

Weak Forms of Open and Closed Functions via $\theta(\tau_1, \tau_2)$ -Open Sets

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Abstract. This paper presents new classes of functions defined between bitopological spaces, namely weakly $\theta(\tau_1, \tau_2)$ -open functions and weakly $\theta(\tau_1, \tau_2)$ -closed functions. Moreover, several characterizations of weakly $\theta(\tau_1, \tau_2)$ -open functions and weakly $\theta(\tau_1, \tau_2)$ -closed functions are established.

1. INTRODUCTION

In topology, there has been recently significant interest in characterizing and investigating the characterizations of some weak forms of open functions and closed functions. Semi-open sets, preopen sets, α -open sets, β -open sets, δ -open sets and θ -open sets play an important role in the research of generalizations of open functions and closed functions. By using these sets, many authors introduced and studied various types of open functions and closed functions. In 1984, Rose [17] introduced and studied the notions of weakly open functions and almost open functions. Rose and Janković [16] investigated some of the fundamental properties of weakly closed functions. Caldas and Navalagi [8] introduced and studied the notions of weakly semi-open functions and weakly semi-closed functions as a new generalization of weakly open functions and weakly closed functions. In 1987, Di Maio and Noiri [11] investigated the concepts of semi- θ -open sets and semi- θ -closed sets which provide a formulation of semi- θ -closure of a set in a topological space. Noiri [15] introduced and studied the concept of θ -semicontinuous functions by involving these sets. In 1991, Mukherjee and Basu [14] continued the work of Di Maio and Noiri [11] and defined the notions of semi- θ -connectedness, semi- θ -components and

Received: Jun. 11, 2025.

2020 *Mathematics Subject Classification.* 54C10, 54E55.

Key words and phrases. $\theta(\tau_1, \tau_2)$ -open set; weakly $\theta(\tau_1, \tau_2)$ -open function; weakly $\theta(\tau_1, \tau_2)$ -closed function.

semi- θ -quasi components. In 2006, Caldas et al. [9] introduced and studied two new classes of functions by utilizing the notions of semi- θ -open sets and the semi- θ -closure operator called weakly semi- θ -open functions and weakly semi- θ -closed functions. The class of weakly semi- θ -openness (resp. weakly semi- θ -closedness) as a new generalization of semi- θ -openness (semi- θ -closedness). Recently, Klanarong and Boonpok [13] studied the notions of weakly $s(\Lambda, p)$ -open functions and weakly $s(\Lambda, p)$ -closed functions by utilizing $s(\Lambda, p)$ -open sets and the $s(\Lambda, p)$ -closure operator. Chutiman and Boonpok [10] studied some properties of weakly $b(\Lambda, p)$ -open functions. On the other hand, the present authors introduced and studied the concepts of (m, n) -weakly open functions [7], $\theta p(\Lambda, p)$ -open functions [4], $\theta p(\Lambda, p)$ -closed functions [4], weakly $\delta(\Lambda, p)$ -open functions [18], weakly $\delta(\Lambda, p)$ -closed functions [12], weakly $\beta(\Lambda, p)$ -open functions [3], weakly $\beta(\Lambda, p)$ -closed functions [3], weakly $p(\Lambda, p)$ -open functions [2], weakly $p(\Lambda, p)$ -closed functions [2], weakly $\theta s(\Lambda, p)$ -open functions [1] and weakly $\theta s(\Lambda, p)$ -closed functions [1]. In this paper, we introduce the notions of weakly $\theta(\tau_1, \tau_2)s$ -open functions and weakly $\theta(\tau_1, \tau_2)s$ -closed functions. Furthermore, several characterizations and some properties concerning weakly $\theta(\tau_1, \tau_2)s$ -open functions and weakly $\theta(\tau_1, \tau_2)s$ -closed functions are investigated.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2) . The closure of A and the interior of A with respect to τ_i are denoted by $\tau_i\text{-Cl}(A)$ and $\tau_i\text{-Int}(A)$, respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ -closed [6] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$ -closed set is called $\tau_1\tau_2$ -open. Let A be a subset of a bitopological space (X, τ_1, τ_2) . The intersection of all $\tau_1\tau_2$ -closed sets of X containing A is called the $\tau_1\tau_2$ -closure [6] of A and is denoted by $\tau_1\tau_2\text{-Cl}(A)$. The union of all $\tau_1\tau_2$ -open sets of X contained in A is called the $\tau_1\tau_2$ -interior [6] of A and is denoted by $\tau_1\tau_2\text{-Int}(A)$.

Lemma 2.1. [6] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2) . For the $\tau_1\tau_2$ -closure, the following properties hold:*

- (1) $A \subseteq \tau_1\tau_2\text{-Cl}(A)$ and $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A)$.
- (2) If $A \subseteq B$, then $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(B)$.
- (3) $\tau_1\tau_2\text{-Cl}(A)$ is $\tau_1\tau_2$ -closed.
- (4) A is $\tau_1\tau_2$ -closed if and only if $A = \tau_1\tau_2\text{-Cl}(A)$.
- (5) $\tau_1\tau_2\text{-Cl}(X - A) = X - \tau_1\tau_2\text{-Int}(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$ -open [20] (resp. $(\tau_1, \tau_2)s$ -open [5], $(\tau_1, \tau_2)p$ -open [5], $(\tau_1, \tau_2)\beta$ -open [5]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$ -open (resp. $(\tau_1, \tau_2)s$ -open, $(\tau_1, \tau_2)p$ -open, $(\tau_1, \tau_2)\beta$ -open) set is called $(\tau_1, \tau_2)r$ -closed (resp. $(\tau_1, \tau_2)s$ -closed, $(\tau_1, \tau_2)p$ -closed, $(\tau_1, \tau_2)\beta$ -closed). A subset A of a bitopological space

(X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$ -open [19] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$ -open set is said to be $\alpha(\tau_1, \tau_2)$ -closed. For a subset A of a bitopological space (X, τ_1, τ_2) , a point $x \in X$ is called $(\tau_1, \tau_2)\theta$ -cluster point [20] of A if $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$ -open set U containing x . The set of all $(\tau_1, \tau_2)\theta$ -cluster points of A is called the $(\tau_1, \tau_2)\theta$ -closure [20] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Cl}(A)$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$ -closed [20] if $A = (\tau_1, \tau_2)\theta\text{-Cl}(A)$. The complement of a $(\tau_1, \tau_2)\theta$ -closed set is said to be $(\tau_1, \tau_2)\theta$ -open. The union of all $(\tau_1, \tau_2)\theta$ -open sets contained in A is called the $(\tau_1, \tau_2)\theta$ -interior [20] of A and is denoted by $(\tau_1, \tau_2)\theta\text{-Int}(A)$.

Lemma 2.2. [20] *For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties hold:*

- (1) *If A is $\tau_1\tau_2$ -open in X , then $\tau_1\tau_2\text{-Cl}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A)$.*
- (2) *$(\tau_1, \tau_2)\theta\text{-Cl}(A)$ is $\tau_1\tau_2$ -closed in X .*

Let A be a subset of a bitopological space (X, τ_1, τ_2) . A point $x \in X$ is called a $\theta(\tau_1, \tau_2)$ -s-cluster point of A if $(\tau_1, \tau_2)\text{-sCl}(U) \cap A \neq \emptyset$ for every (τ_1, τ_2) -s-open set U of X containing x . The set of all $\theta(\tau_1, \tau_2)$ -s-cluster points of A is called the $\theta(\tau_1, \tau_2)$ -s-closure of A and is denoted by $\theta(\tau_1, \tau_2)\text{-sCl}(A)$. If $A = \theta(\tau_1, \tau_2)\text{-sCl}(A)$, then A is called $\theta(\tau_1, \tau_2)$ -s-closed. The complement of a $\theta(\tau_1, \tau_2)$ -s-closed set is called $\theta(\tau_1, \tau_2)$ -s-open. The $\theta(\tau_1, \tau_2)$ -s-interior of A is defined by the union of all $\theta(\tau_1, \tau_2)$ -s-open sets of X contained in A and is denoted by $\theta(\tau_1, \tau_2)\text{-sInt}(A)$.

Lemma 2.3. *For subsets A and B of a bitopological space (X, τ_1, τ_2) , the following properties hold:*

- (1) $X - \theta(\tau_1, \tau_2)\text{-sCl}(A) = \theta(\tau_1, \tau_2)\text{-sInt}(X - A)$ and $X - \theta(\tau_1, \tau_2)\text{-sInt}(A) = \theta(\tau_1, \tau_2)\text{-sCl}(X - A)$.
- (2) A is $\theta(\tau_1, \tau_2)$ -s-open if and only if $A = \theta(\tau_1, \tau_2)\text{-sInt}(A)$.
- (3) $A \subseteq (\tau_1, \tau_2)\text{-sCl}(A) \subseteq \theta(\tau_1, \tau_2)\text{-sCl}(A)$ and $\theta(\tau_1, \tau_2)\text{-sInt}(A) \subseteq (\tau_1, \tau_2)\text{-sInt}(A)$.
- (4) If $A \subseteq B$, then $\theta(\tau_1, \tau_2)\text{-sCl}(A) \subseteq \theta(\tau_1, \tau_2)\text{-sCl}(B)$ and $\theta(\tau_1, \tau_2)\text{-sInt}(A) \subseteq \theta(\tau_1, \tau_2)\text{-sInt}(B)$.
- (5) If A is (τ_1, τ_2) -s-open, then $(\tau_1, \tau_2)\text{-sCl}(A) = \theta(\tau_1, \tau_2)\text{-sCl}(A)$.

3. WEAKLY $\theta(\tau_1, \tau_2)$ -s-OPEN FUNCTIONS

In this section, we introduce the notion of weakly $\theta(\tau_1, \tau_2)$ -s-open functions. Furthermore, some characterizations of weakly $\theta(\tau_1, \tau_2)$ -s-open functions are discussed.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be weakly $\theta(\tau_1, \tau_2)$ -s-open if*

$$f(U) \subseteq \theta(\sigma_1, \sigma_2)\text{-sInt}(f(\tau_1\tau_2\text{-Cl}(U)))$$

for each $\tau_1\tau_2$ -open set U of X .

Theorem 3.1. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) *f is weakly $\theta(\tau_1, \tau_2)$ -s-open;*
- (2) *$f((\tau_1, \tau_2)\theta\text{-Int}(A)) \subseteq \theta(\sigma_1, \sigma_2)\text{-sInt}(f(A))$ for every subset A of X ;*
- (3) *$(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)\text{-sInt}(B))$ for every subset B of Y ;*

- (4) $f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Cl}(B)) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))$ for every subset B of Y ;
- (5) for each $x \in X$ and each $\tau_1\tau_2$ -open set U of X containing x , there exists a $\theta(\sigma_1, \sigma_2)s$ -open set V of Y containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$;
- (6) $f(\tau_1\tau_2\text{-Int}(K)) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(K))$ for every $\tau_1\tau_2$ -closed set K of X ;
- (7) $f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ for every $\tau_1\tau_2$ -open set U of X ;
- (8) $f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ for every $(\tau_1, \tau_2)p$ -open set U of X ;
- (9) $f(U) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ for every $\alpha(\tau_1, \tau_2)$ -open set U of X .

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in (\tau_1, \tau_2)\theta\text{-Int}(A)$. Then, there exists a $\tau_1\tau_2$ -open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq A$. Then, $f(x) \in f(U) \subseteq f(\tau_1\tau_2\text{-Cl}(U)) \subseteq f(A)$. Since f is weakly $\theta(\tau_1, \tau_2)s$ -open, $f(U) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(A))$. It implies that

$$f(x) \in \theta(\sigma_1, \sigma_2)s\text{-Int}(f(A)).$$

Thus, $x \in f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(f(A)))$ and hence $(\tau_1, \tau_2)\theta\text{-Int}(A) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(f(A)))$. This shows that $f((\tau_1, \tau_2)\theta\text{-Int}(A)) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(A))$.

(2) $\Rightarrow$ (1): Let U be any $\tau_1\tau_2$ -open set of X . As $U \subseteq (\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$ implies

$$f(U) \subseteq f((\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))).$$

Thus, f is weakly $\theta(\tau_1, \tau_2)s$ -open.

(2) $\Rightarrow$ (3): Let B be any subset of Y . Then by (2), $f((\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B))) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(B)$. Thus, $(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(B))$.

(3) $\Rightarrow$ (2): The proof is obvious.

(3) $\Rightarrow$ (4): Let B be any subset of Y . Using (3), we have

$$\begin{aligned} X - (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B)) &= (\tau_1, \tau_2)\theta\text{-Int}(X - f^{-1}(B)) \\ &= (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(Y - B)) \\ &= f^{-1}(Y - \theta(\sigma_1, \sigma_2)s\text{-Cl}(B)) \\ &= X - f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Cl}(B)) \end{aligned}$$

and hence $f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Cl}(B)) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))$.

(4) $\Rightarrow$ (3): Let B be any subset of Y . By (4),

$$X - f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(B)) \subseteq X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)).$$

Thus, $(\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1, \sigma_2)s\text{-Int}(B))$.

(1) $\Rightarrow$ (5): Let $x \in X$ and U be any $\tau_1\tau_2$ -open set of X containing x . Since f is weakly $\theta(\tau_1, \tau_2)s$ -open,

$$f(x) \in f(U) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))).$$

Put $V = \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. Then, V is a $\theta(\sigma_1, \sigma_2)s$ -open set of Y containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$.

(5) $\Rightarrow$ (1): Let U be any $\tau_1\tau_2$ -open set of X and $y \in f(U)$. It follows from (5) that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for some $\theta(\sigma_1, \sigma_2)s$ -open set V of Y containing y . Thus, $y \in V \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ and hence $f(U) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that f is weakly $\theta(\tau_1, \tau_2)s$ -open.

(1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1): This is obvious. $\square$

Theorem 3.2. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a bijective function. Then, the following properties are equivalent:

- (1) f is weakly $\theta(\tau_1, \tau_2)s$ -open;
- (2) $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $\tau_1\tau_2$ -open set U of X ;
- (3) $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for every $\tau_1\tau_2$ -closed set K of X .

Proof. (1) $\Rightarrow$ (3): Let K be any $\tau_1\tau_2$ -closed set of X . Then, we have

$$f(X - K) = Y - f(K) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(X - K)))$$

and hence $Y - f(K) \subseteq Y - \theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Thus, $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$.

(3) $\Rightarrow$ (2): Let U be any $\tau_1\tau_2$ -open set of X . Since $\tau_1\tau_2\text{-Cl}(U)$ is $\tau_1\tau_2$ -closed and $U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$, by (3) we have $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(U)) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$.

(2) $\Rightarrow$ (1): Let U be any $\tau_1\tau_2$ -open set of X . By (2), we have

$$\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))).$$

Since f is bijective, $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = Y - \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ and

$$f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = f(X - \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq f(X - U) = Y - f(U).$$

Thus, $f(U) \subseteq \theta(\sigma_1, \sigma_2)s\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U)))$ and hence f is weakly $\theta(\tau_1, \tau_2)s$ -open. $\square$

4. WEAKLY $\theta(\tau_1, \tau_2)s$ -CLOSED FUNCTIONS

In this section, we introduce the notion of weakly $\theta(\tau_1, \tau_2)s$ -closed functions. Moreover, some characterizations of weakly $\theta(\tau_1, \tau_2)s$ -closed functions are investigated.

Definition 4.1. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be weakly $\theta(\tau_1, \tau_2)s$ -closed if

$$\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$$

for each $\tau_1\tau_2$ -closed set K of X .

Theorem 4.1. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is weakly $\theta(\tau_1, \tau_2)s$ -closed;
- (2) $\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for each $(\tau_1, \tau_2)r$ -open set U of X ;

- (3) for each subset B of Y and each $\tau_1\tau_2$ -open set U of X with $f^{-1}(B) \subseteq U$, there exists a $\theta(\sigma_1, \sigma_2)$ - s -open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(U)$;
- (4) for each point $y \in Y$ and each $\tau_1\tau_2$ -open set U of X with $f^{-1}(y) \subseteq U$, there exists a $\theta(\sigma_1, \sigma_2)$ - s -open set V of Y containing y such that $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(U)$;
- (5) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for each $\tau_1\tau_2$ -open set U of X ;
- (6) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}((\tau_1, \tau_2)\theta\text{-Cl}(U)))) \subseteq f((\tau_1, \tau_2)\theta\text{-Cl}(U))$ for each $\tau_1\tau_2$ -open set U of X ;
- (7) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for each (τ_1, τ_2) - p -open set U of X .

Proof. It is clear that $(1) \Rightarrow (5) \Rightarrow (7) \Rightarrow (2) \Rightarrow (1)$, $(1) \Rightarrow (6)$ and $(3) \Rightarrow (4)$. To show that $(2) \Rightarrow (3)$: Let B be any subset of Y and U be any $\tau_1\tau_2$ -open set of X with $f^{-1}(B) \subseteq U$. Then,

$$f^{-1}(B) \cap \tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U)) = \emptyset$$

and $B \cap f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$. Since $X - \tau_1\tau_2\text{-Cl}(U)$ is (τ_1, τ_2) - r -open, we have

$$B \cap \theta(\sigma_1, \sigma_2)\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$$

by (2). Put $V = Y - \theta(\sigma_1, \sigma_2)\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))$. Then, V is a $\theta(\sigma_1, \sigma_2)$ - s -open set of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq X - f^{-1}(\theta(\sigma_1, \sigma_2)\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U)))) \subseteq X - f^{-1}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq \tau_1\tau_2\text{-Cl}(U)$.

$(6) \Rightarrow (1)$: It suffices see that $(\tau_1, \tau_2)\theta\text{-Cl}(U) = \tau_1\tau_2\text{-Cl}(U)$ for every $\tau_1\tau_2$ -open set U of X .

$(4) \Rightarrow (1)$: Let K be any $\tau_1\tau_2$ -closed set of X and $y \in Y - f(K)$. Since $f^{-1}(y) \subseteq X - K$, there exists a $\theta(\sigma_1, \sigma_2)$ - s -open set V of Y such that $y \in V$ and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(X - K) = X - \tau_1\tau_2\text{-Int}(K)$ by (4). Thus, $V \cap f(\tau_1\tau_2\text{-Int}(K)) = \emptyset$ and hence $y \in Y - \theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. Therefore, $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$. This shows that f is weakly $\theta(\tau_1, \tau_2)$ - s -closed.

$(6) \Rightarrow (7)$: This is obvious since $(\tau_1, \tau_2)\theta\text{-Cl}(U) = \tau_1\tau_2\text{-Cl}(U)$ for every (τ_1, τ_2) - p -open set U of X . □

The proof of the following result is mostly straightforward and is therefore omitted.

Theorem 4.2. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is weakly $\theta(\tau_1, \tau_2)$ - s -closed;
- (2) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $\tau_1\tau_2$ -open set U of X ;
- (3) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for every (τ_1, τ_2) - p -closed set K of X ;
- (4) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for every $\alpha(\tau_1, \tau_2)$ -closed set K of X .

Theorem 4.3. For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (1) f is weakly $\theta(\tau_1, \tau_2)$ - s -closed;
- (2) $\theta(\sigma_1, \sigma_2)\text{-Cl}(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $\tau_1\tau_2$ -open set U of X .

Proof. (1) $\Rightarrow$ (2): Let U be any $\tau_1\tau_2$ -open set of X . By (1), we have

$$\begin{aligned}\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(U)) &= \theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(U))) \\ &\subseteq \theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)).\end{aligned}$$

(2) $\Rightarrow$ (1): Let K be any $\tau_1\tau_2$ -closed set of X . Using (2), we have

$$\begin{aligned}\theta(\sigma_1, \sigma_2)s\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(K)) \\ &= f(K).\end{aligned}$$

This shows that f is weakly $\theta(\tau_1, \tau_2)s$ -closed. $\square$

Acknowledgements: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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