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New Study on $\mathfrak{D}-$Paracompactness in Bitopological Spaces

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Abstract. The new concept in the paper [1] of $\mathfrak{D}$-paracompact spaces is a well-studied type of topological space with rich structural properties and many applications in topological structures. That is, $\mathfrak{D}$-paracompact spaces are a natural extension of paracompact spaces in topology and provide a strong basis for understanding the relationship among local countableness, refinements, and $\mathfrak{D}$-covers. In this paper, we introduce the concept of pairwise $\mathfrak{D}$-paracompact spaces and their features relating to bitopological theory. Additionally, we will discuss the notion of $\mathfrak{D}$-paracompact spaces and describe a generalisation of bitopological space characteristics called pairwise $\mathfrak{D}$-paracompact spaces. This paper extends the theorems related to $\mathfrak{D}$-paracompact spaces in bitopological spaces and gives a number of theoretical findings, definitions, and properties that are thoroughly demonstrated.

1. Introduction

In recent years, several authors have investigated new extensions and refinements of compactness and related covering properties in both topological and bitopological settings. Oudetallah and Batiha explored foundational aspects such as almost expandability and finite products of pairwise expandable spaces in bitopological structures [2, 3]. Further contributions on generalized compactness were presented by Oudetallah *et al.*, who analyzed $\mathfrak{D}$-metacompactness [4],

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r-compactness [5], and *c*-compactness [6] within various topological frameworks. More recently, several studies have extended these notions toward higher-dimensional and generalized settings, including refined normality in regionally compact domains [7], nearly Lindelöfness in both N^{th}-topological and bitopological spaces [8,9], and σ-compactness in N^{th}-topological structures [10]. The study of multi-topological frameworks continues to expand, as seen in the recent work on tri-topological spaces and their interrelations [11]. Complementary developments addressing fault-tolerant structures in topological networks further emphasize the growing interdisciplinary relevance of topological analysis [12].

Bitopological spaces are a key basis for the general topology notions that have been expanded since they were first introduced, as they have been extensively generalized and developed in numerous fields of study. In 1963, Kelly [13] introduced the concept of bitopological spaces; after that, various topological features in singular topology were generalized into bitopological spaces. Later, in 1971, using filter base and sequence, Pahk and Choi [14] provided several characterizations about pairwise Hausdorffness, pairwise countable compactness, and pairwise compactness. They also presented a notion of pairwise local compactness that deviates from that of R. A. Stoenberg. They then established certain characterizations of it and examined certain associated features. Furthermore, in 1992, Ganster and Reilly [15] answered a recent question about the connection between the two concepts of paracompactness and bitopological spaces. In 2013, Mukharjee and Bose [16] presented a generalized concept of pairwise compactness, called nearly pairwise compactness. It is a generalization of near compactness as well. In 2016, Mukharjee et al. [17] developed concepts weaker than pairwise paracompactness and discovered several of their features. Furthermore, in 2018, Qoqazeh et al. [18] introduced both notions of pairwise metacompact and pairwise locally metacompact spaces, and examined their characteristics and relations to other topological spaces. Numerous well-known theorems about metacompact spaces were generalized and numerous instances were examined. Also, in 2018, Qoqazeh et al. [19] defined the notion of pairwise $\mathfrak{D}$-metacompact spaces and examined their characteristics and how they relate to other topological spaces. Numerous well-known theorems about metacompact spaces were generalized and several cases were addressed. In 2024, Al-Rababah et al. [1] presented new notions called $\mathfrak{D}$-paracompact spaces and countably $\mathfrak{D}$-paracompact spaces using a special type of covering and difference sets (also known as $\mathfrak{D}$-sets). They discussed these notions and their features. In 2025, Al-Rababah et al. [20] introduced the concept of pairwise $\mathfrak{D}$-paralindelof spaces and explained their significance and properties linked to bitopological theory. Moreover, in 2025, Atoom et al. [21] introduced the notion of $\mathfrak{D}$-paralindelof spaces. In this paper, we introduce and generalize the notion of $\mathfrak{D}$-paracompact spaces presented in [1] within the framework of bitopological spaces. We also investigate additional properties, definitions, and theorems of such significant spaces.

2. Preliminaries and Basic Notions

Within this paper, we have to keep looking at these significant characters as we go through the paper. We need $\mathbb{Q}$, $\mathbb{N}$, and $\mathbb{R}$ to represent the sets of rational, natural, and real numbers, respectively, in order to provide an illustration. In addition, γ_{ind}, γ_{dis}, and γ_{cof} will denote the indiscrete, discrete, and co-finite topologies, respectively. Furthermore, we will represent the simple shortcut *pair-* to denote *pairwise*, such as pair-paracompact for pairwise paracompact. In a similar way, pair-$\mathfrak{D}$- will represent pairwise $\mathfrak{D}$-, such as pair-$\mathfrak{D}$-paracompact for pairwise $\mathfrak{D}$-paracompact. In this section, we introduce known basic notions and theorems related to pair-$\mathfrak{D}$-paracompact spaces.

Definition 2.1. *[22] Let A be a subset of the space (G, γ). Then A is called a $\mathfrak{D}$-set if there exist two open sets M and N such that $M \neq G$ and $A = M - N$. That is, A is generated by M and N. Observe that any open set $M \neq G$ is a $\mathfrak{D}$-set if $A = M$ and $N = \emptyset$.*

Definition 2.2. *[23] A cover $\tilde{m} = \{m_\mu : \mu \in \Omega\}$ of the topological space (G, γ) is called a $\mathfrak{D}$-cover if each m_μ is a $\mathfrak{D}$-set for each $\mu \in \Omega$.*

Definition 2.3. *[19] Let (G, γ_1, γ_2) be a space. Then the $\mathfrak{D}$-sets are said to be pair-$\mathfrak{D}$-sets if they are generated by open sets of (G, γ_1, γ_2), and are denoted by D_{γ_i} for all $i = 1, 2$.*

Definition 2.4. *[19] A cover $\tilde{m}$ of the space (G, γ_1, γ_2) is called a pair-$\mathfrak{D}$-cover if $\tilde{m} \subset D_{\gamma_1} \cup D_{\gamma_2}$. Furthermore, if $\tilde{m}$ contains at least one pair-$\mathfrak{D}$-set, then it is called a pair-$\mathfrak{D}$-cover.*

Theorem 2.1. *[19] Any pair-cover is a pair-$\mathfrak{D}$-cover.*

Definition 2.5. *[24] Let (G, γ_1, γ_2) be a space. It is called pair-compact if each open pair-cover of G has a finite subcover.*

Definition 2.6. *[24] Let (G, γ_1, γ_2) be a space. It is called pair-Lindelöf if every open pair-cover of G has a countable subcover.*

Definition 2.7. *[19] Let (G, γ_1, γ_2) be a space. Then it is called pair-$\mathfrak{D}$-Lindelöf if every pair-$\mathfrak{D}$-cover of G has a countable subcover.*

Definition 2.8. *[1] The topological space (G, γ) is called $\mathfrak{D}$-paracompact if any $\mathfrak{D}$-cover of G has an open locally finite refinement.*

Definition 2.9. *[1] The space (G, γ) is called countably $\mathfrak{D}$-paracompact if any countable $\mathfrak{D}$-cover of G has an open locally finite refinement.*

Definition 2.10. *[20] Let (G, γ_1, γ_2) be a space. Then it is called pair-$\mathfrak{D}$-paralindelöf if every pair-$\mathfrak{D}$-cover of G has a pairwise open locally countable refinement.*

Definition 2.11. *[20] Let (G, γ_1, γ_2) be a space. Then it is called pair-paralindelöf if every pair-cover of G has a pairwise open locally countable refinement.*

Definition 2.12. *[19] Let (G, γ_1, γ_2) be a space. Then it is called locally indiscrete if every open set is clopen. Furthermore, every $\mathfrak{D}$-set is clopen in a locally indiscrete space.*

Definition 2.13. *[19] Let C be a subset of (G, γ_1, γ_2). Then it is called pair-$\mathfrak{D}$-dense if every $\mathfrak{D}$-set D_g containing g satisfies $D_g \cap C \neq \emptyset$ for any $g \in G$.*

Definition 2.14. *[19] Let (G, γ_1, γ_2) be a bitopological space. Then it is called pair-$\mathfrak{D}$-separable if it contains a pair-$\mathfrak{D}$-dense countable subset C.*

Definition 2.15. *[24] A function $\Upsilon : (G, \gamma_1, \gamma_2) \longrightarrow (T, \theta_1, \theta_2)$ is called pair-continuous if the functions $\Upsilon_1 : (G, \gamma_1) \longrightarrow (T, \theta_1)$ and $\Upsilon_2 : (G, \gamma_2) \longrightarrow (T, \theta_2)$ are continuous.*

Definition 2.16. *[24] A function $\Upsilon : (G, \gamma_1, \gamma_2) \longrightarrow (T, \theta_1, \theta_2)$ is called pair-closed if the functions $\Upsilon_1 : (G, \gamma_1) \longrightarrow (T, \theta_1)$ and $\Upsilon_2 : (G, \gamma_2) \longrightarrow (T, \theta_2)$ are closed.*

Definition 2.17. *[19] A function $\Upsilon : (G, \gamma_1, \gamma_2) \longrightarrow (T, \theta_1, \theta_2)$ is called pair-perfect if Υ is pair-continuous, pair-closed, and for every $t \in T$, we have $\Upsilon^{-1}(t)$ is pair-compact.*

Definition 2.18. *[19] Let (G, γ_1, γ_2) and (T, θ_1, θ_2) be spaces. Then the Cartesian product of (G, γ_1, γ_2) and (T, θ_1, θ_2) is the bitopological space $(G \times T, \gamma_1 \times \theta_1, \gamma_2 \times \theta_2)$.*

3. Pairwise of Difference Paracompact Spaces

This section presents the new concepts of pairwise $\mathfrak{D}$-paracompact spaces on bitopological spaces, studies their relations with other spaces, and presents several of their characteristics and theorems.

Definition 3.1. *A bitopological space (G, γ_1, γ_2) is said to be pair-paracompact if any pair-cover of the space has a pairwise open locally finite refinement.*

Definition 3.2. *A bitopological space (G, γ_1, γ_2) is said to be pair-$\mathfrak{D}$-paracompact if any pair-$\mathfrak{D}$-cover of the space has a pairwise open locally finite refinement.*

Theorem 3.1. *Any pair-$\mathfrak{D}$-paracompact space (G, γ_1, γ_2) is pair-paracompact.*

Proof. Let (G, γ_1, γ_2) be a pair-$\mathfrak{D}$-paracompact space, and let $\tilde{m} = \{m_\mu : \mu \in \Omega\}$ be an open pair-cover of (G, γ_1, γ_2). By Theorem 2.5, we have that $\tilde{m}$ is an open pair-$\mathfrak{D}$-cover, so it contains an open pairwise locally finite refinement. □

The following example provides an explanation of why the converse of the previous theorem might not be true.

Example 3.1. *Let $\gamma_1 = \{C \subset \mathbb{R} : \mathbb{Q} \subset C\} \cup \{\emptyset\}$ and $\gamma_2 = \{C \subset \mathbb{R} : \mathbb{R} - C$ is uncountable$\} \cup \{\emptyset\}$. Therefore, the space $(\mathbb{R}, \gamma_1, \gamma_2)$ is pair-paracompact, but it is not pair-$\mathfrak{D}$-paracompact, because $\{\{p\} : p \in \mathbb{R} - \mathbb{Q}\} \cup \{(-k, k) : k \in \mathbb{N}\}$ is an open pair-$\mathfrak{D}$-cover that has no open locally finite refinement.*

To show that the converse of Theorem 3.2 holds under specific assumptions, we present the following theorem.

Theorem 3.2. *If the bitopological space (G, γ_1, γ_2) is locally indiscrete and pair-paracompact, then the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact.*

Proof. Let $\tilde{m}$ be an open pair-$\mathfrak{D}$-cover of (G, γ_1, γ_2). Then $\tilde{m} = \{m_\mu : \mu \in \Omega\} \cup \{l_\nu : \nu \in \Lambda\}$, where $m_\mu \in H_{\gamma_1}$ for each $\mu \in \Omega$ and $l_\nu \in H_{\gamma_2}$ for each $\nu \in \Lambda$. Since the space (G, γ_1, γ_2) is locally indiscrete, each m_μ is a clopen set for all $\mu \in \Omega$, and each l_ν is clopen for all $\nu \in \Lambda$. Hence $\tilde{m} = \{m_\mu : \mu \in \Omega\} \cup \{l_\nu : \nu \in \Lambda\}$ is an open pair-cover. Thus, $\tilde{m}$ contains a pairwise open locally finite refinement. Hence proved. □

Example 3.2. *Observe that the space $(\mathbb{R}, \gamma_{ind}, \gamma_{dis})$ is locally indiscrete and pair-$\mathfrak{D}$-paracompact; hence, it is pair-paracompact.*

Theorem 3.3. *If the bitopological space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact and pair-$\mathfrak{D}$-separable, then the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-Lindelöf.*

Proof. Let the set P have an open pair-$\mathfrak{D}$-cover, denoted by $\tilde{m} = \{m_\mu : \mu \in \Omega\}$. Suppose that $\tilde{m}$ contains no countable subcover. Now, let $\tilde{l} = \{l_\nu : \nu \in \Lambda\}$ be an open locally finite refinement of $\tilde{m}$. Take a countable subset E of P such that E is $\mathfrak{D}$-dense. Therefore, for any $\nu \in \Lambda$, we have $l_\nu \cap E \neq \emptyset$. Hence $\tilde{l}$ is countable, and we can write $\tilde{l} = \{l_k : k \in \mathbb{N}\}$. Moreover, we have $l_k \subseteq m_{\mu_k}$ for some $\mu_k \in \Omega$. Consequently,

$$P = \bigcup_{k \in \mathbb{N}} l_k \subseteq \bigcup_{k \in \mathbb{N}} m_{\mu_k} \subseteq P.$$

Thus, the set $\{m_{\mu_k} : k \in \mathbb{N}\}$ is a countable subcover of $\tilde{m}$, which contradicts the supposition. Hence proved. □

By considering the fact that every open pair-cover is a pair-$\mathfrak{D}$-cover, we obtain the following corollary.

Corollary 3.1. *If the bitopological space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact and pair-$\mathfrak{D}$-separable, then the space (G, γ_1, γ_2) is pair-Lindelöf.*

Definition 3.3. *A bitopological space (G, γ_1, γ_2) is said to be countably pair-$\mathfrak{D}$-paracompact if any countable pair-$\mathfrak{D}$-cover of the space has a pairwise open locally finite refinement.*

Definition 3.4. *Let (G, γ_1, γ_2) be a space. Then the space is said to be pair-$\mathfrak{D}$-compact if each pair-$\mathfrak{D}$-cover of G has a finite subcover.*

Theorem 3.4. *If the bitopological space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-Lindelöf and countably pair-$\mathfrak{D}$-paracompact, then the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact.*

Proof. Let the space G be pair-Lindelöf, and let $\tilde{m} = \{m_\gamma : \gamma \in \Omega\}$ be a pair-$\mathfrak{D}$-cover of G. Then $\tilde{m}$ contains a countable subcover $\tilde{l} = \{l_k\}_{k=1}^{\infty}$. Since the space G is countably pair-$\mathfrak{D}$-paracompact, $\tilde{l}$

contains an open pairwise locally finite refinement $\tilde{R}$ of $\tilde{m}$. Consequently, the space G is pair-$\mathfrak{D}$-paracompact. □

We can present the following corollaries through similar reasoning.

Corollary 3.2. *Any countably pair-$\mathfrak{D}$-paracompact and pair-$\mathfrak{D}$-Lindelöf space (G, γ_1, γ_2) is pair-paracompact.*

Corollary 3.3. *Any countably pair-$\mathfrak{D}$-paracompact and pair-Lindelöf space (G, γ_1, γ_2) is pair-paracompact.*

Example 3.3. *Observe that the space $(G, \gamma_{dis}, \gamma_{ind})$ is pair-$\mathfrak{D}$-Lindelöf and countably pair-$\mathfrak{D}$-paracompact. Hence, the space is pair-$\mathfrak{D}$-paracompact.*

Theorem 3.5. *If the bitopological space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paralindelöf and countably pair-$\mathfrak{D}$-paracompact, then the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact.*

Proof. Let the space G be pair-$\mathfrak{D}$-paralindelöf, and let $\tilde{m} = \{m_\mu : \mu \in \Omega\}$ be a pair-$\mathfrak{D}$-cover of G, where $\tilde{m}$ contains a pairwise open locally countable refinement $\tilde{l} = \{l_{\mu_k}\}_{k=1}^{\infty}$, which is a pair-$\mathfrak{D}$-cover of (G, γ_1, γ_2). Since the space G is countably pair-$\mathfrak{D}$-paracompact, $\tilde{l}$ contains a pairwise open locally finite refinement $\tilde{E}$ of $\tilde{m}$. Therefore, the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact. □

Using a similar approach, the following corollary can be derived.

Corollary 3.4. *Any countably pair-$\mathfrak{D}$-paracompact and pair-$\mathfrak{D}$-paralindelöf space (G, γ_1, γ_2) is pair-paracompact.*

Corollary 3.5. *Any countably pair-$\mathfrak{D}$-paracompact and pair-paralindelöf space (G, γ_1, γ_2) is pair-paracompact.*

Corollary 3.6. *Any countably pair-paracompact and pair-paralindelöf space (G, γ_1, γ_2) is pair-paracompact.*

Theorem 3.6. *If the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact, then any γ_1-closed subset of G is γ_2-$\mathfrak{D}$-paracompact, and any γ_2-closed subset of G is γ_1-$\mathfrak{D}$-paracompact.*

Proof. Let $F \neq \varnothing$ be a γ_1-closed subset of G, and let $\tilde{m} = \{m_\mu : \mu \in \Omega\}$ be a γ_2-$\mathfrak{D}$-cover of F. Then

$$\tilde{l} = \{G - F\} \cup \{m_\mu : \mu \in \Omega\}$$

is a pair-$\mathfrak{D}$-cover of G. Since the space (G, γ_1, γ_2) is pair-$\mathfrak{D}$-paracompact, $\tilde{l}$ has a pairwise open locally finite refinement, say

$$\{n_\nu : \nu \in \Lambda\} \cup \{\tilde{m}^*_\mu : \mu \in \Omega\},$$

where each n_ν is a γ_1-$\mathfrak{D}$-set for all $\nu \in \Lambda$, and each $\tilde{m}^*_\mu$ is a γ_2-$\mathfrak{D}$-set for all $\mu \in \Omega$. Thus, $\{\tilde{m}^*_\mu : \mu \in \Omega\}$ is a locally finite open refinement of $\tilde{m}$. Hence, F is γ_2-$\mathfrak{D}$-paracompact in G. The other case is proved similarly. □

Employing similar reasoning, we can establish the following corollary.

Corollary 3.7. *If the space (G, γ_1, γ_2) is pair-paracompact, then any γ_1-closed subset of G is γ_2-$\mathfrak{D}$-paracompact, and any γ_2-closed subset of G is γ_1-$\mathfrak{D}$-paracompact.*

4. Product of $\mathfrak{D}$-Paracompact Spaces in Bitopological Spaces

This section presents various theoretical findings regarding the notion of mappings as the pairwise product of $\mathfrak{D}$-paracompact spaces.

Theorem 4.1. *Let (G, γ_1, γ_2) and (T, θ_1, θ_2) be bitopological spaces. If the space G is locally indiscrete and the function $\Upsilon : G \longrightarrow T$ is pair-perfect, then the space G is pair-$\mathfrak{D}$-paracompact whenever T is pair-$\mathfrak{D}$-paracompact.*

Proof. Let $\tilde{m} = \{m_\mu : \mu \in \Omega\} \cup \{l_\nu : \nu \in \Lambda\}$ be any pair-$\mathfrak{D}$-cover of G, where $\{m_\mu : \mu \in \Omega\}$ are γ_1-$\mathfrak{D}$-members of $\tilde{m}$ and $\{l_\nu : \nu \in \Lambda\}$ are γ_2-$\mathfrak{D}$-members of $\tilde{m}$. Since the function Υ is pair-perfect, for every $t \in T$, the subset $\Upsilon^{-1}(t)$ is pair-$\mathfrak{D}$-compact in G. Thus, there exist finite subsets $\Omega_1 \subseteq \Omega$ and $\Omega_2 \subseteq \Lambda$ such that

$$\Upsilon^{-1}(t) \subseteq \bigcup_{\mu_k \in \Omega_1} m_{\mu_k} \cup \bigcup_{\nu_k \in \Omega_2} l_{\nu_k}.$$

Now, define

$$F_{t1} = T - \Upsilon\left(G - \bigcup_{\mu_k \in \Omega_1} m_{\mu_k}\right), \quad F_{t2} = T - \Upsilon\left(G - \bigcup_{\nu_k \in \Omega_2} l_{\nu_k}\right).$$

Then F_{t1} is a θ_1-open subset of T with $\Upsilon^{-1}(F_{t1}) \subseteq \bigcup_{\mu_k \in \Omega_1} m_{\mu_k}$, and F_{t2} is a θ_2-open subset of T with $\Upsilon^{-1}(F_{t2}) \subseteq \bigcup_{\nu_k \in \Omega_2} l_{\nu_k}$. Since $t \in F_{t1} \cup F_{t2}$, we have

$$\tilde{F} = \{F_{t1} : t \in T\} \cup \{F_{t2} : t \in T\},$$

which is an open pair-cover of T. Because T is pair-$\mathfrak{D}$-paracompact, $\tilde{F}$ contains a pairwise open locally finite refinement

$$\tilde{F}^* = \{F^*_{t1} : t \in T\} \cup \{F^*_{t2} : t \in T\}.$$

Each F^*_{t1} is θ_1-$\mathfrak{D}$-open, and each F^*_{t2} is θ_2-$\mathfrak{D}$-open. Since Υ is pair-perfect, the collection

$$\{\Upsilon^{-1}(F^*_{t1}) : t \in T\} \cup \{\Upsilon^{-1}(F^*_{t2}) : t \in T\}$$

is a pairwise open locally finite refinement of G. Hence, G is pair-$\mathfrak{D}$-paracompact. □

Theorem 4.2. *Let $\Upsilon : G \longrightarrow T$ be a pair-perfect function. Then the space G is pair-paracompact whenever T is pair-$\mathfrak{D}$-paracompact.*

Proof. This theorem can be proven easily using an approach similar to that of Theorem 4.1. □

Theorem 4.3. *Let (G, γ_1, γ_2) and (T, θ_1, θ_2) be two bitopological spaces. If G is compact Hausdorff and T is pair-$\mathfrak{D}$-paracompact, then $G \times T$ is pair-$\mathfrak{D}$-paracompact.*

Proof. Since the projection function $P : G \times T \longrightarrow T$ is pair-continuous and $P^{-1}\{t\} = G \times \{t\} \simeq G$ is pair-$\mathfrak{D}$-compact for any $t \in T$, the function $P : G \times T \longrightarrow T$ is pair-perfect. Hence, by Theorem 4.1, since T is pair-$\mathfrak{D}$-paracompact, it follows that $G \times T$ is pair-$\mathfrak{D}$-paracompact. □

Example 4.1. *Notice that the bitopological space* $(\mathbb{R}^2, \gamma_{cof} \times \gamma_{cof}, \gamma_{dis} \times \gamma_{dis})$ *is pair-*$\mathfrak{D}$*-paracompact, but it is neither pair-compact nor pair-Lindelöf. This is because the open pair-cover*

$$\{\mathbb{R} \times (\mathbb{R} - \{0\})\} \cup \{(q, 0) : q \in \mathbb{R}\}$$

of $\mathbb{R}^2$ *has no finite subcover. Thus, the space* $(\mathbb{R}^2, \gamma_{cof} \times \gamma_{cof}, \gamma_{dis} \times \gamma_{dis})$ *is not pair-*$\mathfrak{D}$*-compact nor pair-*$\mathfrak{D}$*-Lindelöf.*

Theorem 4.4. *Let* $\Upsilon : (G, \gamma_1, \gamma_2) \longrightarrow (T, \theta_1, \theta_2)$ *be an onto, pair-closed, and pair-continuous function. If* T *is a locally indiscrete space, then* T *is pair-*$\mathfrak{D}$*-paracompact whenever* G *is pair-*$\mathfrak{D}$*-paracompact.*

Proof. Let $\tilde{l} = \{m_\mu : \mu \in \Omega\} \cup \{l_\nu : \nu \in \Lambda\}$ be a pair-$\mathfrak{D}$-cover of T, where $\{m_\mu : \mu \in \Omega\}$ are θ_1-$\mathfrak{D}$-members of $\tilde{l}$ and $\{l_\nu : \nu \in \Lambda\}$ are θ_2-$\mathfrak{D}$-members of $\tilde{l}$. Because Υ is onto and pair-continuous, the collection

$$\tilde{m} = \{\Upsilon^{-1}(m_\mu) : \mu \in \Omega\} \cup \{\Upsilon^{-1}(l_\nu) : \nu \in \Lambda\}$$

is an open pair-cover of G. Since the space G is pair-$\mathfrak{D}$-paracompact, there exists a pairwise open locally finite refinement of $\tilde{m}$, denoted by

$$\tilde{m}^* = \{\Upsilon^{-1}(m^*_\mu) : \mu \in \Omega\} \cup \{\Upsilon^{-1}(l^*_\nu) : \nu \in \Lambda\}.$$

Consequently,

$$\tilde{l}^* = \{m^*_\mu : \mu \in \Omega\} \cup \{l^*_\nu : \nu \in \Lambda\}$$

is a pairwise open locally countable refinement of $\tilde{l}$. Therefore, T is pair-$\mathfrak{D}$-paracompact. □

Using similar reasoning, we can obtain the following corollary.

Corollary 4.1. *Let* $\Upsilon : (G, \gamma_1, \gamma_2) \longrightarrow (T, \theta_1, \theta_2)$ *be an onto, pair-closed, and pair-continuous function. If* T *is a locally indiscrete space, then* T *is pair-*$\mathfrak{D}$*-paracompact whenever* G *is pair-paracompact.*

5. Conclusions

In this paper, we presented a new concept called pair-$\mathfrak{D}$-paracompact spaces, which we established within bitopological spaces as a generalization of the study in [1]. As a result, new facts and generalizations about pair-$\mathfrak{D}$-paracompact spaces have been developed, along with the concept of countably pair-$\mathfrak{D}$-paracompact spaces. In addition, we obtained several theoretical results that show how these spaces relate to other bitopological structures. Moreover, since γ_1, γ_2, and γ_3 are topologies on the same space G, $\mathfrak{D}$-paracompactness in tritopological spaces $(G, \gamma_1, \gamma_2, \gamma_3)$ can be investigated using the same technique. This concept therefore provides a foundation for deducing and generating additional properties and results that may be explored in future studies.

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