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On $\theta(\tau_1, \tau_2)p$-Open Functions and $\theta(\tau_1, \tau_2)p$-Closed Functions

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Abstract. This paper deals with the concepts of $\theta(\tau_1, \tau_2)p$-open functions and $\theta(\tau_1, \tau_2)p$-closed functions. Moreover, several characterizations and some properties concerning $\theta(\tau_1, \tau_2)p$-open functions and $\theta(\tau_1, \tau_2)p$-closed functions are considered.

1. Introduction

Openness and closedness are fundamental with respect to the investigation of general topological spaces. Various types of generalizations of open functions and closed functions have been researched by many mathematicians. Mashhour et al. [18] introduced and studied the notion of preopen functions. Noiri [21] introduced and investigated the concept of semi-open functions. Mashhour et al. [17] studied some characterizations of α-open functions. El-Monsef et al. [1] introduced and investigated the notions of β-open functions. The concept of weakly open functions was first introduced by Rose [24]. Rose and Janković [23] investigated some of the fundamental properties of weakly closed functions. Caldas and Navalagi [11] introduced two new classes of functions called weakly preopen functions and weakly preclosed functions as a generalization of weak openness and weak closedness due to [24] and [23], respectively. Moreover, Caldas and Navalagi [10] introduced and investigated the concepts of weakly semi-open functions and weakly semi-closed functions as a new generalization of weakly open functions and weakly closed functions, respectively. Noiri and Popa [19] studied a new class of functions called M-closed functions as functions defined between sets satisfying some conditions. Pal et al. [22] introduced and studied

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the notion of pre-θ-closed sets in topological spaces. Caldas et al. [9] introduced the notions of pre-θ-derided, pre-θ-border, pre-θ-frontier and pre-θ-exterior of a set. Noiri [20] introduced and investigated the notion of θ-precontinuous functions. Furthermore, Caldas et al. [9] defined the concepts of θ-preopenness and θ-preclosedness as a natural dual to the θ-precontinuity due to Noiri [20]. Khampakdee and Boonpok [14] investigated some characterizations of (Λ, p)-closed functions. Klanarong and Boonpok [16] studied the notion of weakly $s(\Lambda, p)$-open functions and weakly $s(\Lambda, p)$-closed functions by utilizing $s(\Lambda, p)$-open sets and the $s(\Lambda, p)$-closure operator. On the other hand, the present authors introduced and studied the concepts of $\theta p(\Lambda, p)$-open functions [5], $\theta p(\Lambda, p)$-closed functions [5], (m, n)-weakly open functions [8], weakly $\delta(\Lambda, p)$-open functions [25], weakly $\delta(\Lambda, p)$-closed functions [15], weakly $\beta(\Lambda, p)$-open functions [4], weakly $\beta(\Lambda, p)$-closed functions [4], weakly $p(\Lambda, p)$-open functions [3], weakly $p(\Lambda, p)$-closed functions [3], weakly $\theta s(\Lambda, p)$-open functions [2], weakly $\theta s(\Lambda, p)$-closed functions [2] and weakly $b(\Lambda, p)$-open functions [13]. In this paper, we introduce the concepts of $\theta(\tau_1, \tau_2)p$-open functions and $\theta(\tau_1, \tau_2)p$-closed functions. We also investigate several characterizations of $\theta(\tau_1, \tau_2)p$-open functions and $\theta(\tau_1, \tau_2)p$-closed functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [7] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [27] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space

(X,τ_1,τ_2) is said to be *$\alpha(\tau_1,\tau_2)$-open* [26] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1,\tau_2)$-open set is said to be *$\alpha(\tau_1,\tau_2)$-closed*. For a subset A of a bitopological space (X,τ_1,τ_2), a point $x \in X$ is called *$(\tau_1,\tau_2)\theta$-cluster point* [27] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1,\tau_2)\theta$-cluster points of A is called the *$(\tau_1,\tau_2)\theta$-closure* [27] of A and is denoted by $(\tau_1,\tau_2)\theta$-Cl(A). A subset A of a bitopological space (X,τ_1,τ_2) is said to be *$(\tau_1,\tau_2)\theta$-closed* [27] if $A = (\tau_1,\tau_2)\theta$-Cl$(A)$. The complement of a $(\tau_1,\tau_2)\theta$-closed set is said to be *$(\tau_1,\tau_2)\theta$-open*. The union of all $(\tau_1,\tau_2)\theta$-open sets contained in A is called the *$(\tau_1,\tau_2)\theta$-interior* [27] of A and is denoted by $(\tau_1,\tau_2)\theta$-Int(A).

Lemma 2.2. [27] *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) *If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1,\tau_2)\theta$-Cl$(A)$.*
(2) *$(\tau_1,\tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.*

Let A be a subset of a bitopological space (X,τ_1,τ_2). A point $x \in X$ is called a *$\theta(\tau_1,\tau_2)p$-cluster point* of A if (τ_1,τ_2)-sCl$(U) \cap A \neq \emptyset$ for every $(\tau_1,\tau_2)p$-open set U of X containing x. The set of all $\theta(\tau_1,\tau_2)p$-cluster points of A is called the *$\theta(\tau_1,\tau_2)p$-closure* of A and is denoted by $\theta(\tau_1,\tau_2)p$-Cl(A). If $A = \theta(\tau_1,\tau_2)p$-Cl$(A)$, then A is called *$\theta(\tau_1,\tau_2)p$-closed*. The complement of a $\theta(\tau_1,\tau_2)p$-closed set is called *$\theta(\tau_1,\tau_2)p$-open*. The *$\theta(\tau_1,\tau_2)p$-interior* of A is defined by the union of all $\theta(\tau_1,\tau_2)p$-open sets of X contained in A and is denoted by $\theta(\tau_1,\tau_2)p$-Int(A).

Lemma 2.3. *For subsets A and B of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) *$X - \theta(\tau_1,\tau_2)p$-Cl$(A) = \theta(\tau_1,\tau_2)p$-Int$(X - A)$ and $X - \theta(\tau_1,\tau_2)p$-Int$(A) = \theta(\tau_1,\tau_2)p$-Cl$(X - A)$.*
(2) *A is $\theta(\tau_1,\tau_2)p$-open if and only if $A = \theta(\tau_1,\tau_2)p$-Int$(A)$.*
(3) *$A \subseteq (\tau_1,\tau_2)$-pCl$(A) \subseteq \theta(\tau_1,\tau_2)p$-Cl$(A)$ and $\theta(\tau_1,\tau_2)p$-Int$(A) \subseteq (\tau_1,\tau_2)$-pInt$(A)$.*
(4) *If $A \subseteq B$, then $\theta(\tau_1,\tau_2)p$-Cl$(A) \subseteq \theta(\tau_1,\tau_2)p$-Cl$(B)$ and $\theta(\tau_1,\tau_2)p$-Int$(A) \subseteq \theta(\tau_1,\tau_2)p$-Int$(B)$.*
(5) *If A is $(\tau_1,\tau_2)p$-open, then (τ_1,τ_2)-pCl$(A) = \theta(\tau_1,\tau_2)p$-Cl$(A)$.*

3. On $\theta(\tau_1,\tau_2)p$-open functions

In this section, we introduce the concept of $\theta(\tau_1,\tau_2)p$-open functions. Moreover, some characterizations of $\theta(\tau_1,\tau_2)p$-open functions are discussed.

Definition 3.1. *A functions $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be $\theta(\tau_1,\tau_2)p$-open if*

$$f(U) \subseteq \theta(\sigma_1,\sigma_2)p\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$$

for each $\tau_1\tau_2$-open set U of X.

Theorem 3.1. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *f is $\theta(\tau_1,\tau_2)p$-open;*
(2) *$f((\tau_1,\tau_2)\theta$-Int$(A)) \subseteq \theta(\sigma_1,\sigma_2)p$-Int$(f(A))$ for every subset A of X;*
(3) *$(\tau_1,\tau_2)\theta$-Int$(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1,\sigma_2)p$-Int$(B))$ for every subset B of Y;*

(4) $f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-}Cl(B)) \subseteq (\tau_1,\tau_2)\theta\text{-}Int(f^{-1}(B))$ *for every subset B of Y;*
(5) $f(\tau_1\tau_2\text{-}Int(K)) \subseteq \theta(\sigma_1,\sigma_2)p\text{-}Int(f(K))$ *for every $\tau_1\tau_2$-closed set K of X;*
(6) $f(\tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(U))) \subseteq \theta(\sigma_1,\sigma_2)p\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for every $\tau_1\tau_2$-open set U of X;*
(7) $f(U) \subseteq \theta(\sigma_1,\sigma_2)p\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for every $(\tau_1,\tau_2)r$-open set U of X;*
(8) $f(U) \subseteq \theta(\sigma_1,\sigma_2)p\text{-}Int(f(\tau_1\tau_2\text{-}Cl(U)))$ *for every $\alpha(\tau_1,\tau_2)$-open set U of X.*

Proof. The proofs of (5) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (1) are straightforward and are omitted.

(1) ⇒ (2): Let A be any subset of X and $x \in (\tau_1,\tau_2)\theta\text{-Int}(A)$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq U$. Then, $f(x) \in f(U) \subseteq f(\tau_1\tau_2\text{-Cl}(U)) \subseteq f(A)$. Since f is $\theta(\tau_1,\tau_2)p$-open, $f(U) \subseteq \theta(\sigma_1,\sigma_2)p\text{-Int}(f(\tau_1\tau_2\text{-Cl}(U))) \subseteq \theta(\sigma_1,\sigma_2)p\text{-Int}(f(A))$. It implies that $f(x) \in \theta(\sigma_1,\sigma_2)p\text{-Int}(f(A))$. Therefore, $x \in f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(f(A)))$. Thus, $(\tau_1,\tau_2)\theta\text{-Int}(A) \subseteq f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(f(A)))$ and hence

$$f((\tau_1,\tau_2)\theta\text{-Int}(A)) \subseteq \theta(\sigma_1,\sigma_2)p\text{-Int}(f(A)).$$

(2) ⇒ (3): Let B be any subset of Y. Then by (2), $f((\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(B))) \subseteq \theta(\sigma_1,\sigma_2)p\text{-Int}(B)$. Thus, $(\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(B))$.

(3) ⇒ (4): Let B be any subset of Y. Using (3), we have

$$\begin{aligned} X-(\tau_1,\tau_2)\theta\text{-Cl}(f^{-1}(B)) &= (\tau_1,\tau_2)\theta\text{-Int}(X-f^{-1}(B)) \\ &= (\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(Y-B)) \\ &\subseteq f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(Y-B)) \\ &= f^{-1}(Y-\theta(\sigma_1,\sigma_2)p\text{-Cl}(B)) \\ &= X-f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Cl}(B)) \end{aligned}$$

and hence $f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Cl}(B)) \subseteq (\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(B))$.

(4) ⇒ (5): Let K be any $\tau_1\tau_2$-closed set of X. Thus by (4),

$$f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Cl}(Y-f(K))) \subseteq (\tau_1,\tau_2)\theta\text{-Cl}(f^{-1}(Y-f(K))).$$

We have
$f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Cl}(Y-f(K))) = f^{-1}(Y-\theta(\sigma_1,\sigma_2)p\text{-Int}(f(K))) = X-f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(f(K)))$.
On the other hand,

$$\begin{aligned} (\tau_1,\tau_2)\theta\text{-Cl}(f^{-1}(Y-f(K))) &= (\tau_1,\tau_2)\theta\text{-Cl}(X-f^{-1}(f(K))) \\ &\subseteq (\tau_1,\tau_2)\theta\text{-Cl}(X-K) \\ &= X-(\tau_1,\tau_2)\theta\text{-Int}(K) \\ &= X-\tau_1\tau_2\text{-Int}(K), \end{aligned}$$

since K is $\tau_1\tau_2$-closed. Thus, $\tau_1\tau_2\text{-Int}(K) \subseteq f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Int}(f(K)))$ and so

$$f(\tau_1\tau_2\text{-Int}(K)) \subseteq \theta(\sigma_1,\sigma_2)p\text{-Int}(f(K)).$$

□

Theorem 3.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bijective function. Then, the following properties are equivalent:*

(1) *f is $\theta(\tau_1, \tau_2)p$-open;*
(2) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(U)) \subseteq f(\tau_1\tau_2$-$Cl(U))$ for every $\tau_1\tau_2$-open set U of X;*
(3) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(\tau_1\tau_2$-$Int(K))) \subseteq f(K)$ for every $\tau_1\tau_2$-closed set K of X.*

Proof. (1) ⇒ (3): Let K be any $\tau_1\tau_2$-closed set of X. Then, we have

$$Y - f(K) = f(X - K) \subseteq \theta(\sigma_1, \sigma_2)p\text{-Int}(f(\tau_1\tau_2\text{-Cl}(X - K)))$$

and hence $Y - f(K) \subseteq Y - \theta(\sigma_1, \sigma_2)p$-Cl$(f(\tau_1\tau_2$-Int$(K)))$. Thus, $\theta(\sigma_1, \sigma_2)p$-Cl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$.

(3) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Since $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-closed and $U \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))$, by (3) we have $\theta(\sigma_1, \sigma_2)p$-Cl$(f(U)) \subseteq \theta(\sigma_1, \sigma_2)p$-Cl$(f(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U)))) \subseteq f(\tau_1\tau_2$-Cl$(U))$.

(2) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X. By (2), we have

$$\theta(\sigma_1, \sigma_2)p\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))).$$

Since f is bijective, $\theta(\sigma_1, \sigma_2)p$-Cl$(f(X - \tau_1\tau_2$-Cl$(U))) = Y - \theta(\sigma_1, \sigma_2)p$-Int$(f(\tau_1\tau_2$-Cl$(U)))$ and

$$f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = f(X - \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq f(X - U) = Y - f(U).$$

Thus, $f(U) \subseteq \theta(\sigma_1, \sigma_2)p$-Int$(f(\tau_1\tau_2$-Cl$(U)))$. This shows that f is $\theta(\tau_1, \tau_2)p$-open. □

4. On $\theta(\tau_1, \tau_2)p$-closed functions

In this section, we introduce the notion of $\theta(\tau_1, \tau_2)p$-closed functions. Furthermore, several characterizations of $\theta(\tau_1, \tau_2)p$-closed functions are investigated.

Definition 4.1. *A functions $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $\theta(\tau_1, \tau_2)p$-closed if*

$$\theta(\sigma_1, \sigma_2)p\text{-}Cl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$$

for each $\tau_1\tau_2$-closed set K of X.

Theorem 4.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is $\theta(\tau_1, \tau_2)p$-closed;*
(2) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(U)) \subseteq f(\tau_1\tau_2$-$Cl(U))$ for every $\tau_1\tau_2$-open set U of X;*
(3) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(U)) \subseteq f(\tau_1\tau_2$-$Cl(U))$ for every $(\tau_1, \tau_2)p$-open set U of X;*
(4) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(\tau_1\tau_2$-$Int(K))) \subseteq f(K)$ for every $(\tau_1, \tau_2)p$-closed set K of X;*
(5) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(\tau_1\tau_2$-$Int(K))) \subseteq f(K)$ for every $\alpha(\tau_1, \tau_2)$-closed set K of X;*
(6) *$\theta(\sigma_1, \sigma_2)p$-$Cl(f(\tau_1\tau_2$-$Int(\tau_1\tau_2$-$Cl(A)))) \subseteq f(\tau_1\tau_2$-$Cl(A))$ for every subset A of X.*

Proof. (1) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Then by (1),

$$\begin{aligned}\theta(\sigma_1,\sigma_2)p\text{-Cl}(f(U)) &= \theta(\sigma_1,\sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(U)))\\ &\subseteq \theta(\sigma_1,\sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))\\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)).\end{aligned}$$

(2) ⇒ (3): Let U be any $(\tau_1,\tau_2)p$-open set of X. Using (2), we have

$$\begin{aligned}\theta(\sigma_1,\sigma_2)p\text{-Cl}(f(U)) &\subseteq \theta(\sigma_1,\sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))\\ &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))\\ &\subseteq f(\tau_1\tau_2\text{-Cl}(U)).\end{aligned}$$

(3) ⇒ (4): Let K be any $(\tau_1,\tau_2)p$-closed set of X. Then, we have

$$\theta(\sigma_1,\sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K).$$

It is clear that (4) ⇒ (5) ⇒ (6) ⇒ (1). □

Definition 4.2. [12] *A bitopological space (X,τ_1,τ_2) is said to be (τ_1,τ_2)-regular if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Lemma 4.1. [12] *A bitopological space (X,τ_1,τ_2) is (τ_1,τ_2)-regular if and only if for each $x \in X$ and each $\tau_1\tau_2$-open set U with $x \in U$, there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Theorem 4.2. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is (σ_1,σ_2)-regular, the following properties are equivalent:*

(1) *f is $\theta(\tau_1,\tau_2)p$-closed;*
(2) *$\theta(\sigma_1,\sigma_2)p\text{-}Cl(f(U)) \subseteq f(\tau_1\tau_2\text{-}Cl(U))$ for each $(\tau_1,\tau_2)r$-open set U of X;*
(3) *for each subset B of Y and each $\tau_1\tau_2$-open set U of X with $f^{-1}(B) \subseteq U$, there exists a $\theta(\sigma_1,\sigma_2)p$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Cl(U)$;*
(4) *for each point $y \in Y$ and each $\tau_1\tau_2$-open set U of X with $f^{-1}(y) \subseteq U$, there exists a $\theta(\sigma_1,\sigma_2)p$-open set V of Y containing y and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Cl(U)$.*

Proof. (1) ⇒ (2) and (3) ⇒ (4): The proofs are obvious.

(2) ⇒ (3): Let B be any subset of Y and U be any $\tau_1\tau_2$-open set of X with $f^{-1}(B) \subseteq U$. Then, we have $f^{-1}(B) \cap \tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U)) = \emptyset$ and hence $B \cap f(\tau_1\tau_2\text{-Cl}(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$. Since $X - \tau_1\tau_2\text{-Cl}(U)$ is $(\tau_1,\tau_2)r$-open, $B \cap \theta(\sigma_1,\sigma_2)p\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))) = \emptyset$ by (2). Put

$$V = Y - \theta(\sigma_1,\sigma_2)p\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))).$$

Then, V is a $\theta(\sigma_1,\sigma_2)p$-open set of Y such that $B \subseteq V$ and

$$\begin{aligned}f^{-1}(V) &\subseteq X - f^{-1}(\theta(\sigma_1,\sigma_2)p\text{-Cl}(f(X - \tau_1\tau_2\text{-Cl}(U))))\\ &\subseteq X - f^{-1}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq \tau_1\tau_2\text{-Cl}(U).\end{aligned}$$

$(4) \Rightarrow (1)$: Let K be any $\tau_1\tau_2$-closed set of Y and $y \in Y - f(K)$. Since $f^{-1}(y) \subseteq X - K$, there exists a $\theta(\sigma_1, \sigma_2)p$-open set V of Y such that $y \in V$ and $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Cl}(X - K) = X - \tau_1\tau_2\text{-Int}(K)$ by (4). Thus, $V \cap f(\tau_1\tau_2\text{-Int}(K)) = \emptyset$ and hence $y \in Y - \theta(\sigma_1, \sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K)))$. It implies that $\theta(\sigma_1, \sigma_2)p\text{-Cl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$. This shows that f is $\theta(\tau_1, \tau_2)p$-closed. □

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