^n \binom{n}{k}}{C_k(\eta)} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{([n-k+j+1](\theta y)^\beta+\lambda)^{\lambda+1}}. \end{aligned}$$

For a fixed sample size $N=n$, the conditional CDF of the k-th order statistic $X_{(k)}$ is:

$$F_{X_{(k)}}(y \mid Z=z) = \sum_{j=k}^{n} \binom{n}{j} \left(1-e^{-z(\theta y)^\beta}\right)^j e^{-z(n-j)(\theta y)^\beta} \tag{5.1}$$

To derive the CDF, use the binomial theorem::

$$(1-e^{-z(\theta y)^\beta})^j = \sum_{m=0}^{j} \binom{j}{m} (-1)^m e^{-mz(\theta y)^\beta}.$$

Thus, The unconditional CDF integrates over the frailty distribution $Z \sim \text{Gamma}(\lambda,\lambda)$:

$$F_Y(y) = \mathbb{E}_Z\left[F_{X_{(k)}}(y \mid Z)\right] \tag{5.2}$$

$$= \sum_{n=k}^{\infty} \frac{a_n\eta^n}{C_k(\eta)} \int_0^\infty F_{X_{(k)}}(y \mid z) \frac{\lambda^\lambda}{\Gamma(\lambda)} z^{\lambda-1} e^{-\lambda z} dz \tag{5.3}$$

$$= \sum_{n=k}^{\infty} \frac{a_n\eta^n}{C_k(\eta)} \sum_{j=k}^{n} \binom{n}{j} \sum_{m=0}^{j} \binom{j}{m} (-1)^m \int_0^\infty e^{-[(m+n-j)(\theta y)^\beta+\lambda]z} \frac{\lambda^\lambda}{\Gamma(\lambda)} z^{\lambda-1} dz \tag{5.4}$$

Substitute into the integral for the CDF of the order statistic, and apply:

$$\int_0^\infty z^{\lambda-1} e^{-([m+n-j](\theta y)^\beta+\lambda)z} dz = \frac{\Gamma(\lambda)}{([m+n-j](\theta y)^\beta+\lambda)^\lambda}.$$

This yields:

$$F_Y(y) = \lambda^\lambda \sum_{n=k}^{\infty} \frac{a_n\eta^n}{C_k(\eta)} \sum_{j=k}^{n} \binom{n}{j} \sum_{m=0}^{j} \binom{j}{m} \frac{(-1)^m}{([m+n-j](\theta y)^\beta+\lambda)^\lambda}.$$

The integration of the marginal PDF of the k-th order statistic results in an alternative, but equivalent, expression for the same marginal CDF:

$$F(y) = \frac{k}{C_k(\eta)} \sum_{n=k}^{\infty} a_n \eta^n \binom{n}{k} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+n-k+1} \left(1-\left(\frac{\lambda}{(j+n-k+1)(\theta y)^\beta+\lambda}\right)^\lambda\right).$$

A2. Proof: $F_Y(y) \sim C\,y^{\beta}$ as $y \downarrow 0$ for $k = 1$. Define

$$A_{n,j,m}(y) = \Big([m+n-j]\,(\theta y)^{\beta} + \lambda\Big)^{-\lambda}.$$

A Taylor expansion at $(\theta y)^{\beta} = 0$ gives

$$A_{n,j,m}(y) = \lambda^{-\lambda}\Big(1 + \tfrac{m+n-j}{\lambda}(\theta y)^{\beta}\Big)^{-\lambda} = \lambda^{-\lambda}\Big(1 - (m+n-j)\,(\theta y)^{\beta} + O(y^{2\beta})\Big).$$

$$\sum_{m=0}^{j}\binom{j}{m}(-1)^m A_{n,j,m}(y) = \lambda^{-\lambda}\sum_{m=0}^{j}\binom{j}{m}(-1)^m\Big[1 - (m+n-j)(\theta y)^{\beta} + O(y^{2\beta})\Big].$$

Since $\sum_{m=0}^{j}\binom{j}{m}(-1)^m = (1-1)^j = 0$, the constant term vanishes. The remaining term is

$$-\lambda^{-\lambda}(\theta y)^{\beta}\sum_{m=0}^{j}\binom{j}{m}(-1)^m\,(m+n-j) + O(y^{2\beta}).$$

A standard binomial identity shows $\sum_{m=0}^{j}\binom{j}{m}(-1)^m(m+n-j) = n\,(1-1)^{j-1}$, which is nonzero only when $j = 1$. Thus for $k = 1$,

$$\sum_{j=1}^{n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}(-1)^m A_{n,j,m}(y) = \lambda^{-\lambda}n^2(\theta y)^{\beta} + O(y^{2\beta}).$$

$$F_Y(y) = \sum_{n=1}^{\infty}\frac{a_n\,\eta^n}{C_1(\eta)}\Big[\lambda^{-\lambda}n^2(\theta y)^{\beta} + O(y^{2\beta})\Big].$$

Since $\sum n^2 a_n \eta^n < \infty$, the $O(y^{2\beta})$ terms remain higher order. Therefore, as $y \to 0$,

$$F_Y(y) = \lambda^{-\lambda}\,\theta^{\beta}\left(\sum_{n=1}^{\infty}\frac{a_n\,\eta^n}{C_1(\eta)}\,n^2\right)y^{\beta} + o(y^{\beta}) \;=\; C\,y^{\beta} + o(y^{\beta}),$$

where

$$C = \lambda^{-\lambda}\,\theta^{\beta}\sum_{n=1}^{\infty}\frac{a_n\,\eta^n}{C_1(\eta)}\,n^2.$$

This establishes the approximation $F_Y(y) \approx C\,y^{\beta}$ as $y \to 0$.

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References

[1] G. Mudholkar, D. Srivastava, Exponentiated Weibull Family for Analyzing Bathtub Failure-Rate Data, IEEE Trans. Reliab. 42 (1993), 299–302. https://doi.org/10.1109/24.229504.

[2] A.W. Marshall, I. Olkin, A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families, Biometrika 84 (1997), 641–652. https://www.jstor.org/stable/2337585.

[3] M.H. Tahir, G.M. Cordeiro, Compounding of Distributions: A Survey and New Generalized Classes, J. Stat. Distrib. Appl. 3 (2016), 13. https://doi.org/10.1186/s40488-016-0052-1.

[4] M. Rahmouni, A New Compound Family of Weibull Order Statistics and Left K-Truncated Power Series Distributions, Int. J. Anal. Appl. 23 (2025), 106. https://doi.org/10.28924/2291-8639-23-2025-106.

[5] P. Hougaard, Analysis of Multivariate Survival Data, Springer New York, 2000. https://doi.org/10.1007/978-1-4612-1304-8.

[6] L. Duchateau, P. Janssen, The Frailty Model, Springer, (2008).

[7] A.L. Morais, W. Barreto-Souza, A Compound Class of Weibull and Power Series Distributions, Comput. Stat. Data Anal. 55 (2011), 1410–1425. https://doi.org/10.1016/j.csda.2010.09.030.

[8] M. Rahmouni, D. Ziedan, The Weibull-Generalized Shifted Geometric Distribution: Properties, Estimation, and Applications, AIMS Math. 10 (2025), 9773–9804. https://doi.org/10.3934/math.2025448.

[9] H.A. David, H.N. Nagaraja, Order Statistics, Wiley, 2003.

[10] A. Rényi, On the Theory of Order Statistics, Acta Math. Acad. Sci. Hung. 4 (1953), 191–231. https://doi.org/10.1007/bf02127580.