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Weak Forms of Open and Closed Functions Defined Between Bitopological Spaces

Prapart Pue-on[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1]*Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*

[2]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

*Corresponding author: chawalit.b@msu.ac.th

Abstract. This paper presents new classes of open and closed functions defined between bitopological spaces, called weakly $(\tau_1, \tau_2)p$-open functions and weakly $(\tau_1, \tau_2)p$-closed functions. Furthermore, several characterizations of weakly $(\tau_1, \tau_2)p$-open functions and weakly $(\tau_1, \tau_2)p$-closed functions are investigated.

1. Introduction

In 1984, Rose [16] introduced and studied the notions of weakly open functions and almost open functions. Rose and Janković [15] investigated some of the fundamental properties of weakly closed functions. In 2004, Caldas and Navalagi [10] introduced two new classes of functions called weakly preopen functions and weakly preclosed functions as a generalization of weak openness and weak closedness due to [16] and [15], respectively. Moreover, Caldas and Navalagi [8] introduced and investigated the concepts of weakly semi-open functions and weakly semi-closed functions as a new generalization of weakly open functions and weakly closed functions, respectively. In 2006, Caldas et al. [9] presented the class of weakly semi-θ-openness (resp. weakly semi-θ-closedness) as a new generalization of semi-θ-openness (resp. semi-θ-closedness). In 2009, Noiri et al. [14] introduced and studied two new classes of functions called weakly b-θ-open functions and weakly b-θ-closed functions by utilizing the notions of b-θ-open sets and the b-θ-closure operator. Weak b-θ-openness (resp. b-θ-closedness) is a generalization of both θ-preopenness and weak semi-θ-openness (resp. θ-preclosedness and weak semi-θ-closedness). Recently, Chutiman and Boonpok [11] studied some properties of weakly $b(\Lambda, p)$-open functions. Klanarong and

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Boonpok [13] introduced and investigated the notions of weakly $s(\Lambda, p)$-open functions and weakly $s(\Lambda, p)$-closed functions by utilizing $s(\Lambda, p)$-open sets and the $s(\Lambda, p)$-closure operator. On the other hand, the present authors introduced and studied the concepts of $\theta p(\Lambda, p)$-open (resp. $\theta p(\Lambda, p)$-closed) functions [4], semi-$(\mathscr{I}, \mathscr{J})$-open (resp. semi-$(\mathscr{I}, \mathscr{J})$-closed) functions [5], weakly $\delta(\Lambda, p)$-open functions [17], weakly $\delta(\Lambda, p)$-closed functions [12], weakly $\beta(\Lambda, p)$-open (resp. weakly $\beta(\Lambda, p)$-closed) functions [3], weakly $p(\Lambda, p)$-open (resp. weakly $p(\Lambda, p)$-closed) functions [2] and weakly $\theta s(\Lambda, p)$-open (resp. weakly $\theta s(\Lambda, p)$-closed) functions [1]. In this paper, we introduce the concepts of weakly $(\tau_1, \tau_2)p$-open functions and weakly $(\tau_1, \tau_2)p$-closed functions. Moreover, some characterizations of weakly $(\tau_1, \tau_2)p$-open functions and weakly $(\tau_1, \tau_2)p$-closed functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 2.1. [7] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [20] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [19] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. The intersection of all $(\tau_1, \tau_2)p$-closed sets of X containing A is called the $(\tau_1, \tau_2)p$-*closure* [18] of A and is denoted by (τ_1, τ_2)-pCl(A). The union of all $(\tau_1, \tau_2)p$-open sets of X contained in A is called the $(\tau_1, \tau_2)p$-*interior* [18] of A and is denoted by (τ_1, τ_2)-pInt(A). For a subset A of a bitopological space (X, τ_1, τ_2), a point $x \in X$ is called

$(\tau_1, \tau_2)\theta$*-cluster point* of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* if $A = (\tau_1, \tau_2)\theta$-Cl$(A)$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A) [20].

Lemma 2.2. [20] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.*
(2) *$(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.*

3. On weakly $(\tau_1, \tau_2)p$-open functions

In this section, we introduce the concept of weakly $(\tau_1, \tau_2)p$-open functions. Moreover, some characterizations of weakly $(\tau_1, \tau_2)p$-open functions are discussed.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $(\tau_1, \tau_2)p$-open if $f(U) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U)))$ for each $\tau_1\tau_2$-open set U of X.*

Theorem 3.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly $(\tau_1, \tau_2)p$-open;*
(2) *$f((\tau_1, \tau_2)\theta$-Int$(A)) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(A))$ for every subset A of X;*
(3) *$(\tau_1, \tau_2)\theta$-Int$(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)$-pInt$(B))$ for every subset B of Y;*
(4) *$f^{-1}((\sigma_1, \sigma_2)$-pCl$(B)) \subseteq (\tau_1, \tau_2)\theta$-Cl$(f^{-1}(B))$ for every subset B of Y;*
(5) *for each $x \in X$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $(\sigma_1, \sigma_2)p$-open set V of Y containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2$-Cl$(U))$;*
(6) *$f(\tau_1\tau_2$-Int$(K)) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(K))$ for each $\tau_1\tau_2$-closed set K of X;*
(7) *$f(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U))) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U)))$ for every $\tau_1\tau_2$-open set U of X;*
(8) *$f(U) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U)))$ for every $(\tau_1, \tau_2)p$-open set U of X;*
(9) *$f(U) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U)))$ for every $\alpha(\tau_1, \tau_2)$-open set U of X.*

Proof. (1) ⇒ (2): Let A be any subset of X and $x \in (\tau_1, \tau_2)\theta$-Int$(A)$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq A$. Then, $f(x) \in f(U) \subseteq f(\tau_1\tau_2$-Cl$(U)) \subseteq f(A)$. Since f is weakly $(\tau_1, \tau_2)p$-open, $f(U) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U))) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(A))$. It implies that $f(x) \in (\sigma_1, \sigma_2)$-pInt$(f(A))$. This shows that $x \in f^{-1}((\sigma_1, \sigma_2)$-pInt$(f(A)))$. Thus, $(\tau_1, \tau_2)\theta$-Int$(A) \subseteq f^{-1}((\sigma_1, \sigma_2)$-pInt$(f(A)))$ and hence $f((\tau_1, \tau_2)\theta$-Int$(A)) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(A))$.

(2) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X. As $U \subseteq (\tau_1, \tau_2)\theta$-Int$(\tau_1\tau_2$-Cl$(U))$ implies

$$f(U) \subseteq f((\tau_1, \tau_2)\theta\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq (\sigma_1, \sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U))).$$

Thus, f is weakly $(\tau_1, \tau_2)p$-open.

(2) ⇒ (3): Let B be any subset of Y. Then by (2), $f((\tau_1, \tau_2)\theta$-Int$(f^{-1}(B))) \subseteq (\sigma_1, \sigma_2)$-pInt$(B)$. Thus, $(\tau_1, \tau_2)\theta$-Int$(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)$-pInt$(B))$.

(3) ⇒ (2): Let A be any subset of X. By (3), we have

$$(\tau_1,\tau_2)\theta\text{-Int}(A) \subseteq (\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(f(A))) \subseteq f^{-1}((\sigma_1,\sigma_2)\text{-pInt}(f(A)))$$

and so $f((\tau_1,\tau_2)\theta\text{-Int}(A)) \subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(A))$.

(3) ⇒ (4): Let B be any subset of Y. Using (3), we have

$$\begin{aligned} X - (\tau_1,\tau_2)\theta\text{-Cl}(f^{-1}(B)) &= (\tau_1,\tau_2)\theta\text{-Int}(X - f^{-1}(B)) \\ &= (\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}((\sigma_1,\sigma_2)\text{-pInt}(Y - B)) \\ &= f^{-1}(Y - (\sigma_1,\sigma_2)\text{-pCl}(B)) \\ &= X - f^{-1}((\sigma_1,\sigma_2)\text{-pCl}(B)) \end{aligned}$$

and hence $f^{-1}((\sigma_1,\sigma_2)\text{-pCl}(B)) \subseteq (\tau_1,\tau_2)\theta\text{-Cl}(f^{-1}(B))$.

(4) ⇒ (3): Let B be any subset of Y. Using (4), we have

$$X - f^{-1}((\sigma_1,\sigma_2)\text{-pInt}(B)) \subseteq X - (\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(B))$$

and so $(\tau_1,\tau_2)\theta\text{-Int}(f^{-1}(B)) \subseteq f^{-1}((\sigma_1,\sigma_2)\text{-pInt}(B))$.

(1) ⇒ (5): Let $x \in X$ and U be any $\tau_1\tau_2$-open set of X containing x. Since f is weakly $(\tau_1,\tau_2)p$-open, $f(x) \in f(U) \subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$. Put $V = (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$. Then, V is $(\sigma_1,\sigma_2)p$-open in Y containing $f(x)$ such that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$.

(5) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X and $y \in f(U)$. It follows from (5) that $V \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for some $(\sigma_1,\sigma_2)p$-open set V of Y containing y. Thus, $y \in V \subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$ and hence $f(U) \subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$. This shows that f is weakly $(\tau_1,\tau_2)p$-open.

(1) ⇒ (6): Let K be any $\tau_1\tau_2$-closed set of X. Then, $\tau_1\tau_2\text{-Int}(K)$ is $\tau_1\tau_2$-open in X. Thus by (1),

$$\begin{aligned} f(\tau_1\tau_2\text{-Int}(K)) &\subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K)))) \\ &\subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(K))) \\ &= (\sigma_1,\sigma_2)\text{-pInt}(f(K)). \end{aligned}$$

(6) ⇒ (7): Let U be any $\tau_1\tau_2$-open set of X. Then, we have $\tau_1\tau_2\text{-Cl}(U)$ is $\tau_1\tau_2$-closed in X and by (6), $f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$.

(7) ⇒ (8): Let U be any $\tau_1\tau_2$-open set of X. Then, we have $U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))$. By (7),

$$\begin{aligned} f(U) &\subseteq f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))) \\ &= f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))) \\ &\subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U))))) \\ &\subseteq (\sigma_1,\sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U))). \end{aligned}$$

(8) ⇒ (9): This is obvious since every $\alpha(\tau_1,\tau_2)$-open set is $(\tau_1,\tau_2)p$-open.

(9) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X. Then, U is $\alpha(\tau_1, \tau_2)$-open in X. Thus by (9), we have

$$f(U) \subseteq (\sigma_1, \sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U)))$$

and hence f is weakly $(\tau_1, \tau_2)p$-open. □

Theorem 3.2. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bijective function. Then, the following properties are equivalent:*

(1) *f is weakly $(\tau_1, \tau_2)p$-open;*
(2) *(σ_1, σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2$-Cl$(U))$ for every $\tau_1\tau_2$-open set U of X;*
(3) *(σ_1, σ_2)-pCl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$ for every $\tau_1\tau_2$-closed set K of X.*

Proof. (1) ⇒ (3): Let K be any $\tau_1\tau_2$-closed set of X. By (1), we have

$$f(X - K) = Y - f(K) \subseteq (\sigma_1, \sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(X - K)))$$

and hence $Y - f(K) \subseteq Y - (\sigma_1, \sigma_2)$-pCl$(f(\tau_1\tau_2$-Int$(K)))$. Thus, (σ_1, σ_2)-pCl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$.

(3) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Since $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-closed and

$$U \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)).$$

Thus by (3), (σ_1, σ_2)-pCl$(f(U)) \subseteq (\sigma_1, \sigma_2)$-pCl$(f(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U)))) \subseteq f(\tau_1\tau_2$-Cl$(U))$.

(2) ⇒ (3): Let K be any $\tau_1\tau_2$-closed set of X. Since $\tau_1\tau_2$-Int(K) is $\tau_1\tau_2$-open in X and by (2),

$$\begin{aligned}(\sigma_1, \sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(K))) &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(K)) = f(K).\end{aligned}$$

(3) ⇒ (1): Let U be any $\tau_1\tau_2$-open set of X. By (3), we have

$$\begin{aligned}Y - (\sigma_1, \sigma_2)\text{-pInt}(f(\tau_1\tau_2\text{-Cl}(U))) &= (\sigma_1, \sigma_2)\text{-pCl}(Y - f(\tau_1\tau_2\text{-Cl}(U))) \\ &\subseteq f(X - U) = Y - f(U)\end{aligned}$$

and hence $f(U) \subseteq (\sigma_1, \sigma_2)$-pInt$(f(\tau_1\tau_2$-Cl$(U)))$. Thus, f is weakly $(\tau_1, \tau_2)p$-open. □

4. On weakly $(\tau_1, \tau_2)p$-closed functions

We begin this section by introducing the concept of weakly $(\tau_1, \tau_2)p$-closed functions.

Definition 4.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly $(\tau_1, \tau_2)p$-closed if*

$$(\sigma_1, \sigma_2)\text{-}pCl(f(\tau_1\tau_2\text{-}Int(K))) \subseteq f(K)$$

for each $\tau_1\tau_2$-closed set K of X.

Theorem 4.1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is weakly $(\tau_1, \tau_2)p$-closed;*
(2) *(σ_1, σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2$-Cl$(U))$ for every $\tau_1\tau_2$-open set U of X.*

Proof. (1) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Thus by (1), we have

$$(\sigma_1,\sigma_2)\text{-pCl}(f(U)) = (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(U)))$$
$$\subseteq (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U)).$$

(2) ⇒ (1): Let K be any $\tau_1\tau_2$-closed set of X. Using (2), we have

$$(\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K)))$$
$$\subseteq f(\tau_1\tau_2\text{-Cl}(K)) = f(K).$$

This shows that f is weakly $(\tau_1,\tau_2)p$-closed. □

Theorem 4.2. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *f is weakly $(\tau_1,\tau_2)p$-closed;*
(2) *(σ_1,σ_2)-pCl$(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for every $(\tau_1,\tau_2)p$-closed set K of X;*
(3) *(σ_1,σ_2)-pCl$(f(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K)$ for every $\alpha(\tau_1,\tau_2)$-closed set K of X.*

Proof. (1) ⇒ (2): Let K be any $(\tau_1,\tau_2)p$-closed set of X. Then, $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(K)) \subseteq K$. Thus by (1),

$$(\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(K))) \subseteq (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K)))))$$
$$\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(K))) \subseteq f(K).$$

(2) ⇒ (3): Let K be any $\alpha(\tau_1,\tau_2)$-closed set of X. Then, K is $(\tau_1,\tau_2)p$-closed in X. Using (2), we have (σ_1,σ_2)-pCl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$.

(3) ⇒ (1): Let K be any $\tau_1\tau_2$-closed set of X. Then, we have K is $\alpha(\tau_1,\tau_2)$-closed in X. By (3), (σ_1,σ_2)-pCl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$. Thus, f is weakly $(\tau_1,\tau_2)p$-closed. □

Theorem 4.3. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *f is weakly $(\tau_1,\tau_2)p$-closed;*
(2) *(σ_1,σ_2)-pCl$(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $\tau_1\tau_2$-open set U of X;*
(3) *(σ_1,σ_2)-pCl$(f(\tau_1\tau_2\text{-Int}((\tau_1,\tau_2)\theta\text{-Cl}(U)))) \subseteq f((\tau_1,\tau_2)\theta\text{-Cl}(U))$ for every $\tau_1\tau_2$-open set U of X;*
(4) *(σ_1,σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $(\tau_1,\tau_2)p$-open set U of X;*
(5) *(σ_1,σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2\text{-Cl}(U))$ for every $(\tau_1,\tau_2)r$-open set U of X;*
(6) *for each subset B of Y and each $\tau_1\tau_2$-open set U of X with $f^{-1}(B) \subseteq U$, there exists a $(\sigma_1,\sigma_2)p$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq \tau_1\tau_2$-Cl$(U)$;*
(7) *for each point $y \in Y$ and each $\tau_1\tau_2$-open set U of X with $f^{-1}(y) \subseteq U$, there exists a $(\sigma_1,\sigma_2)p$-open set V of Y containing y and $f^{-1}(V) \subseteq \tau_1\tau_2$-Cl$(U)$.*

Proof. (1) ⇒ (2): Let U be any $\tau_1\tau_2$-open set of X. Then, $\tau_1\tau_2$-Cl(U) is $\tau_1\tau_2$-closed in X. Since f is weakly $(\tau_1,\tau_2)p$-closed, (σ_1,σ_2)-pCl$(f(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(U)))) \subseteq f(\tau_1\tau_2$-Cl$(U))$.

(2) ⇒ (3): It is suffices see that $(\tau_1,\tau_2)\theta$-Cl$(U) = \tau_1\tau_2$-Cl(U) for every $\tau_1\tau_2$-open set U of X.

(3) $\Rightarrow$ (4): It is suffices see that $(\tau_1,\tau_2)\theta$-Cl$(U) = \tau_1\tau_2$-Cl(U) for every $(\tau_1,\tau_2)p$-open set U of X.

(4) $\Rightarrow$ (5): Let U be any $(\tau_1,\tau_2)r$-open set of X. Then, U is $(\tau_1,\tau_2)p$-open in X. Using (4), we have (σ_1,σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2$-Cl$(U))$.

(5) $\Rightarrow$ (6): Let B be any subset of Y and U be any $\tau_1\tau_2$-open set of X with $f^{-1}(B) \subseteq U$. Then, $f^{-1}(B) \cap \tau_1\tau_2$-Cl$(X - \tau_1\tau_2$-Cl$(U)) = \emptyset$ and hence $B \cap f(\tau_1\tau_2$-Cl$(X - \tau_1\tau_2$-Cl$(U))) = \emptyset$. Since $X - \tau_1\tau_2$-Cl(U) is $(\tau_1,\tau_2)r$-open, $B \cap (\sigma_1,\sigma_2)$-pCl$(f(X - \tau_1\tau_2$-Cl$(U))) = \emptyset$ by (5). Put $V = Y - (\sigma_1,\sigma_2)$-pCl$(f(X - \tau_1\tau_2$-Cl$(U)))$. Then, we have V is $(\sigma_1,\sigma_2)p$-open such that $B \subseteq V$ and

$$\begin{aligned} f^{-1}(V) &\subseteq X - f^{-1}((\sigma_1,\sigma_2)\text{-pCl}(f(X - \tau_1\tau_2\text{-Cl}(U)))) \\ &\subseteq X - f^{-1}(f(X - \tau_1\tau_2\text{-Cl}(U))) \subseteq \tau_1\tau_2\text{-Cl}(U). \end{aligned}$$

(6) $\Rightarrow$ (7): This is obvious.

(7) $\Rightarrow$ (1): Let K be any $\tau_1\tau_2$-closed set of X and $y \in Y - f(K)$. Since $f^{-1}(y) \subseteq X - K$, there exists a $(\sigma_1,\sigma_2)p$-open set V of Y such that $y \in V$ and $f^{-1}(V) \subseteq \tau_1\tau_2$-Cl$(X - K) = X - \tau_1\tau_2$-Int$(K)$ by (7). Thus, $V \cap f(\tau_1\tau_2$-Int$(K)) = \emptyset$ and hence $y \in Y - (\sigma_1,\sigma_2)$-pCl$(f(\tau_1\tau_2$-Int$(K)))$. It implies that (σ_1,σ_2)-pCl$(f(\tau_1\tau_2$-Int$(K))) \subseteq f(K)$. This shows that f is weakly $(\tau_1,\tau_2)p$-closed. □

Theorem 4.4. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) *f is weakly $(\tau_1,\tau_2)p$-closed;*
(2) *(σ_1,σ_2)-pCl$(f(U)) \subseteq f(\tau_1\tau_2$-Cl$(U))$ for every $(\tau_1,\tau_2)r$-open set U of X.*

Proof. (1) $\Rightarrow$ (2): Let U be any $(\tau_1,\tau_2)r$-open set of X. Then, U is $\tau_1\tau_2$-open in X. Thus by Theorem 4.3,

$$(\sigma_1,\sigma_2)\text{-pCl}(f(U)) = (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(U)))) \subseteq f(\tau_1\tau_2\text{-Cl}(U)).$$

(2) $\Rightarrow$ (1): Let K be any $\tau_1\tau_2$-closed set of X. Then, $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(K))$ is $(\tau_1,\tau_2)r$-open in X. By (2), we have

$$\begin{aligned} (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(K))) &\subseteq (\sigma_1,\sigma_2)\text{-pCl}(f(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(K))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(K)))) \\ &\subseteq f(\tau_1\tau_2\text{-Cl}(K)) = f(K). \end{aligned}$$

This shows that f is weakly $(\tau_1,\tau_2)p$-closed. □

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