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Characterization of Level Set Through Multi-Polar Extension of Intuitionistic Fuzzy Different Ideals on Regular Ordered Ternary Semigroups

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Abstract. In this paper, we investigate several algebraic properties of regular ordered ternary semigroups through the lens of multi-polar intuitionistic fuzzy sets. We introduce a novel approach to defining Q-anti-fuzzy different types of ideals—namely, left ideals, right ideals, lateral ideals, and bi-ideals. These generalized ideals are systematically extended within the framework of ordered ternary semigroups, revealing new structural insights.

1. Introduction

Languages used in computer programming have partially additive semantics. Because functional compositions and partial functions under disjoint-domain sums do not fit the field specification, linear algebra cannot be applied in these situations. Since they are algebraic structures, they may be thought of as partial ternary semirings that are capable of processing ternary multiplications, infinite partial additions, and both natural and partial ternary semirings. Rings, ternary semirings, and other ideal types have all been covered by mathematical structures like semirings [1]. Furthermore, Lajos used generalized bi-ideals and quasi-ideals to study semigroups both regularly and intra-regularly. Bi-ideals are frequently employed in various types of semigroups.

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Lajos discussed the bi-ideals of associative rings. The left and right ideals, which are particular instances of bi-ideals, can be generalized to become quasi-ideals. The idea that rings can be generalized to semirings. Lehmer [5] first presented a triplex as a ternary algebraic system. He researched triplexes, a kind of ternary algebraic structure that is a commutative ternary group. Hestenes [3] used linear transformations and matrices as examples to study ternary algebra.

Various types of ideals in mathematical structures, including rings and semirings, have been the subject of several research [1, 6]. Dedekind introduced the concept of ideals into the theory of algebraic numbers, which involved associative rings. Lajos employed generalized BIDs and quasi-ideals to investigate regular and intra-regular semigroups. Lehmer first proposed the triplex structures, also known as ternary algebraic systems, in 1932 [5]. Hestenes [3] introduced the idea of ternary algebra in 1962 using matrices and linear transformations as examples. As fuzzy set (FS) theory advances rapidly, more and more hybrid fuzzy models are being developed. The uncertainties have led to the development of several theories of uncertainty, including fuzzy sets (FS), intuitionistic fuzzy sets (IFS), and Pythagorean fuzzy sets (PFS) [10]. Since non-membership grades (NMG) may only have a value of 1, an FS is made up of MG sets, or sets with grades between 0 and 1. IFS is categorized as MG. The total of MGs and NMGs during a decision-making process may occasionally approach 1. Yager [10] employed PFS logic to construct the generalized MG and NMG logic, which is based on the square of the MGs and NMGs and has a value of no more than 1. These theories are unable to characterize the neutral state since it is neither positive nor negative. Palanikumar et al. [7] proposed an intuitionistic fuzzy normal subbisemiring. Palanikumar et al. [8] introduced bisemiring by utilizing bipolar-valued neutrosophic normal sets. Hila et al. [4] investigated bi-ideals in ordered semigroups. Recently, the extension of neutrosophic ideals of ordered ternary semigroups was studied by Rajalakshmi et al. [9]. Additionally, Hatamleh et al. [2] used several ideals of bisemirings to play with the concept of a complex cubic intuitionistic fuzzy set. Recently, Rajalakshmi et al. [9] discussed the extension of neutrosophic ideals of ordered ternary semigroups. Also, Hatamleh et al. [2] interacted with the concept of Complex cubic intuitionistic fuzzy set via different ideals of bisemirings.

2. $(\imath_1, \imath_2)$-Intuitionistic multi-polar Q-anti-fuzzy ideals

Table 1: Summary of Abbreviations

Abbreviation	**Full Meaning**
OTS	Ordered Ternary Semigroup
IFS	Intuitionistic Fuzzy Set
FS	Fuzzy Set
PFS	Pythagorean Fuzzy Set
MPIFS	Multi-Polar Intuitionistic Fuzzy Set

Continued on next page

Table 1 (continued)

Abbreviation	Full Meaning
MPIQAFSS	Multi-Polar Intuitionistic Q-Anti-Ternary Subsemigroup
MPIQAFLI	Multi-Polar Intuitionistic Q-Anti-Fuzzy Left Ideal
MPIQAFRI	Multi-Polar Intuitionistic Q-Anti-Fuzzy Right Ideal
MPIQAFLATI	Multi-Polar Intuitionistic Q-Anti-Fuzzy Lateral Ideal
MPIQAFBI	Multi-Polar Intuitionistic Q-Anti-Fuzzy Bi-Ideal
SS	Ternary Subsemigroup
LI	Left Ideal
RI	Right Ideal
LATI	Lateral Ideal
BI	Bi-Ideal
MG	Membership Grade
NMG	Non-Membership Grade

Here Ξ denotes the ordered ternary semigroup and $\imath_1, \imath_2 \in [0,1]$ and $0 \leq \imath_1 < \imath_2 \leq 1$, $(\imath_1, \imath_2)$ an arbitrary fixed.

Definition 2.1. *An MPIFS $X = [\Pi^k_X, \Psi^k_X]$, the pair (X, Q), where Q is a non-empty set over X, is called an $(\imath_1, \imath_2)$-MPIQAFSS of Ξ if*

(1) if $\varkappa \leq \varphi$, then $\Pi^k(\varkappa, \lambda) \leq \Pi^k(\varphi, \lambda)$ and $\Psi^k(\varkappa, \lambda) \geq \Psi^k(\varphi, \lambda)$,

(2) $\min\{\Pi^k(\varkappa\varpi\varphi, \lambda), \imath_1\} \leq \max\{\Pi^k(\varkappa, \lambda), \Pi^k(\varpi, \lambda), \Pi^k(\varphi, \lambda), \imath_2\}$

(3) $\max\{\Psi^k(\varkappa\varpi\varphi, \lambda), \imath_1\} \geq \min\{\Psi^k(\varkappa, \lambda), \Psi^k(\varpi, \lambda), \Psi^k(\varphi, \lambda), \imath_2\}$ for all $\varkappa, \varpi, \varphi \in \Xi$, $\lambda \in Q$ and $k \in \{1, 2, ..., n\}$.

Example 2.1. *Let $\Xi = \{\natural_1, \natural_2, \natural_3, \natural_4\}$ be an ordered ternary semigroup defined by:*

$\cdot$	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
$\natural_1$	c	c	c	c
$\natural_2$	c	d	e	f
$\natural_3$	c	e	e	e
$\natural_4$	c	e	e	e

$\cdot$	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
c	$\natural_1$	$\natural_1$	$\natural_1$	$\natural_1$
d	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
e	$\natural_1$	$\natural_3$	$\natural_3$	$\natural_3$
f	$\natural_1$	$\natural_3$	$\natural_3$	$\natural_3$

$$\leq = \{(\natural_1,\natural_1), (\natural_1,\natural_2), (\natural_1,\natural_3), (\natural_1,\natural_4), (\natural_2,\natural_2), (\natural_2,\natural_3), (\natural_2,\natural_4), (\natural_3,\natural_3), (\natural_4,\natural_3), (\natural_4,\natural_4)\}.$$

Define the MPIFS $X = [\Pi^k_X, \Psi^k_X]$ as follows: $(\Pi^k, \Psi^k)(\natural_1, \lambda) = (0.34, 0.66)$, $(\Pi^k, \Psi^k)(\natural_2, \lambda) = (0.41, 0.46)$, $(\Pi^k, \Psi^k)(\natural_3, \lambda) = (0.51, 0.16)$, *and* $(\Pi^k, \Psi^k)(\natural_4, \lambda) = (0.46, 0.26)$. *Then X is a* $(0.56, 0.71)$-*MPIQAFSS of Ξ.*

Definition 2.2. *An MPIFS $X = [\Pi^k_X, \Psi^k_X]$ of Ξ is called an $(\imath_1, \imath_2)$-MPIQAFBI of Ξ if*

(1) if $\varkappa \leq \varphi$, then $\Pi^k(\varkappa, \lambda) \leq \Pi^k(\varphi, \lambda)$ and $\Psi^k(\varkappa, \lambda) \geq \Psi^k(\varphi, \lambda)$,

(2) $\min\{\Pi^k(\varkappa\varpi_1\varphi, \lambda), \imath_1\} \leq \max\{\Pi^k(\varkappa, \lambda), \Pi^k(\varphi, \lambda), \imath_2\}$,
$\max\{\Psi^k(\varkappa\varpi_1\varphi, \lambda), \imath_1\} \geq \min\{\Psi^k(\varkappa, \lambda), \Psi^k(\varphi, \lambda), \imath_2\}$,

(3) $\min\{\Pi^k(\varkappa\varpi_1\varphi\varpi_2\rho,\lambda),\iota_1\} \leq \max\{\Pi^k(\varkappa,\lambda),\Pi^k(\rho,\lambda),\iota_2\}$,
$\max\{\Psi^k(\varkappa\varpi_1\varphi\varpi_2\rho,\lambda),\iota_1\} \geq \min\{\Psi^k(\varkappa,\lambda),\Psi^k(\rho,\lambda),\iota_2\}$, *for* $\varkappa,\varphi,\rho,\varpi_1,\varpi_2 \in \Xi$, $q \in Q$ *and* $k \in \{1,2,...,n\}$.

Example 2.2. *Let* $\Xi = \{\natural_1,\natural_2,\natural_3,\natural_4\}$ *be an ordered ternary semigroup defined by:*

$\cdot$	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
$\natural_1$	c	c	c	c
$\natural_2$	c	d	e	d
$\natural_3$	c	e	e	e
$\natural_4$	c	d	e	f

$\cdot$	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
c	$\natural_1$	$\natural_1$	$\natural_1$	$\natural_1$
d	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$
e	$\natural_1$	$\natural_3$	$\natural_3$	$\natural_3$
f	$\natural_1$	$\natural_2$	$\natural_3$	$\natural_4$

$$\leq = \{(\natural_1,\natural_1),(\natural_1,\natural_2),(\natural_1,\natural_3),(\natural_1,\natural_4),(\natural_2,\natural_2),(\natural_2,\natural_3),(\natural_2,\natural_4),(\natural_3,\natural_3),(\natural_4,\natural_3),(\natural_4,\natural_4)\}.$$

Define the MPIFS $\Lambda^k = [\Pi^k,\Psi^k]$ *as follows:* $(\Pi^k,\Psi^k)(\natural_1,\lambda) = (0.39,0.56)$, $(\Pi^k,\Psi^k)(\natural_2,\lambda) = (0.44,0.37)$, $(\Pi^k,\Psi^k)(\natural_3,\lambda) = (0.54,0.09)$, *and* $(\Pi^k,\Psi^k)(\natural_4,\lambda) = (0.49,0.18)$. *Then* Λ^k *is a* $(0.42,0.57)$*-MPIQAFBI of* Ξ.

Theorem 2.1. *Let* $\Lambda^k_{\iota_1}$ *be an* (ι_1,ι_2)*-MPIQAFSS (MPIQAFLI, MPIQAFLATI, MPIQAFRI, MPIQAFBI) of* Ξ*. Then the lower level set* $\Pi^k_{\iota_1}$ *is an SS (LI, LATI, RI, BI) of* Ξ*, where* $\Pi^k_{\iota_1} = \{\varkappa \in \Xi \mid \Pi^k(\varkappa,\lambda) < \iota_1\}$ *and* $\Psi^k_{\iota_1} = \{\varkappa \in \Xi \mid \Psi^k(\varkappa,\lambda) > \iota_1\}$ *and* $k \in \{1,2,...,n\}$.

Proof. Let $\Lambda^k_{\iota_1}$ be an (ι_1,ι_2)-MPIQAFSS of Ξ. Let $\varkappa,\varpi,\varphi \in \Xi$ be such that $\varkappa,\varpi,\varphi \in \Pi^k_{\iota_1}$. Then $\Pi^k(\varkappa,\lambda) < \iota_1, \Pi^k(\varpi,\lambda) < \iota_1, \Pi^k(\varphi,\lambda) < \iota_1$.
Therefore, $\min\{\Pi^k(\varkappa\varpi\varphi,\lambda),\iota_1\} \leq \max\{\Pi^k(\varkappa,\lambda),\Pi^k(\varpi,\lambda),\Pi^k(\varphi,\lambda),\iota_2\} < \max\{\iota_1,\iota_1,\iota_1,\iota_2\} = \iota_2$. Hence, $\Pi^k(\varkappa\varpi\varphi,\lambda) < \iota_1$. It shows that $\varkappa\varpi\varphi \in \Pi^k_{\iota_1}$. Therefore, $\Pi^k_{\iota_1}$ is an SS of Ξ. Let $\varkappa,\varpi,\varphi \in \Xi$ such that $\varkappa,\varpi,\varphi \in \Psi^k_{\iota_1}$. Then $\Psi^k(\varkappa,\lambda) > \iota_1, \Psi^k(\varpi,\lambda) > \iota_1 \Psi^k(\varphi,\lambda) > \iota_1$. Therefore, $\max\{\Psi^k(\varkappa\varpi\varphi,\lambda),\iota_1\} \geq \min\{\Psi^k(\varkappa,\lambda),\Psi^k(\varpi,\lambda),\Psi^k(\varphi,\lambda),\iota_2\} > \min\{\iota_1,\iota_1,\iota_1,\iota_2\} = \iota_1$. Hence, $\Psi^k(\varkappa\varpi\varphi,\lambda) > \iota_1$. It shows that $\varkappa\varpi\varphi \in \Psi^k_{\iota_1}$. Therefore, $\Psi^k_{\iota_1}$ is an SS of Ξ. Therefore, $\Lambda^k_{\iota_1}$ is an SS of Ξ. □

Theorem 2.2. *A non-empty subset* F *of* Ξ *is an SS (LI, LATI, RI, BI) of* Ξ *if and only if the MPIFS* $\Lambda^k = [\Pi^k,\Psi^k]$ *of* Ξ *is defined as*

$$\Pi^k(\varkappa,\lambda) = \begin{cases} \leq \iota_2 & \text{for all } \varkappa \in (F] \\ \iota_1 & \text{for all } \varkappa \notin (F] \end{cases} \quad \Psi^k(\varkappa,\lambda) = \begin{cases} \geq \iota_2 & \text{for all } \varkappa \in (F] \\ \iota_1 & \text{for all } \varkappa \notin (F] \end{cases}$$

is an (ι_1,ι_2)*-MPIQAFSS (MPIQAFLI, MPIQAFLATI, MPIQAFRI, MPIQAFBI) of* Ξ.

Proof. Suppose that F is an SS of Ξ. Let $\varkappa,\varpi,\varphi \in \Xi$ and $\varkappa,\varpi,\varphi \in (F]$. Then $\varkappa\varpi\varphi \in (F]$. Hence, $\Pi^k(\varkappa\varpi\varphi,\lambda) \leq \iota_2$ and $\Psi^k(\varkappa\varpi\varphi,\lambda) \geq \iota_2$. Thus,

$$\min\{\Pi^k(\varkappa\varpi\varphi,\lambda),\iota_1\} \leq \iota_2 = \max\{\Pi^k(\varkappa,\lambda),\Pi^k(\varpi,\lambda),\Pi^k(\varphi,\lambda),\iota_2\}$$

and

$$\max\{\Psi^k(\varkappa\varpi\varphi,\lambda),\iota_1\} \geq \iota_2 = \min\{\Psi^k(\varkappa,\lambda),\Psi^k(\varpi,\lambda),\Psi^k(\varphi,\lambda),\iota_2\}.$$

If $\varkappa \notin (F]$ or $\varpi \notin (F]$ or $\varphi \notin (F]$, then

$$\max\{\Pi^k(\varkappa,\lambda),\Pi^k(\varpi,\lambda),\Pi^k(\varphi,\lambda),\imath_2\} = \imath_1$$

and

$$\min\{\Psi^k(\varkappa,\lambda),\Psi^k(\varpi,\lambda),\Psi^k(\varphi,\lambda),\imath_2\} = \imath_2.$$

That is,

$$\min\{\Pi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \leq \max\{\Pi^k(\varkappa,\lambda),\Pi^k(\varpi,\lambda),\Pi^k(\varphi,\lambda),\imath_2\}$$

and

$$\max\{\Psi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \geq \min\{\Psi^k(\varkappa,\lambda),\Psi^k(\varpi,\lambda),\Psi^k(\varphi,\lambda),\imath_2\}.$$

Therefore, Λ^k is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ.

Conversely, assume that $\Lambda^k = [\Pi^k,\Psi^k]$ is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ. Let $\varkappa\varpi\varphi \in (F]$. Then $\Pi^k(\varkappa,\lambda) \leq \imath_2, \Pi^k(\varpi,\lambda) \leq \imath_2, \Pi^k(\varphi,\lambda) \leq \imath_2$ and $\Psi^k(\varkappa,\lambda) \geq \imath_2, \Psi^k(\varpi,\lambda) \geq \imath_2, \Psi^k(\varphi,\lambda) \geq \imath_2$. Now $\Lambda^k = [\Pi^k,\Psi^k]$ is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ. Therefore,

$$\min\{\Pi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \leq \max\{\Pi^k(\varkappa,\lambda),\Pi^k(\varpi,\lambda),\Pi^k(\varphi,\lambda),\imath_2\} \leq \max\{\imath_2,\imath_2,\imath_2,\imath_2\} = \imath_2$$

and

$$\max\{\Psi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \geq \min\{\Psi^k(\varkappa,\lambda),\Psi^k(\varpi,\lambda),\Psi^k(\varphi,\lambda),\imath_2\} \geq \min\{\imath_2,\imath_2,\imath_2,\imath_2\} = \imath_2.$$

It follows that $\varkappa\varpi\varphi \in (F]$. Therefore, F is an SS of Ξ. □

Theorem 2.3. *An MPIFS $\Lambda^k = [\Pi^k,\Psi^k]$ is an $(\imath_1,\imath_2)$-MPIQAFSS (MPIQAFLI, MPIQAFLATI, MPIQAFRI, MPIQAFBI) of Ξ if and only if each non-empty level subset Λ^k_t is an SS (LI, LATI, RI, BI) of Ξ for all $t \in (\imath_1,\imath_2]$.*

Proof. Assume that Λ^k_t is an SS of Ξ for each $t \in [0,1]$ and $k \in \{1,2,...,n\}$. Let

$$t = \max\{\Pi^k(\varkappa_l,\lambda),\Pi^k(\varkappa_m,\lambda),\Pi^k(\varkappa_n,\lambda)\}.$$

Then $\varkappa_l,\varkappa_m,\varkappa_n \in \Pi^k_t$ for each $\varkappa_l,\varkappa_m,\varkappa_n \in \Xi$. Thus,

$$\min\{\Pi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \leq t = \max\{\Pi^k(\varkappa_l,\lambda),\Pi^k(\varkappa_m,\lambda),\Pi^k(\varkappa_n,\lambda),\imath_2\}.$$

Let

$$t = \min\{\Psi^k(\varkappa_l,\lambda),\Psi^k(\varkappa_m,\lambda),\Psi^k(\varkappa_n,\lambda)\}.$$

Then $\varkappa_l,\varkappa_m,\varkappa_n \in \Psi^k_t$ for each $\varkappa_l,\varkappa_m,\varkappa_n \in \Xi$. Thus,

$$\max\{\Psi^k(\varkappa\varpi\varphi,\lambda),\imath_1\} \geq t = \min\{\Psi^k(\varkappa_l,\lambda),\Psi^k(\varkappa_m,\lambda),\Psi^k(\varkappa_n,\lambda),\imath_2\}.$$

Hence, Λ^k is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ.

Conversely, let us assume that Λ^k is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ. For $t \in [0,1]$ and $\varkappa_l,\varkappa_m,\varkappa_n \in \Pi^k_t$. We have $\Pi^k(\varkappa_l,\lambda) \leq t, \Pi^k(\varkappa_m,\lambda) \leq t, \Pi^k(\varkappa_n,\lambda) \leq t$. Since Π^k is an SS of Ξ, $\min\{\Pi^k(\varkappa_l\varkappa_m\varkappa_n,\lambda),\imath_1\} \leq \max\{\Pi^k(\varkappa_l,\lambda),\Pi^k(\varkappa_m,\lambda),\Pi^k(\varkappa_n,\lambda),\imath_2\} \leq t$. This implies that $\varkappa_l\varkappa_m\varkappa_n \in \Pi^k_t$. We have $\Psi^k(\varkappa_l,\lambda) \geq t, \Psi^k(\varkappa_m,\lambda) \geq t, \Psi^k(\varkappa_n,\lambda) \geq t$. Since Ψ^k is an SS of Ξ, we have

$$\max\{\Psi^k(\varkappa_l\varkappa_m\varkappa_n,\lambda),\imath_1\} \geq \min\{\Psi^k(\varkappa_l,\lambda),\Psi^k(\varkappa_m,\lambda),\Psi^k(\varkappa_n,\lambda),\imath_2\} \geq t.$$

This implies that $\varkappa_l\varkappa_m\varkappa_n \in \Psi^k_t$. Therefore, Λ^k_t is an SS of Ξ for each $t \in (\imath_1,\imath_2]$. □

Example 2.3. *Every MPIQAFSS Λ^k of Ξ is a $(\imath_1,\imath_2)$-MPIQAFSS of Ξ, but reverse implication is need not be true.*

For Example 2.1, $(\Pi^k,\Psi^k)(\natural_1,\lambda) = (0.27, 0.42)$, $(\Pi^k,\Psi^k)(\natural_2,\lambda) = (0.32, 0.35)$, $(\Pi^k,\Psi^k)(\natural_3,\lambda) = (0.42, 0.25)$, and $(\Pi^k,\Psi^k)(\natural_4,\lambda) = (0.37, 0.30)$. Then Λ^k is a $(0.33, 0.47)$-MPIQAFSS of Ξ, but not an MPIQAFSS. Since $\Pi^k(\natural_4\varpi\natural_4,\lambda) = 0.42 \not\leq \max\{\Pi^k(\natural_4,q),\Pi^k(\natural_4,q)\} = 0.37$ and $\Psi^k(\natural_4\varpi\natural_4,\lambda) = 0.25 \not\geq \min\{\Psi^k(\natural_4,q),\Psi^k(\natural_4,q)\} = 0.30$.

Example 2.4. *Every MPIQAFBI $\Lambda^k = [\Pi^k,\Psi^k]$ of Ξ is a $(\imath_1,\imath_2)$-MPIQAFBI of Ξ, but reverse implication is need not be true.*

For Example 2.2, $(\Pi^k,\Psi^k)(\natural_1,\lambda) = (0.14, 0.48)$, $(\Pi^k,\Psi^k)(\natural_2,\lambda) = (0.29, 0.29)$, $(\Pi^k,\Psi^k)(\natural_3,\lambda) = (0.39, 0.07)$, and $(\Pi^k,\Psi^k)(\natural_4,\lambda) = (0.34, 0.10)$. Then Λ^k is a $(0.24, 0.49)$-MPIQAFBI, but not an MPIQAFBI. Since $\Pi^k(\natural_4\varpi_1\natural_4\varpi_2\natural_4,\lambda) = \Pi^k(\natural_3,\lambda) = 0.39 \not\leq \max\{\Pi^k(\natural_4,\lambda),\Pi^k(\natural_4,\lambda)\} = 0.34$ and $\Psi^k(\natural_4\varpi_1\natural_4\varpi_2\natural_4,\lambda) = \Psi^k(\natural_3,\lambda) = 0.07 \not\geq \min\{\Pi^k(\natural_4,\lambda),\Pi^k(\natural_4,\lambda)\} = 0.10$.

Definition 2.3. *The characteristic function $(\delta^k_F)^{\imath_2}_{\imath_1}$ is defined as*

$$(\eta^k_F)^{\imath_2}_{\imath_1}(\varkappa,\lambda) = \begin{cases} \imath_2 & \text{if } \varkappa \in (F] \\ \imath_1 & \text{otherwise} \end{cases} \qquad (\psi^k_F)^{\imath_2}_{\imath_1}(\varkappa,\lambda) = \begin{cases} \imath_1 & \text{if } \varkappa \in (F] \\ \imath_2 & \text{otherwise} \end{cases}$$

Theorem 2.4. *Let F be a non-empty subset of Ξ is an SS (LI, LATI, RI, BI) of Ξ if and only if $\delta^k_{(F]}$ is an $(\imath_1,\imath_2)$-MPIQAFSS (MPIQAFLI, MPIQAFLATI, MPIQAFRI, MPIQAFBI).*

Proof. Let F be an SS of Ξ and hence $\delta^k_{(F]}$ is an MPIQAFSS of Ξ which is a $\delta^k_{(F]}$ is an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ.

Conversely, let $\delta^k_{(F]}$ be an $(\imath_1,\imath_2)$-MPIQAFSS of Ξ. Let $\varkappa,\varpi,\varphi \in \Xi$ and $\varkappa,\varpi,\varphi \in (F]$. Then $\eta^k_{(F]}(\varkappa,\lambda) = \imath_2, \eta^k_{(F]}(\varpi,\lambda) = \imath_2, \eta^k_{(F]}(\varphi,\lambda) = \imath_2$. Since $\eta^k_{(F]}$ is an $(\imath_1,\imath_2)$-MPIQAFSS, we have

$$\begin{aligned} \min\{\eta^k_{(F]}(\varkappa\varpi\varphi,\lambda),\imath_1\} &\leq \max\{\eta^k_{(F]}(\varkappa,\lambda),\eta^k_{(F]}(\varpi,\lambda),\eta^k_{(F]}(\varphi,\lambda),\imath_2\} \\ &= \max\{\imath_2,\imath_2,\imath_2,\imath_2\} \\ &= \imath_2 \end{aligned}$$

as $\imath_1 < \imath_2$, this implies that $\eta^k_{(F]}(\varkappa\varpi\varphi,\lambda) \leq \imath_2$. Thus, $\varkappa\varpi\varphi \in (F]$.

Let $\varkappa, \varpi, \varphi \in \Xi$ and $\varkappa, \varpi, \varphi \in (F]$. Then $\psi^k_{(F]}(\varkappa, \lambda) = \imath_1, \psi^k_{(F]}(\varpi, \lambda) = \imath_1, \psi^k_{(F]}(\varphi, \lambda) = \imath_1$. Since $\psi^k_{(F]}$ is an $(\imath_1, \imath_2)$-MPIQAFSS, we have

$$\begin{aligned}\max\{\psi^k_{(F]}(\varkappa\varpi\varphi, \lambda), \imath_1\} &\geq \min\{\psi^k_{(F]}(\varkappa, \lambda), \psi^k_{(F]}(\varpi, \lambda), \psi^k_{(F]}(\varphi, \lambda), \imath_2\}\\ &= \min\{\imath_1, \imath_1, \imath_1, \imath_2\}\\ &= \imath_1\end{aligned}$$

as $\imath_1 < \imath_2$, this implies that $\psi^k_{(F]}(\varkappa\varpi\varphi, \lambda) \geq \imath_1$. Thus, $\varkappa\varpi\varphi \in (F]$. Therefore, F is an SS of Ξ.

Let $\varkappa, \varpi, \varphi \in \Xi$ and $\varkappa, \varpi, \varphi \notin (F]$. Then $\eta^k_{(F]}(\varkappa, \lambda) = \imath_1, \eta^k_{(F]}(\varpi, \lambda) = \imath_1, \eta^k_{(F]}(\varphi, \lambda) = \imath_1$. Since $\eta^k_{(F]}$ is an $(\imath_1, \imath_2)$-MPIQAFSS, we have

$$\begin{aligned}\min\{\eta^k_{(F]}(\varkappa\varpi\varphi, \lambda), \imath_1\} &\leq \max\{\eta^k_{(F]}(\varkappa, \lambda), \eta^k_{(F]}(\varpi, \lambda), \eta^k_{(F]}(\varphi, \lambda), \imath_2\}\\ &= \max\{\imath_1, \imath_1, \imath_1, \imath_2\}\\ &= \imath_2\end{aligned}$$

as $\imath_1 < \imath_2$, this implies that $\eta^k_{(F]}(\varkappa\varpi\varphi, \lambda) \leq \imath_1$. Thus, $\varkappa\varpi\varphi \notin (F]$.

Let $\varkappa, \varpi, \varphi \in \Xi$ and $\varkappa, \varpi, \varphi \notin (F]$. Then $\psi^k_{(F]}(\varkappa, \lambda) = \imath_2, \psi^k_{(F]}(\varpi, \lambda) = \imath_2, \psi^k_{(F]}(\varphi, \lambda) = \imath_2$. Since $\psi^k_{(F]}$ is an $(\imath_1, \imath_2)$-MPIQAFSS, we have

$$\begin{aligned}\max\{\psi^k_{(F]}(\varkappa\varpi\varphi, \lambda), \imath_1\} &\geq \min\{\psi^k_{(F]}(\varkappa, \lambda), \psi^k_{(F]}(\varpi, \lambda), \psi^k_{(F]}(\varphi, \lambda), \imath_2\}\\ &= \min\{\imath_2, \imath_2, \imath_2, \imath_2\}\\ &= \imath_2\end{aligned}$$

as $\imath_1 < \imath_2$, this implies that $\psi^k_{(F]}(\varkappa\varpi\varphi, \lambda) \geq \imath_2$. Thus, $\varkappa\varpi\varphi \notin (F]$.

Therefore, F is an SS of Ξ. □

Definition 2.4. *For three MPIQAFSs Λ^k, ϖ^k, and $\flat^k$ of Ξ. Then*

$$(\Lambda^k_{\mathscr{T}} \cdot \varpi^k_{\mathscr{T}} \cdot \flat^k_{\mathscr{T}})(\varkappa, \lambda) = \begin{cases} \inf\limits_{(r,s,t)\in F_\varkappa} \{\Lambda^k_{\mathscr{T}}(r, \lambda) \boxminus \varpi^k_{\mathscr{T}}(s, \lambda) \boxminus \flat^k_{\mathscr{T}}(t, \lambda)\} & if \quad F_\varkappa \neq 0 \\ (1, 1, ..., 1)(n \quad times) & otherwise \end{cases}$$

$$(\Lambda^k_{\mathscr{F}} \cdot \varpi^k_{\mathscr{F}} \cdot \flat^k_{\mathscr{F}})(\varkappa, \lambda) = \begin{cases} \sup\limits_{(r,s,t)\in F_\varkappa} \{\Lambda^k_{\mathscr{F}}(r, \lambda) \boxplus \varpi^k_{\mathscr{F}}(s, \lambda) \boxplus \flat^k_{\mathscr{F}}(t, \lambda)\} & if \quad F_\varkappa \neq 0 \\ (0, 0, ..., 0)(n \quad times) & otherwise \end{cases}$$

Definition 2.5. *We define the subset $(\Pi^k)^{\imath_2}_{\imath_1}(\varkappa, \lambda) = \{\Pi^k(\varkappa, \lambda) \boxminus \imath_2\} \boxplus \imath_1$ and $(\Psi^k)^{\imath_2}_{\imath_1}(\varkappa, \lambda) = \{\Psi^k(\varkappa, \lambda) \boxplus \imath_2\} \boxminus \imath_1$, for all $\varkappa \in \Xi$ and $k \in \{1, 2, ..., n\}$.*

Lemma 2.1. *Let F, F_1, and F_2 be MPIFSs of Ξ. Then*
(i) $(\delta^k_{(F]} \boxminus \delta^k_{(F_1]} \boxminus \delta^k_{(F_2]})^{\imath_2}_{\imath_1} = (\delta^k_{(F \Cup F_1 \Cup F_2]})^{\imath_2}_{\imath_1}$,
(ii) $(\delta^k_{(F]} \boxplus \delta^k_{(F_1]} \boxplus \delta^k_{(F_2]})^{\imath_2}_{\imath_1} = (\delta^k_{(F \Cap F_1 \Cap F_2]})^{\imath_2}_{\imath_1}$,
(iii) $(\delta^k_{(F]} \cdot \delta^k_{(F_1]} \cdot \delta^k_{(F_2]})^{\imath_2}_{\imath_1} = (\delta^k_{(F F_1 F_2]})^{\imath_2}_{\imath_1}$.

Proof. (i) Let $x \in \Xi$. Assume $x \in (\delta^k_{(F \Cup F_1 \Cup F_2]})^{\imath_2}_{\imath_1}$. Then, by definition of level set:

$$\mu_k(\delta^k(F \Cup F_1 \Cup F_2)(x)) \geq \imath_1 \quad \text{and} \quad \nu_k(\delta^k(F \Cup F_1 \Cup F_2)(x)) \leq \imath_2.$$

Since μ_k uses maximum, and ν_k uses minimum over their respective domains, we get:

$$\min\{\mu_k(F(x)), \mu_k(F_1(x)), \mu_k(F_2(x))\} \geq \imath_1, \quad \max\{\nu_k(F(x)), \nu_k(F_1(x)), \nu_k(F_2(x))\} \leq \imath_2.$$

Thus $x \in (\delta^k_{(F]} \boxminus \delta^k_{(F_1]} \boxminus \delta^k_{(F_2]})^{\imath_2}_{\imath_1}$.

The converse follows similarly using the inclusion of x in each component and the definitions of $\boxminus$ and $\Cup$ under fuzzy operations.

(ii) Analogously, assume $x \in (\delta^k_{(F \Cap F_1 \Cap F_2]})^{\imath_2}_{\imath_1}$, then

$$\mu_k(\delta^k(F \Cap F_1 \Cap F_2)(x)) \geq \imath_1 \quad \text{and} \quad \nu_k(\delta^k(F \Cap F_1 \Cap F_2)(x)) \leq \imath_2.$$

Since the intersection of fuzzy sets corresponds to the max of membership and the min of non-membership:

$$\max\{\mu_k(F(x)), \mu_k(F_1(x)), \mu_k(F_2(x))\} \geq \imath_1 \quad \text{and} \quad \min\{\nu_k(F(x)), \nu_k(F_1(x)), \nu_k(F_2(x))\} \leq \imath_2.$$

Thus x belongs to the boxplus composition on the left side.

(iii) Let $\varkappa \in \Xi$. If $\varkappa \in (FF_1F_2]$, then $(\delta^k_{(FF_1F_2]})(\varkappa, \lambda) = \imath_2$. Since $\varkappa \leq \kappa_1\kappa_2\kappa_3$ for some $\kappa_1 \in (F]$, $\kappa_2 \in (F_1]$ and $\kappa_3 \in (F_2]$, we have $(\kappa_1, \kappa_2, \kappa_3) \in F_\varkappa$ and $F_\varkappa \neq 0$. Thus,

$$\begin{aligned}
(\eta^k_{(F]} \cdot \eta^k_{(F_1]} \cdot \eta^k_{(F_2]})(\varkappa, \lambda) &= \inf_{\varkappa = \nu_1\nu_2\nu_3} \max\{\eta^k_{(F]}(\nu_1, \lambda), \eta^k_{(F_1]}(\nu_2, \lambda), \eta^k_{(F_2]}(\nu_3, \lambda)\} \\
&\leq \max\{\eta^k_{(F]}(\kappa_1, \lambda), \eta^k_{(F_1]}(\kappa_2, \lambda), \eta^k_{(F_2]}(\kappa_3, \lambda)\} \\
&= \imath_2, \\
(\psi^k_{(F]} \cdot \psi^k_{(F_1]} \cdot \psi^k_{(F_2]})(\varkappa, \lambda) &= \sup_{\varkappa = \nu_1\nu_2\nu_3} \min\{\psi^k_{(F]}(\nu_1, \lambda), \psi^k_{(F_1]}(\nu_2, \lambda), \psi^k_{(F_2]}(\nu_3, \lambda)\} \\
&\geq \min\{\psi^k_{(F]}(\kappa_1, \lambda), \psi^k_{(F_1]}(\kappa_2, \lambda), \psi^k_{(F_2]}(\kappa_3, \lambda)\} \\
&= \imath_1.
\end{aligned}$$

Therefore, $(\delta^k_{(F]} \cdot \delta^k_{(F_1]} \cdot \delta^k_{(F_2]})(\varkappa, \lambda) = (\delta^k_{(FF_1F_2]})(\varkappa, \lambda)$.

If $\varkappa \notin (FF_1F_2]$ then $(\eta^k_{(FF_1F_2]})(\varkappa, \lambda) = \imath_1$ and $(\psi^k_{(FF_1F_2]})(\varkappa, \lambda) = \imath_2$. Since $\varkappa \leq \kappa_1\kappa_2\kappa_3$ for some $\kappa_1 \notin (F]$, $\kappa_2 \notin (F_1]$ and $\kappa_3 \notin (F_2]$, we have

$$\begin{aligned}
(\eta^k_{(F]} \cdot \eta^k_{(F_1]} \cdot \eta^k_{(F_2]})(\varkappa, \lambda) &= \inf_{\varkappa = \nu_1\nu_2\nu_3} \max\{\eta^k_{(F]}(\nu_1, \lambda), \eta^k_{(F_1]}(\nu_2, \lambda), \eta^k_{(F_2]}(\nu_3, \lambda)\} \\
&\leq \max\{\eta^k_{(F]}(\kappa_1, \lambda), \eta^k_{(F_1]}(\kappa_2, \lambda), \eta^k_{(F_2]}(\kappa_3, \lambda)\} \\
&= \imath_1, \\
(\psi^k_{(F]} \cdot \psi^k_{(F_1]} \cdot \psi^k_{(F_2]})(\varkappa, \lambda) &= \sup_{\varkappa = \nu_1\nu_2\nu_3} \min\{\psi^k_{(F]}(\nu_1, \lambda), \psi^k_{(F_1]}(\nu_2, \lambda), \psi^k_{(F_2]}(\nu_3, \lambda)\} \\
&\geq \min\{\psi^k_{(F]}(\kappa_1, \lambda), \psi^k_{(F_1]}(\kappa_2, \lambda), \psi^k_{(F_2]}(\kappa_3, \lambda)\} \\
&= \imath_2.
\end{aligned}$$

Hence, $(\delta^k_{(F]} \cdot \delta^k_{(F_1]} \cdot \delta^k_{(F_2]})(\varkappa, \lambda) = (\delta^k_{(FF_1F_2]})(\varkappa, \lambda)$. □

Theorem 2.5. *For an MPIFS F of Ξ and $\{F_j \mid j \in J\}$ be a collection of MPIFSs of Ξ. Then*
(i) $(F] \subseteq (F_1]$ if and only if $(\delta^k_{(F]})^{\imath_2}_{\imath_1} \leq (\delta^k_{(F_1]})^{\imath_2}_{\imath_1}$.
(ii) $(\Cap_{j\in J}\delta^k_{(F_j]})^{\imath_2}_{\imath_1} = (\delta^k_{\Cap_{j\in J}(F_j]})^{\imath_2}_{\imath_1}$.
(iii) $(\Cup_{j\in J}\delta^k_{(F_j]})^{\imath_2}_{\imath_1} = (\delta^k_{\Cup_{j\in J}(F_j]})^{\imath_2}_{\imath_1}$.

Theorem 2.6. *Let F be an $(\imath_1, \imath_2)$-MPIQAFRI, F_1 be an $(\imath_1, \imath_2)$-MPIQAFLATI, and F_2 be an $(\imath_1, \imath_2)$-MPIQAFLI of Ξ. Then $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} \subseteq ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$.*

Proof. Let $F = [\Pi^k_F, \Psi^k_F]$ be an $(\imath_1, \imath_2)$-MPIQAFRI, $F_1 = [\Pi^k_{F_1}, \Psi^k_{F_1}]$ be an $(\imath_1, \imath_2)$-MPIQAFLATI, and $F_2 = [\Pi^k_{F_2}, \Psi^k_{F_2}]$ be an $(\imath_1, \imath_2)$-MPIQAFLI of Ξ. Let $(\varkappa, \varpi, \varphi) \in X_\rho$. If $X_\rho \neq \emptyset$, then $\rho \leq \varkappa\varpi\varphi$. Thus, $\Pi^k_F(\rho, \lambda) \leq \Pi^k_F(\varkappa\varpi\varphi, \lambda) \leq \Pi^k_F(\varkappa, \lambda)$ and $\Psi^k_F(\rho, \lambda) \geq \Psi^k_F(\varkappa\varpi\varphi, \lambda) \geq \Psi^k_F(\varkappa, \lambda)$.

Similarly, $\Pi^k_{F_1}(\rho, \lambda) \leq \Pi^k_{F_1}(\varkappa\varpi\varphi, \lambda) \leq \Pi^k_{F_1}(\varpi, \lambda)$ and $\Psi^k_{F_1}(\rho, \lambda) \geq \Psi^k_{F_1}(\varkappa\varpi\varphi, \lambda) \geq \Psi^k_{F_1}(\varpi, \lambda)$.

Similarly, $\Pi^k_{F_2}(\rho, \lambda) \leq \Pi^k_{F_2}(\varkappa\varpi\varphi, \lambda) \leq \Pi^k_{F_2}(\varphi, \lambda)$ and $\Psi^k_{F_2}(\rho, \lambda) \geq \Psi^k_{F_2}(\varkappa\varpi\varphi, \lambda) \geq \Psi^k_{F_2}(\varphi, \lambda)$. Thus,

$$\begin{aligned}
&(\Pi^k_{(F\cdot F_1\cdot F_2]})^{\imath_2}_{\imath_1}(\rho, \lambda)\\
&= (\Pi^k_{(F\cdot F_1\cdot F_2]}(\rho, \lambda) \boxminus \imath_2) \boxplus \imath_1\\
&= \left[[\inf_{\rho\leq\varkappa\varpi\varphi}\{\Pi^k_F(\varkappa, \lambda) \boxminus \Pi^k_{F_1}(\varpi, \lambda) \boxminus \Pi^k_{F_2}(\varphi, \lambda)\} \boxminus \imath_2]\right] \boxplus \imath_1\\
&= \left[\inf_{\rho\leq\varkappa\varpi\varphi}\{\Pi^k_F(\varkappa, \lambda) \boxminus \Pi^k_{F_1}(\varpi, \lambda) \boxminus \Pi^k_{F_2}(\varphi, \lambda)\} \boxminus \imath_2 \boxminus \imath_2 \boxminus \imath_2 \boxminus \imath_2\right] \boxplus \imath_1\\
&= \left[\inf_{\rho\leq\varkappa\varpi\varphi}\{(\Pi^k_F(\varkappa, \lambda) \boxminus \imath_2) \boxminus (\Pi^k_{F_1}(\varpi, \lambda) \boxminus \imath_2) \boxminus (\Pi^k_{F_2}(\varphi, \lambda) \boxminus \imath_2)\} \boxminus \imath_2\right] \boxplus \imath_1\\
&\geq (\{(\Pi^k_F(\rho, \lambda) \boxplus \imath_1) \boxminus (\Pi^k_{F_1}(\rho, \lambda) \boxplus \imath_1) \boxminus (\Pi^k_{F_2}(\rho, \lambda) \boxplus \imath_1)\} \boxminus \imath_2) \boxplus \imath_1\\
&= \{((\Pi^k_F(\rho, \lambda) \boxminus \Pi^k_{F_1}(\rho, \lambda) \boxminus \Pi^k_{F_2}(\rho, \lambda)) \boxplus \imath_1) \boxminus \imath_2\} \boxplus \imath_1\\
&= \{((\Pi^k_F \boxminus \Pi^k_{F_1} \boxminus \Pi^k_{F_2})(\rho, \lambda) \boxminus \imath_2\} \boxplus \imath_1\\
&= (\Pi^k_{F\Cap F_1\Cap F_2})^{\imath_2}_{\imath_1}(\rho, \lambda),
\end{aligned}$$

$$\begin{aligned}
&(\Psi^k_{(F\cdot F_1\cdot F_2]})^{\imath_2}_{\imath_1}(\rho, \lambda)\\
&= (\Psi^k_{(F\cdot F_1\cdot F_2]}(\rho, \lambda) \boxplus \imath_2) \boxminus \imath_1\\
&= \left[[\sup_{\rho\leq\varkappa\varpi\varphi}\{\Psi^k_F(\varkappa, \lambda) \boxplus \Psi^k_{F_1}(\varpi, \lambda) \boxplus \Psi^k_{F_2}(\varphi, \lambda)\} \boxplus \imath_2]\right] \boxminus \imath_1\\
&= \left[\sup_{\rho\leq\varkappa\varpi\varphi}\{\Psi^k_F(\varkappa, \lambda) \boxplus \Psi^k_{F_1}(\varpi, \lambda) \boxplus \Psi^k_{F_2}(\varphi, \lambda)\} \boxplus \imath_2 \boxplus \imath_2 \boxplus \imath_2 \boxplus \imath_2\right] \boxminus \imath_1\\
&= \left[\sup_{\rho\leq\varkappa\varpi\varphi}\{(\Psi^k_F(\varkappa, \lambda) \boxplus \imath_2) \boxplus (\Psi^k_{F_1}(\varpi, \lambda) \boxplus \imath_2) \boxplus (\Psi^k_{F_2}(\varphi, \lambda) \boxplus \imath_2)\} \boxplus \imath_2\right] \boxminus \imath_1\\
&\leq (\{(\Psi^k_F(\rho, \lambda) \boxminus \imath_1) \boxplus (\Psi^k_{F_1}(\rho, \lambda) \boxminus \imath_1) \boxplus (\Psi^k_{F_2}(\rho, \lambda) \boxminus \imath_1)\} \boxplus \imath_2) \boxminus \imath_1\\
&= \{((\Psi^k_F(\rho, \lambda) \boxplus \Psi^k_{F_1}(\rho, \lambda) \boxplus \Psi^k_{F_2}(\rho, \lambda)) \boxminus \imath_1) \boxplus \imath_2\} \boxminus \imath_1
\end{aligned}$$

$$= \{((\Psi^k_F \boxplus \Psi^k_{F_1} \boxplus \Psi^k_{F_2})(\rho,\lambda) \boxplus \imath_2\} \boxminus \imath_1$$
$$= (\Psi^k_{F \Cup F_1 \Cup F_2})^{\imath_2}_{\imath_1}(\rho,\lambda).$$

Let $\varkappa, \varpi, \varphi \notin X_\rho$. If $X_\rho = \emptyset$, then $(\Pi^k_F \cdot F_1 \cdot \Pi^k_{F_2})(\rho,\lambda) = 1$ and $(\Psi^k_F \cdot F_1 \cdot \Psi^k_{F_2})(\rho,\lambda) = 0$ such that $\rho \leq \varkappa\varpi\varphi$. Thus,

$$\begin{aligned}(\Pi^k_{(F \cdot F_1 \cdot F_2]})^{\imath_2}_{\imath_1}(\rho,\lambda) &= (\Pi^k_{(F \cdot F_1 \cdot F_2]}(\rho,\lambda) \boxminus \imath_2) \boxplus \imath_1 \\ &= 1 \boxplus \imath_1 \\ &\geq (\Pi^k_{F \Cap F_1 \Cap F_2}(\rho,\lambda) \boxminus \imath_2) \boxplus \imath_1 \\ &= (\Pi^k_{F \Cap F_1 \Cap F_2}(\rho,\lambda) \boxminus \imath_2),\end{aligned}$$

$$\begin{aligned}(\Psi^k_{(F \cdot F_1 \cdot F_2]})^{\imath_2}_{\imath_1}(\rho,\lambda) &= (\Psi^k_{(F \cdot F_1 \cdot F_2]}(\rho,\lambda) \boxplus \imath_2) \boxminus \imath_1 \\ &= 0 \boxminus \imath_1 \\ &= \imath_1 \\ &\leq (\Psi^k_{F \Cup F_1 \Cup F_2}(\rho,\lambda) \boxplus \imath_2) \boxminus \imath_1 \\ &= (\Psi^k_{F \Cup F_1 \Cup F_2}(\rho,\lambda) \boxplus \imath_2).\end{aligned}$$

Therefore, $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} \subseteq ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$. □

Corollary 2.1. *Ξ is regular if and only if every RI F, every LATI F_1 and every LI F_2 of Ξ, $(F \Cap F_1 \Cap F_2] = (F \cdot F_1 \cdot F_2]$.*

Theorem 2.7. *Let F be an $(\imath_1,\imath_2)$-MPIQAFRI, F_1 be an $(\imath_1,\imath_2)$-MPIQAFLATI, and F_2 be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. Then Ξ be regular if and only if $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} = ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$.*

Proof. Let F be an $(\imath_1,\imath_2)$-MPIQAFRI, F_1 be an $(\imath_1,\imath_2)$-MPIQAFLATI, and F_2 be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. Let $(\varkappa,\varphi) \in X_\rho$. If $X_\rho \neq \emptyset$, then $\rho \leq \varkappa\varpi\varphi$. Thus, $\Pi^k_F(\rho,\lambda) \leq \Pi^k_F(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_F(\varkappa,\lambda)$ and $\Psi^k_F(\rho,\lambda) \geq \Psi^k_F(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_F(\varkappa,\lambda)$.

Similarly, $\Pi^k_{F_1}(\rho,\lambda) \leq \Pi^k_{F_1}(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_{F_1}(\varpi,\lambda)$ and $\Psi^k_{F_1}(\rho,\lambda) \geq \Psi^k_{F_1}(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_{F_1}(\varpi,\lambda)$.

Similarly, $\Pi^k_{F_2}(\rho,\lambda) \leq \Pi^k_{F_2}(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_{F_2}(\varphi,\lambda)$ and $\Psi^k_{F_2}(\rho,\lambda) \geq \Psi^k_{F_2}(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_{F_2}(\varphi,\lambda)$.

For $\rho \in \Xi$, there exists $x \in \Xi$ such that $\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho$. Then $\rho \leq (\sigma_1\rho\sigma_2\rho\sigma_3), \rho \in X_\rho$. Thus,

$$\begin{aligned}&(\Pi^k_{(F \cdot F_1 \cdot F_2]})^{\imath_2}_{\imath_1}(\rho,\lambda) \\ &= (\Pi^k_{(F \cdot F_1 \cdot F_2]}(\rho,\lambda) \boxminus \imath_2) \boxplus \imath_1 \\ &= \left[[\inf_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{\Pi^k_F(\varkappa,\lambda) \boxminus \Pi^k_{F_1}(\varpi,\lambda) \boxminus \Pi^k_{F_2}(\varphi,\lambda)\} \boxminus \imath_2]\right] \boxplus \imath_1 \\ &= \left[\inf_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{\Pi^k_F(\varkappa,\lambda) \boxminus \Pi^k_{F_1}(\varpi,\lambda) \boxminus \Pi^k_{F_2}(\varphi,\lambda)\} \boxminus \imath_2 \boxminus \imath_2 \boxminus \imath_2 \boxminus \imath_2\right] \boxplus \imath_1 \\ &= \left[\inf_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{(\Pi^k_F(\varkappa,\lambda) \boxminus \imath_2) \boxminus (\Pi^k_{F_1}(\varpi,\lambda) \boxminus \imath_2) \boxminus (\Pi^k_{F_2}(\varphi,\lambda) \boxminus \imath_2)\} \boxminus \imath_2\right] \boxplus \imath_1 \\ &\leq (\{(\Pi^k_F(\rho,\lambda) \boxplus \imath_1) \boxminus (\Pi^k_{F_1}(\sigma_1\rho\sigma_2\rho\sigma_3) \boxplus \imath_1) \boxminus (\Pi^k_{F_2}(\rho,\lambda) \boxplus \imath_1)\} \boxminus \imath_2) \boxplus \imath_1\end{aligned}$$

$$\leq (\{(\Pi^k_F(\rho,\lambda) \boxplus \imath_1) \boxminus (\Pi^k_{F_1}(\rho,\lambda) \boxplus \imath_1) \boxminus (\Pi^k_{F_2}(\rho,\lambda) \boxplus \imath_1)\} \boxminus \imath_2) \boxplus \imath_1$$
$$= \{((\Pi^k_F(\rho,\lambda) \boxminus \Pi^k_{F_1}(\rho,\lambda) \boxminus \Pi^k_{F_2}(\rho,\lambda)) \boxplus \imath_1) \boxminus \imath_2\} \boxplus \imath_1$$
$$= \{((\Pi^k_F \boxminus \Pi^k_{F_1} \boxminus \Pi^k_{F_2})(\rho,\lambda) \boxminus \imath_2\} \boxplus \imath_1$$
$$= (\Pi^k_{F \Cap F_1 \Cap F_2})^{\imath_2}_{\imath_1}(\rho,\lambda),$$

$$(\Psi^k_{(F \cdot F_1 \cdot F_2]})^{\imath_2}_{\imath_1}(\rho,\lambda)$$
$$= (\Psi^k_{(F \cdot F_1 \cdot F_2]}(\rho,\lambda) \boxplus \imath_2) \boxminus \imath_1$$
$$= \Big[[\sup_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{\Psi^k_F(\varkappa,\lambda) \boxplus \Psi^k_{F_1}(\varpi,\lambda) \boxplus \Psi^k_{F_2}(\varphi,\lambda)\} \boxplus \imath_2]\Big] \boxminus \imath_1$$
$$= \Big[\sup_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{\Psi^k_F(\varkappa,\lambda) \boxplus \Psi^k_{F_1}(\varpi,\lambda) \boxplus \Psi^k_{F_2}(\varphi,\lambda)\} \boxplus \imath_2 \boxplus \imath_2 \boxplus \imath_2 \boxplus \imath_2\Big] \boxminus \imath_1$$
$$= \Big[\sup_{\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho} \{(\Psi^k_F(\varkappa,\lambda) \boxplus \imath_2) \boxplus (\Psi^k_{F_1}(\varpi,\lambda) \boxplus \imath_2) \boxplus (\Psi^k_{F_2}(\varphi,\lambda) \boxplus \imath_2)\} \boxplus \imath_2\Big] \boxminus \imath_1$$
$$\geq (\{(\Psi^k_F(\rho,\lambda) \boxminus \imath_1) \boxplus (\Psi^k_{F_1}(\sigma_1\rho\sigma_2\rho\sigma_3) \boxminus \imath_1) \boxplus (\Psi^k_{F_2}(\rho,\lambda) \boxminus \imath_1)\} \boxplus \imath_2) \boxminus \imath_1$$
$$\geq (\{(\Psi^k_F(\rho,\lambda) \boxminus \imath_1) \boxplus (\Psi^k_{F_1}(\rho,\lambda) \boxminus \imath_1) \boxplus (\Psi^k_{F_2}(\rho,\lambda) \boxminus \imath_1)\} \boxplus \imath_2) \boxminus \imath_1$$
$$= \{((\Psi^k_F(\rho,\lambda) \boxplus \Psi^k_{F_1}(\rho,\lambda) \boxplus \Psi^k_{F_2}(\rho,\lambda)) \boxminus \imath_1) \boxplus \imath_2\} \boxminus \imath_1$$
$$= \{((\Psi^k_F \boxplus \Psi^k_{F_1} \boxplus \Psi^k_{F_2})(\rho,\lambda) \boxplus \imath_2\} \boxminus \imath_1$$
$$= (\Psi^k_{F \Cup F_1 \Cup F_2})^{\imath_2}_{\imath_1}(\rho,\lambda).$$

Thus, $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} \supseteq ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$ and by Theorem 2.6 and hence, $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} = ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$.

Conversely, assume that $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} = ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$. Let $F = (\Pi^k_F, \Psi^k_F)$ be an $(\imath_1,\imath_2)$-MPIQAFRI, $F_1 = (\Pi^k_{F_1}, \Psi^k_{F_1})$ be an $(\imath_1,\imath_2)$-MPIQAFLATI, and $F_2 = (\Pi^k_{F_2}, \Xi_{F_2}, \Psi^k_{F_2})$ be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. Then by Theorem 2.4, δ^k_F is an $(\imath_1,\imath_2)$-MPIQAFRI, $\delta^k_{F_1}$ is an $(\imath_1,\imath_2)$-MPIQAFLATI, and $\delta^k_{F_2}$ be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. By Lemma 2.1 and Theorem 2.5, $(\delta^k_{(F \Cap F_1 \Cap F_2]})^{\imath_2}_{\imath_1} = (\delta^k_F \Cap \delta^k_{F_1} \Cap \delta^k_{F_2})^{\imath_2}_{\imath_1} = (\delta^k_F \cdot \delta^k_{F_1} \cdot \delta^k_{F_2})^{\imath_2}_{\imath_1} = (\delta^k_{(F \cdot F_1 \cdot F_2]})^{\imath_2}_{\imath_1}$. This implies $((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1} = ((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1}$. We appeal to Corollary 2.1, hence, Ξ is regular. □

Theorem 2.8. *Let F be an $(\imath_1,\imath_2)$-MPIQAFBI, F_1 be an $(\imath_1,\imath_2)$-MPIQAFLATI, and F_2 be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. Then Ξ is regular if and only if $((F \cdot F_1 \cdot F_2])^{\imath_2}_{\imath_1} = ((F \Cap F_1 \Cap F_2])^{\imath_2}_{\imath_1}$.*

Proof. Let F be an $(\imath_1,\imath_2)$-MPIQAFBI, F_1 be an $(\imath_1,\imath_2)$-MPIQAFLATI, and F_2 be an $(\imath_1,\imath_2)$-MPIQAFLI of Ξ. Let $(\varkappa,\varphi) \in X_\rho$. If $X_\rho \neq \emptyset$, then $\rho \leq \varkappa\varpi\varphi$. Thus, $\Pi^k_F(\rho,\lambda) \leq \Pi^k_F(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_F(\varkappa,\lambda)$ and $\Psi^k_F(\rho,\lambda) \geq \Psi^k_F(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_F(\varkappa,\lambda)$.

Similarly, $\Pi^k_{F_1}(\rho,\lambda) \leq \Pi^k_{F_1}(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_{F_1}(\varpi,\lambda)$ and $\Psi^k_{F_1}(\rho,\lambda) \geq \Psi^k_{F_1}(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_{F_1}(\varpi,\lambda)$.

Similarly, $\Pi^k_{F_2}(\rho,\lambda) \leq \Pi^k_{F_2}(\varkappa\varpi\varphi,\lambda) \leq \Pi^k_{F_2}(\varphi,\lambda)$ and $\Psi^k_{F_2}(\rho,\lambda) \geq \Psi^k_{F_2}(\varkappa\varpi\varphi,\lambda) \geq \Psi^k_{F_2}(\varphi,\lambda)$.

For $\rho \in \Xi$, there exists $x \in \Xi$ such that $\rho \leq \rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho$. Then $\rho \leq (\rho\sigma_1\rho\sigma_2\rho), (\sigma_3\rho\sigma_4\rho\sigma_5), \rho \in X_\rho$. Thus,

$$
\begin{aligned}
&(\Pi^k_{(F\cdot F_1\cdot F_2]})^{\iota_2}_{\iota_1}(\rho,\lambda)\\
&=(\Pi^k_{(F\cdot F_1\cdot F_2]}(\rho,\lambda)\boxminus\iota_2)\boxplus\iota_1\\
&=\Big[[\inf_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{\Pi^k_F(\varkappa,\lambda)\boxminus\Pi^k_{F_1}(\varpi,\lambda)\boxminus\Pi^k_{F_2}(\varphi,\lambda)\}\boxminus\iota_2]\Big]\boxplus\iota_1\\
&=\Big[\inf_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{\Pi^k_F(\varkappa,\lambda)\boxminus\Pi^k_{F_1}(\varpi,\lambda)\boxminus\Pi^k_{F_2}(\varphi,\lambda)\}\boxminus\iota_2\boxminus\iota_2\boxminus\iota_2\boxminus\iota_2\Big]\boxplus\iota_1\\
&=\Big[\inf_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{(\Pi^k_F(\varkappa,\lambda)\boxminus\iota_2)\boxminus(\Pi^k_{F_1}(\varpi,\lambda)\boxminus\iota_2)\boxminus(\Pi^k_{F_2}(\varphi,\lambda)\boxminus\iota_2)\}\boxminus\iota_2\Big]\boxplus\iota_1\\
&\leq(\{(\Pi^k_F(\rho\sigma_1\rho\sigma_2\rho,\lambda)\boxplus\iota_1)\boxminus(\Pi^k_{F_1}(\sigma_3\rho\sigma_4\rho\sigma_5)\boxplus\iota_1)\boxminus(\Pi^k_{F_2}(\rho,\lambda)\boxplus\iota_1)\}\boxminus\iota_2)\boxplus\iota_1\\
&\leq(\{(\Pi^k_F(\rho,\lambda)\boxplus\iota_1)\boxminus(\Pi^k_{F_1}(\rho,\lambda)\boxplus\iota_1)\boxminus(\Pi^k_{F_2}(\rho,\lambda)\boxplus\iota_1)\}\boxminus\iota_2)\boxplus\iota_1\\
&=\{((\Pi^k_F(\rho,\lambda)\boxminus\Pi^k_{F_1}(\rho,\lambda)\boxminus\Pi^k_{F_2}(\rho,\lambda))\boxplus\iota_1)\boxminus\iota_2\}\boxplus\iota_1\\
&=\{((\Pi^k_F\boxminus\Pi^k_{F_1}\boxminus\Pi^k_{F_2})(\rho,\lambda)\boxminus\iota_2\}\boxplus\iota_1\\
&=(\Pi^k_{F\Cap F_1\Cap F_2})^{\iota_2}_{\iota_1}(\rho,\lambda),
\end{aligned}
$$

$$
\begin{aligned}
&(\Psi^k_{(F\cdot F_1\cdot F_2]})^{\iota_2}_{\iota_1}(\rho,\lambda)\\
&=(\Psi^k_{(F\cdot F_1\cdot F_2]}(\rho,\lambda)\boxplus\iota_2)\boxminus\iota_1\\
&=\Big[[\sup_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{\Psi^k_F(\varkappa,\lambda)\boxplus\Psi^k_{F_1}(\varpi,\lambda)\boxplus\Psi^k_{F_2}(\varphi,\lambda)\}\boxplus\iota_2]\Big]\boxminus\iota_1\\
&=\Big[\sup_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{\Psi^k_F(\varkappa,\lambda)\boxplus\Psi^k_{F_1}(\varpi,\lambda)\boxplus\Psi^k_{F_2}(\varphi,\lambda)\}\boxplus\iota_2\boxplus\iota_2\boxplus\iota_2\boxplus\iota_2\Big]\boxminus\iota_1\\
&=\Big[\sup_{\rho\leq\rho\sigma_1\rho\sigma_2\rho\sigma_3\rho\sigma_4\rho\sigma_5\rho}\{(\Psi^k_F(\varkappa,\lambda)\boxplus\iota_2)\boxplus(\Psi^k_{F_1}(\varpi,\lambda)\boxplus\iota_2)\boxplus(\Psi^k_{F_2}(\varphi,\lambda)\boxplus\iota_2)\}\boxplus\iota_2\Big]\boxminus\iota_1\\
&\geq(\{(\Psi^k_F(\rho\sigma_1\rho\sigma_2\rho,\lambda)\boxminus\iota_1)\boxplus(\Psi^k_{F_1}(\sigma_3\rho\sigma_4\rho\sigma_5)\boxminus\iota_1)\boxplus(\Psi^k_{F_2}(\rho,\lambda)\boxminus\iota_1)\}\boxplus\iota_2)\boxminus\iota_1\\
&\geq(\{(\Psi^k_F(\rho,\lambda)\boxminus\iota_1)\boxplus(\Psi^k_{F_1}(\rho,\lambda)\boxminus\iota_1)\boxplus(\Psi^k_{F_2}(\rho,\lambda)\boxminus\iota_1)\}\boxplus\iota_2)\boxminus\iota_1\\
&=\{((\Psi^k_F(\rho,\lambda)\boxplus\Psi^k_{F_1}(\rho,\lambda)\boxplus\Psi^k_{F_2}(\rho,\lambda))\boxminus\iota_1)\boxplus\iota_2\}\boxminus\iota_1\\
&=\{((\Psi^k_F\boxplus\Psi^k_{F_1}\boxplus\Psi^k_{F_2})(\rho,\lambda)\boxplus\iota_2\}\boxminus\iota_1\\
&=(\Psi^k_{F\Cup F_1\Cup F_2})^{\iota_2}_{\iota_1}(\rho,\lambda).
\end{aligned}
$$

Thus, $((F\cdot F_1\cdot F_2])^{\iota_2}_{\iota_1}\supseteq((F\Cap F_1\Cap F_2])^{\iota_2}_{\iota_1}$ and by Theorem 2.6 and hence $((F\cdot F_1\cdot F_2])^{\iota_2}_{\iota_1}=((F\Cap F_1\Cap F_2])^{\iota_2}_{\iota_1}$.

Conversely, assume that $((F\cdot F_1\cdot F_2])^{\iota_2}_{\iota_1}=((F\Cap F_1\Cap F_2])^{\iota_2}_{\iota_1}$. Let $F=(\Pi^k_F,\Psi^k_F)$ be an (ι_1,ι_2)-MPIQAFBI, $F_1=(\Pi^k_{F_1},\Xi_{F_1},\Psi^k_{F_1})$ be an (ι_1,ι_2)-MPIQAFLATI, and $F_2=(\Pi^k_{F_2},\Psi^k_{F_2})$ be an (ι_1,ι_2)-MPIQAFLI of Ξ. Then by Theorem 2.4, δ^k_F is an (ι_1,ι_2)-MPIQAFBI, $\delta^k_{F_1}$ is an (ι_1,ι_2)-MPIQAFLATI, and $\delta^k_{F_2}$ be an (ι_1,ι_2)-MPIQAFLI of Ξ. By Lemma 2.1 and Theorem 2.5, $(\delta^k_{(F\Cap F_1\Cap F_2]})^{\iota_2}_{\iota_1}=(\delta^k_F\Cap\delta^k_{F_1}\Cap\delta^k_{F_2})^{\iota_2}_{\iota_1}=(\delta^k_F\cdot\delta^k_{F_1}\cdot\delta^k_{F_2})^{\iota_2}_{\iota_1}=(\delta^k_{(F\cdot F_1\cdot F_2]})^{\iota_2}_{\iota_1}$. This implies $((F\Cap F_1\Cap F_2])^{\iota_2}_{\iota_1}=((F\cdot F_1\cdot F_2])^{\iota_2}_{\iota_1}$. We appeal to Corollary 2.1, hence, Ξ is regular. □

3. Conclusion

In this study, we have explored the structural properties of regular ordered ternary semigroups by developing a comprehensive framework for multi-polar intuitionistic Q-anti-fuzzy ideals. Specifically, we introduced and analyzed various forms of these ideals—including left, right, lateral, and bi-ideals—by characterizing their corresponding level sets. The interplay between regularity in ternary semigroups and the behavior of their intuitionistic multi-polar extensions was rigorously investigated, leading to several characterizations and inclusion properties. These results not only generalize existing fuzzy ideal theories but also establish new algebraic insights into uncertainty modeling over ternary operations.

Future research may focus on extending these results to more generalized algebraic structures or on applying the multi-polar fuzzy ideal framework to computational models that involve higher-order operations and degrees of uncertainty.

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