Int. J. Anal. Appl. (2025), 23:189

International Journal of Analysis and Applications

Analytic General Conformable Semigroup

Nur Natasha Lim Boon Chye @ Mohd Hairie Lim[1], Bambang Hendriya Guswanto[2], Ahmad Fadillah Embong[1], Mohd Ariff Admon[1,*]

[1]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia*

[2]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Jenderal Soedirman, Purwokerto, Indonesia*

[*]*Corresponding author:* ariffadmon@utm.my

Abstract. Conformable fractional derivative is introduced by [1] to simplify the definition of fractional derivatives since most of them used an integral form which is difficult to solve real problem. However, [1] defined the conformable fractional derivative by considering a particular conformable fractional function $t^{1-\alpha}$. In this study, general conformable fractional Cauchy problem is considered and solved by using general conformable Laplace transform to obtain the solution operator of general conformable fractional Cauchy problem. Properties of classical semigroup are employed to retrieve the properties of general conformable semigroup from the solution operator of general conformable fractional Cauchy problem. Consequently, general conformable semigroup properties can be used to detemine the regularity of general conformable fractional Cauchy problem including its existence and uniqueness of the solution.

1. Introduction

The Cauchy problem is a foundational aspect of differential equation theory and serves a critical role in the mathematical modeling across various scientific and engineering fields [2]. Traditionally, the Cauchy problem entails determining a function that fulfills a given differential equation, alongside a specified set of initial conditions, usually defined at a single point, commonly time, $t = 0$. Mathematically, this can be formulated as follows:

$$\frac{du}{dt} = Au(t) + f(t), \quad u(0) = u_0, \tag{1.1}$$

Received: Jun. 5, 2025.

2020 *Mathematics Subject Classification.* 35A01, 35A02, 47A60.

Key words and phrases. general conformable; fractional derivative; Cauchy problem; semigroup theory; Laplace transform.

https://doi.org/10.28924/2291-8639-23-2025-189

ISSN: 2291-8639

where A is a sectorial operator, $u(t)$ is the arbitrary function, $f(t)$ defines the dynamics of the system, and u_0 is the initial value of u at $t = 0$. The solution to this problem describes the evolution of $u(t)$ for $t \geq 0$ based on the given initial state.

The Cauchy problem is worth to be investigated which is important to the behavior of the dynamical system prediction, as it includes both the differential laws that govern the system and the initial conditions that are needed to solve the problem [3]. This problem is particularly valuable in the contexts where time-dependent processes are modeled, such as wave propagation and heat diffusion [4, 5]. The solution to Cauchy problems provides insights into how initial states evolve under the given dynamic conditions, making them essential in the fields of a crucial predictive modeling.

A significant challenge in solving Cauchy problems lies in guaranteeing the existence and uniqueness of the solutions. In the absence of existence, a solution may not be attainable for the specified conditions, rendering the model unreliable as it could fail to generate results or predictions in certain scenarios. Similarly, without uniqueness, multiple solutions might emerge, making it impossible to identify a singular and definitive outcome. Ensuring both existence and uniqueness is essential for establishing the reliability and interpretability of the mathematical models. Hence, the findings from this study offer a framework to establish both the existence and uniqueness of solution to the general conformable fractional Cauchy problem (GCFCP), that leads to strengthen the theoretical foundation of Cauchy problems.

Numerous studies are conducted on the Cauchy problem, with a focus on the existence and uniqueness of the solutions for fractional Cauchy problems involving Riemann-Liouville and Caputo fractional derivatives (see [6–16]). The following provides the definition of Riemann-Liouville and Caputo fractional derivative respectively.

Definition 1.1. *[17] The Riemann-Liouville derivative (RL) of fractional order α of function $f(t)$ is defined as*

$$ {}_{RL}D^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-x)^{m-\alpha-1} f(x) dx, $$

where $\Gamma(m-\alpha)$ is gamma function, $m-1 \leq \alpha < m$ with $\alpha, m \in \mathbb{Z}^+$.

Definition 1.2. *[17] The Caputo derivative (C) of fractional order α of function $f(t)$ is defined as*

$$ {}_{C}D^{\alpha}_{0,t} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} f^{(m)}(x) dx, $$

for $m-1 \leq \alpha < m$ and $\alpha, m \in \mathbb{Z}^+$.

While the Cauchy problem is well-established and extensively analyzed, its fractional counterparts introduce unique challenges and opportunities. Fractional calculus broadens the scope of differentiation and integration by incorporating non-integer (or fractional) orders, thereby embedding memory and hereditary effects into the models. This approach is particularly effective for representing systems in which the current state is influenced by the entire history of past states, rather than just the most recent one [18].

However, traditional fractional derivatives, such as the Riemann-Liouville and Caputo formulations, face practical limitations. Their reliance on integral representations over extended intervals introduces non-local properties, which often makes them difficult to interpret and reflect the real-world scenarios.

To address these challenges, Khalil et al. [1] proposed a novel definition of fractional derivatives, termed the conformable fractional derivative (CFD). The following provides the definition of CFD.

Definition 1.3. *[1] Given a function $f : [0,\infty) \to \mathbb{R}$. Then, the CFD of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{h\to 0} \frac{f(t+ht^{1-\alpha}) - f(t)}{h},$$

for all $t > 0, \alpha \in (0,1)$. Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$. If f is differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

This definition overcomes several limitations of classical fractional derivatives, such as those based on the Riemann–Liouville and Caputo formulations, which frequently fail to uphold key properties of classical calculus [19]. The CFD is more closely aligned with classical calculus, making it easier to handle certain computations in fractional differential equations [20]. Khalil et al. [1] demonstrated that this new framework offers a practical and efficient alternative for modeling with fractional derivatives, particularly in solving specific fractional differential equations. Moreover, they suggested that this derivative provides a more intuitive and computationally viable option for applications requiring fractional calculus but without the memory effects.

The theoretical framework of CFD using the definition in [1] is extensively explored, particularly in the context of the Cauchy problem. Numerous studies are delved into the mathematical properties and applications of CFDs within this framework [21–24]. These works established a solid foundation, demonstrating the consistency of Khalil's definition with classical calculus and its applicability in modeling time-evolution problems. Moreover, the results obtained using Khalil's definition are widely applied to various real-world models, highlighting their versatility and effectiveness in addressing complex phenomena [25–27].

Building on the foundational work in [1], researchers developed various extensions and modifications to CFDs in order to enhance their utility and alignment with classical calculus principles. For example, Katugampola [28] introduced fractional integrals that preserve essential classical properties such as linearity, product rule, quotient rule, chain rule, and compatibility with Rolle's and mean value theorems. Subsequently, the N-derivative is proposed as a local fractional derivative to address inconsistencies in existing fractional definitions, particularly in the application of the product, quotient, and chain rules [29]. Further advancements are made by Sharif and Malkawi [30], who introduced a modified CFD to resolve the lack of a fully valid chain rule in fractional calculus. Recently, Kajouni et al. [31] proposed another extension of the CFD using a limit-based approach consistent with classical calculus principles and established a generalized mean value theorem.

These modified and extended definitions of CFDs are widely studied and applied across various domains. For instance, Katugampola's definition [28] incorporates fractional integrals is studied by [32,33] and is extensively used in solving multi-dimensional fractional equations and analyzing stability in the fields such as fluid dynamics and viscoelastic systems [34]. The definition of CFD and its nonlocal properties from [29] are investigated by [35], and is proven effectively in applications involving fractional Schrödinger equations and quantum mechanical systems [36]. Similarly, [30] integrated a tensor-product framework in the definition of CFD, enabling the study of fractional semigroups and nonlinear systems [37], with notable applications in Gardner equations and soliton theory [38]. Finally, limit-based approach to CFD in [31] is employed to explore generalized mean value theorems [39] and soliton stability in wave propagation models, further extending the versatility of fractional calculus in applied mathematics [40].

These advancements highlight the adaptability and growing importance of CFDs in modern mathematics and applied sciences. The properties of CFDs, as defined by Khalil et al. [1], Guzman et al. [29], and Kajouni et al. [31], are extensively analyzed to enhance their versatility and applicability [41]. By retaining fundamental classical calculus properties while addressing the limitations inherent to traditional fractional derivatives, these developments offer intuitive and effective tools for solving fractional differential equations and modeling complex phenomena. Moreover, the theoretical foundations and practical applications of CFDs, particularly those based on [1] and [28] definitions, are explored in [19], with further applications demonstrated in [42].

This study investigates the solution operator for Cauchy problem as in equation (1.1) by replacing the usual derivative with general conformable fractional derivative (GCFD). The following provides the general conformable fractional Cauchy problem (GCFCP) considered in this study,

$$\begin{aligned} D^{\alpha}_{\psi}u(t) &= Au(t) + f(t), \qquad 0 < t \leq T, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

with $\alpha \in (0,1]$, $f : (0,T] \to X$, and $u_0 \in X$, where X is Banach space and ψ is a fractional conformable function. We call the solution operator of the homogeneous case to the problem (1.2) as general conformable semigroup. The definition of GCFD is given as follows,

$$D^{\alpha}_{\psi}u(t) = \lim_{h \to 0} \frac{u(t + h\psi(t,\alpha)) - u(t)}{h},$$

where $\psi(t,\alpha)$ is a fractional conformable function and $\alpha \in (0,1]$.

Since CFDs represent a specific case of GCFD, the results in this paper accomodate all the existing results of Cauchy problem with various CFDs under some conditions. Moreover, the implementation of real application models is significantly more effective when using GCFD compared to CFDs. GCFD also offer a more flexible and robust mathematical framework, accommodating a broader range of dynamic behaviors and complex systems.

This paper consists of four sections. The first section provides the introduction on Cauchy problem with GCFD. Next section is the preliminaries where the foundational concepts, including sectorial operators and classical semigroup theory, as a basis for developing solutions to fractional Cauchy problems are discussed. In the main results section, there are three subsections, which includes the analytic general conformable semigroup, the fractional power of sectorial operators, and solutions to GCFCP by applying the general conformable Laplace transform (GCLT). Finally, the conclusion summarizes the findings, emphasizing the advantages of GCFD for simplifying fractional differential equations while preserving consistency with classical calculus, and suggests its applicability to real-world mathematical modeling.

2. Preliminaries

This section contains two subsections which are semigroup and GCFD respectively. The first subsection explores the classical theory of semigroups and its applications in solving non-homogeneous Cauchy problems involving linear operators. Definitions, properties of sectorial operators, and their relation to analytic semigroups are presented, highlighting their utility in tackling linear evolution equations. This subsection lays a strong mathematical foundation for exploring semigroup dynamics and their applications.

Next subsection provides the properties of GCFD, discusses the limitations of classical fractional derivatives and motivates the use of GCFD for broader applicability. Some key properties of usual derivative such as linearity, product and chain rules, and compatibility with classical calculus are explored. Additionally, the subsection introduces the GCLT, which will be applied to obtain the solution of GCFCP.

2.1. Analytic Semigroup.

The following provides the typical non-homogeneous Cauchy problem of usual derivative.

Let $A : D(A) \subset X \to X$ be a sectorial linear operator,

$$\begin{aligned} \frac{du}{dt} &= Au(t) + f(t), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where $u_0 \in X$, and function, $f : (0, \infty) \to X$, with X is Banach space.

The definition of sectorial operator is provided as follows.

Definition 2.1. *[43] An operator A is called sectorial if A satisfies the properties that there are constant $\theta \in (\frac{\pi}{2}, \pi)$ and constant $M > 0$ such that*

$$\rho(A) \supset \Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \theta\}$$

$$\|R(\lambda; A)\| \leq \frac{M}{|\lambda|}, \quad \lambda \in \Sigma_\theta$$

where $R(\lambda; A) = (\lambda I - A)^{-1}$ and $\rho(A) = \{\lambda \subseteq \mathbb{C} : R(\lambda; A)$ is bounded$\}$ which are called resolvent operator and resolvent set of A respectively. Note that every sectorial operator is closed, because its resolvent set is not empty.

A well-developed theory and a precise definition of semigroup are crucial for understanding its significance. Generally, semigroups are usually applied to solve a wide range of issues related to evolution equations. This section provides the properties of semigroup of a usual derivative and the definition of linear sectorial operator with its properties.

Semigroup theory offers a structured method to derive the solution. When the operator A generates an analytic semigroup $S(t)$, where

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,w}} e^{\lambda t} R(\lambda; A) d\lambda, \tag{2.2}$$

with

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r\},$$

for $r > 0$ and $\frac{\pi}{2} < \omega < \theta$ is oriented counterclockwise. The solution $u(t)$ to the homogeneous part of problem (2.1) can be expressed as:

$$u(t) = S(t)u_0.$$

Semigroup theory is a fundamental area of functional analysis that provides powerful tools for studying the solutions of time-evolution problems, such as the Cauchy problem. Essentially, semigroup theory focuses on families of operators that evolve over time, allowing for the systematic treatment of differential equations within infinite-dimensional spaces, like Banach and Hilbert spaces. This approach is particularly essential when dealing with linear evolution equations, where semigroups describe the progression of states in a dynamic system under the influence of an operator. The following theorem shows the properties of semigroup of a usual derivative.

Theorem 2.1. *[43] Let A be a linear sectorial operator. If $S(t)$ is an analytic semigroup generated by A in equation* (2.2)*, then the following statement holds.*

(i) $S(t) \in B(X)$ *and there exists constant* $C_1 > 0$ *such that for* $t > 0$,

$$\|S(t)\| \leq C_1,$$

where $B(X)$=\{$T : X \to X \mid T$ *is bounded operator*\}.

(ii) $S(t) \in B(X; D(A))$ *and* $S(t)x \in D(A)$ *for* $t > 0$ *and if* $x \in D(A)$ *then,*

$$AS(t)x = S(t)Ax.$$

Moreover, there exists constant $C_2 > 0$ *such that for* $t > 0$,

$$\|AS(t)x\| \leq C_2 t^{-1}.$$

(iii) The function $t \mapsto S(t)$ *belongs to* $C^{\infty}((0,\infty); B(X))$ *and it holds that*

$$S^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^n R(\lambda; A) d\lambda,$$

and there exists $M_n > 0$, *such that for* $t > 0$,

$$\|S^{(n)}(t)\| \leq M_n t^{-n},$$

for $n = 1, 2, 3, ...$ Moreover, it has analytic continuation $S(z)$ to the sector $\Sigma_{\theta-\frac{\pi}{2}}$ and for $z \in \Sigma_{\theta-\frac{\pi}{2}}$, $\eta \in (\frac{\pi}{2}, \theta)$, it holds that

$$S(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda; A) d\lambda.$$

(iv) For $s, t > 0$, and $x \in X$,

$$S(t)S(s)x = S(t+s)x,$$

and,

$$\frac{d}{dt}S(t)x = AS(t)x.$$

Theorem 2.2. *[43] Let A be a linear sectorial operator. If $S(t)$ is an analytic semigroup generated by A in equation* (2.2)*, then the following statement holds.*

(i) If $x \in \overline{D(A)}$ then

$$\lim_{t\to 0^+} S(t)x = x.$$

(ii) For every $x \in X$ and $t \geq 0$,

$$\int_0^t S(\tau)x d\tau \in D(A),$$

$$A\int_0^t S(\tau)x d\tau = S(t)x - x.$$

Moreover, if $\tau \to AS(\tau)x$ is integrable on $(0, \varepsilon)$, for some $\varepsilon > 0$ then, for $t \geq 0$,

$$S(t)x - x = \int_0^t AS(\tau)x d\tau.$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then

$$\lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax.$$

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then

$$\lim_{t\to 0^+} AS(t)x = Ax.$$

Theorem 2.3. *[43] Let $A : D(A) \subseteq X \to X$ be a sectorial linear operator. If $S(t)$ is an analytic semigroup generated by A as expressed in equation* (2.2)*, then for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,*

$$R(\lambda; A) = \int_0^\infty e^{\lambda t} S(t)\, dt.$$

Next, the following theorem shows the properties of fractional power of sectorial operator. Consider the fractional power of an operator A for $x \in X$, is defined by

$$A^{-\beta}x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{-\beta} R(\lambda; A)x\, d\lambda, \beta > 0 \tag{2.3}$$

and for $x \in D(A^\beta)$, and $A^{\beta-1}x \in D(A)$,

$$A^\beta x = A(A^{\beta-1}) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1} AR(\lambda; A)x\, d\lambda, 0 < \beta < 1. \tag{2.4}$$

If A is sectorial linear operator generating the analytic semigroup $S(t)$, one has the following theorem.

Theorem 2.4. *If A is a sectorial linear operator generating the analytic semigroup, $S(t)$, the following statements hold.*

(i) For $t > 0$ and $\beta \geq 0$, $S(t) : X \to D(A^{\beta})$;
(ii) For $x \in D(A^{\beta})$, $S(t)A^{\beta}x = A^{\beta}S(t)x$;
(iii) For $t > 0$, $A^{\beta}S(t)$ is bounded and $\left\|A^{\beta}S(t)\right\| \leq M_{\beta}t^{-\beta}$;
(iv) For $0 < \beta \leq 1$ and $x \in D(A^{\beta})$, $\left\|S(t)x - x\right\| \leq C_{\beta}t^{\beta}\left\|A^{\beta}x\right\|$.

2.2. **General Conformable Fractional Derivative.** The CFD introduced by [1], addresses many limitations of classical fractional derivatives, such as their reliance on non-local integral representations and their inconsistency with classical calculus properties like the product and chain rules. However, the original CFD is limited in its scope, as it defines the fractional behavior through a fixed function $\psi(t) = t^{1-\alpha}$, which restricts its adaptability to various real-world systems.

To overcome this limitation, [20] proposed the GCFD which introduces a more flexible framework by generalizing $\psi(t)$. The motivation for GCFD lies in its ability to retain the simplicity and intuitive nature of CFD while extending its applicability to various fields, such as physics, biology, and finance, where non-integer derivatives offer powerful modeling capabilities. The following provides the definition of fractional conformable function, ψ, for the unique meaning of each order $\alpha \in (0,1)$, $\psi(t,\alpha)$ should differ from different α,

$$\psi(t,1) = 1, \tag{2.5}$$

$$\psi(\cdot,p) \neq \psi(\cdot,q), \text{where } p \neq q \text{ and } p,q \in (0,1). \tag{2.6}$$

Definition 2.2. *[20] (Fractional Conformable Function) Fractional continuous real functions satisfying the equations in equation* (2.5), (2.6) *and constant value function $\psi(t,\alpha) = 1$ are called fractional conformable functions.*

The definition of GCFD is given as follows:

Definition 2.3. *[20] Let $\psi(t,\alpha)$ be a fractional conformable function and $\alpha \in (0,1]$. The GCFD is defined as:*

$$D^{\alpha}_{\psi}u(t) = \lim_{h\to 0}\frac{u(t+h\psi(t,\alpha)) - u(t)}{h},$$

where $t > 0$. If the limit exists, then u is said to be ψ-differentiable for $t > 0$.

Theorem 2.5. *If $u : (0,\infty) \to \mathbb{R}$ is a differentiable function at $t > 0$ then, for $\alpha \in (0,1]$,*

$$D^{\alpha}_{\psi}u(t) = \psi(t,\alpha)\frac{d}{dt}u(t). \tag{2.7}$$

The GCFD retains several classical calculus properties, making it intuitive and easy to apply in practical scenarios [20]:

(1) Linearity:

For any two functions f and g and constants $a, b \in \mathbb{R}$,

$$D_{\psi}^{\alpha}(af+bg)(t) = aD_{\psi}^{\alpha}(f)(t) + bD_{\psi}^{\alpha}(g)(t).$$

(2) Product Rule:

For two functions f and g,

$$D_{\psi}^{\alpha}(fg)(t) = f(t)D_{\psi}^{\alpha}(g)(t) + g(t)D_{\psi}^{\alpha}(f)(t).$$

(3) Quotient Rule:

For two functions f and g where $g(t) \neq 0$,

$$D_{\psi}^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t)D_{\psi}^{\alpha}(f)(t) - f(t)D_{\psi}^{\alpha}(g)(t)}{(g(t))^2}.$$

(4) Chain Rule:

If f is differentiable and g is ψ-differentiable, then

$$D_{\psi}^{\alpha}(f \circ g(t)) = f'(g(t))D_{\psi}^{\alpha}(g(t)).$$

(5) Conformability to Classical Calculus:

When $\psi(t, \alpha) = 1$, the general conformable fractional derivative reduces to the usual derivative. This compatibility with usual derivatives adds to its versatility, allowing it to bridge fractional and integer-order calculus seamlessly.

Next, the Laplace transform is a fundamental technique in mathematics and engineering, frequently employed to solve differential equations by transforming differential operators into algebraic expressions, thereby simplifying the solution process. Building on this approach, the GCLT extends its applicability to fractional calculus, specifically for conformable fractional derivatives. GCLT serves as an effective tool for managing differential equations with fractional orders, enabling more accurate modeling of systems exhibiting memory effects. The theorem below presents the complex inversion formula for the exponential Laplace transform.

We choose a fractional conformable function $\psi(t, \alpha)$ such that for $t \geq 0$,

$$\varepsilon(t) = \int_0^t \frac{1}{\psi(\tau, \alpha)} d\tau \tag{2.8}$$

exists and $\varepsilon(0) = 0$.

Definition 2.4. *[44] General Conformable Laplace Transform (GCLT)*
Given $u : [0, \infty) \to \mathbb{R}$ is piecewise continuous and of the exponential order such that $|u(t)| \leq Me^{c\varepsilon(t)}$ for some constant c, M. The general conformable Laplace transform is given as follows,

$$\mathcal{L}_{\psi}\{u(t); s\} = U_{\psi}(s) = \int_0^{\infty} \varepsilon'(t)e^{-s\varepsilon(t)}u(t)\, dt.$$

Definition 2.5. *[44] Inverse of General Conformable Laplace Transform*
Let $U_\psi(s)$ be analytic function of s, where $\mathrm{Re}(s) = c$. If $U_\psi(s) \to 0$ as $s \to \infty$ through the left plane $Re(s) \leq c$, then inverse of general conformable Laplace transform are defined as follows,

$$\mathcal{L}_\psi^{-1}\{U_\psi(s);t\} = u(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U_\psi(s)e^{s\varepsilon(t)}\,ds.$$

Lemma 2.1. *If $u : [0,\infty) \to \mathbb{R}$ is a Laplace transformable function at $t > 0$, then the following holds,*

$$\mathcal{L}_\psi\{u(t);s\} = \mathcal{L}\{u(\varepsilon^{-1}(t));s\}. \tag{2.9}$$

Proof. By definition of general conformable Laplace transform, one has

$$\mathcal{L}_\psi\{u(t);s\} = \int_0^\infty \varepsilon'(t)e^{-s\varepsilon(t)}u(t)\,dt.$$

It follows that,

$$\mathcal{L}_\psi\{u(t);s\} = \int_0^\infty e^{-st}u(\varepsilon^{-1}(t))\,dt = \mathcal{L}\{u(\varepsilon^{-1}(t));s\}.$$

□

Theorem 2.6. *If $f : [0,\infty) \to \mathbb{R}$ is a ψ-differentiable function at $t > 0$ and $\alpha \in (0,1]$, then*

$$\mathcal{L}_\psi\{D_\psi^\alpha f(t);s\} = sF_\psi(s) - f(0).$$

Proof. From equation (2.9), one gets

$$\begin{aligned}\mathcal{L}_\psi\{D_\psi^\alpha f(t);s\} &= \mathcal{L}\{D_\psi^\alpha f(\varepsilon^{-1}(t));s\}\\ &= s\mathcal{L}\{f(\varepsilon^{-1}(t));s\} - f(0)\\ &= s\mathcal{L}_\psi\{f(t);s\} - f(0)\\ &= sF_\psi(s) - f(0).\end{aligned}$$

□

Next, the theorem of ψ-convolution for functions f and g is provided. The ψ-convolution is defined as follows,

$$(f *_\psi g)(t) = \int_0^t \varepsilon'(\tau)f(\varepsilon^{-1}(\varepsilon(t) - \varepsilon(\tau)))g(\tau)d\tau. \tag{2.10}$$

Note that the convolution of functions f and g is defined as

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau. \tag{2.11}$$

Theorem 2.7. *If $F(s)$ and $G(s)$ are the Laplace tranform of the functions $f(x)$, $g(x)$ respectively, then*

$$F(s)G(s) = \mathcal{L}\{(f * g)(t);s\}.$$

Theorem 2.8. *If f and g are piecewise continuous functions on [0, ∞), then*

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = \mathcal{L}\{(f * g)(\varepsilon^{-1}(t));s\}. \tag{2.12}$$

Proof. The GCLT of convolution function of f and g is

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = \int_0^\infty \varepsilon'(t)e^{-s\varepsilon(t)} \int_0^t \varepsilon'(\tau)f(\varepsilon^{-1}(\varepsilon(t)-\varepsilon(\tau)))g(\tau)d\tau\, dt.$$

By letting $p = \varepsilon(t)$, then $\frac{dp}{dt} = \varepsilon'(t)$, the equation becomes

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = \int_0^\infty e^{-sp} \int_0^{\varepsilon^{-1}(p)} \varepsilon'(\tau)f(\varepsilon^{-1}(p-\varepsilon(\tau)))g(\tau)d\tau\, dp.$$

Now let $q = \varepsilon(\tau)$, then $\frac{dq}{d\tau} = \varepsilon'(\tau)$, then the equation becomes

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = \int_0^\infty e^{-sp} \int_0^p f(\varepsilon^{-1}(p-q))g(\varepsilon^{-1}(q))dq\, dp.$$

By employing equation (2.11), and replacing p by t, one gets

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = \mathcal{L}\{(f * g)(\varepsilon^{-1}(t));s\}.$$

□

Theorem 2.9. *If f and g are piecewise continuous functions on $[0,\infty)$, then the general conformable fractional Laplace transform of the ψ-convolution $f *_\psi g$ of function f and g is given by*

$$\mathcal{L}_\psi\{(f *_\psi g)(t);s\} = F_\psi(s)\cdot G_\psi(s).$$

Proof. By employing equation (2.12),

$$\begin{aligned}\mathcal{L}_\psi\{(f *_\psi g)(t);s\} &= \mathcal{L}\{(f * g)(\varepsilon^{-1}(t));s\}\\ &= \mathcal{L}\{f(\varepsilon^{-1}(t));s\}\cdot \mathcal{L}\{g(\varepsilon^{-1}(t));s\}\\ &= \mathcal{L}_\psi\{f(t);s\}\cdot\mathcal{L}_\psi\{g(t);s\}\\ &= F_\psi(s)\cdot G_\psi(s).\end{aligned}$$

□

3. Main Results

Recall the GCFCP in equation (1.2), let $A : D(A) \subset X \to X$ be a sectorial operator and let $T > 0$,

$$\begin{aligned}D_\psi^\alpha u(t) &= Au(t) + f(t), \qquad 0 < t \le T,\\ u(0) &= u_0,\end{aligned}$$

where X is Banach space, $0 < \alpha < 1$, $f : (0,T] \to X$, and $u_0 \in X$.

3.1. **Analytic General Conformable Semigroup.** This section provides the solution operator of the Cauchy problem (1.2). By applying GCLT to the Cauchy problem (1.2), one has

$$\begin{aligned}
\mathcal{L}_\psi\{D_\psi^\alpha u(t);s\} &= \mathcal{L}_\psi\{Au(t);s\}+\mathcal{L}_\psi\{f(t);s\},\\
s\mathcal{L}_\psi\{u(t);s\}-u(0) &= A\mathcal{L}_\psi\{u(t);s\}+\mathcal{L}_\psi\{f(t);s\},\\
(sI-A)\mathcal{L}_\psi\{u(t);s\} &= u(0)+\mathcal{L}_\psi\{f(t);s\},\\
\mathcal{L}_\psi\{u(t);s\} &= (sI-A)^{-1}u_0+(sI-A)^{-1}\mathcal{L}_\psi\{f(t);s\},\\
U_\psi(s) &= R(s;A)u_0+R(s;A)\mathcal{L}_\psi\{f(t);s\}.
\end{aligned}$$

Then, by applying inverse of general conformable Laplace transform, one obtains

$$\begin{aligned}
\mathcal{L}_\psi^{-1}\{U_\psi(s);t\} &= \mathcal{L}_\psi^{-1}\{R(s;A)u_0;t\}+\mathcal{L}_\psi^{-1}\{R(s;A)\mathcal{L}_\psi\{f(t);s\};t\},\\
u(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s\varepsilon(t)}R(s;A)u_0\,ds.+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{s\varepsilon(t)}F_\psi(s)R(s;A)\,ds.
\end{aligned}$$

By considering homogeneous part of equation (1.2), the operator $S_\psi(t)$ is defined as follows,

$$S_\psi(t)=\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}}e^{\lambda\varepsilon(t)}R(\lambda;A)u_0\,d\lambda, \tag{3.1}$$

where $t>0$.

Next, the following theorems provide the properties of the operator $S_\psi(t)$.

Theorem 3.1. *Let A be a sectorial linear operator, $B(X)$ is the set of bounded linear operator on Banach space X, $D(A)$ is domain of sectorial operator A and is a linear subspace of X, $S_\psi(t)$ is an operator defined in equation* (3.1), *then the following statements hold.*

(i) $S_\psi(t)\in B(X)$, *and there exists constant, C_1 such that for $t>0$,*

$$\left\|S_\psi(t)\right\|\le C_1.$$

(ii) $S_\psi(t)\in B(X:D(A))$, *for $t>0$ and if $x\in D(A)$ then, $AS_\psi(t)x=S_\psi(t)Ax$. Moreover, there exists constant, $C_2>0$ such that for $t>0$,*

$$\left\|AS_\psi(t)x\right\|\le C_2[\varepsilon(t)]^{-1}.$$

(iii) The function $t\mapsto S_\psi(t)$ is differentiable at $(0,\infty)$ and

$$S'_\psi(t)=\frac{1}{\psi(t,\alpha)}AS_\psi(t),$$

and there exists $M_n>0$ such that for

$$\left\|S'_\psi(t)\right\|\le M_n[\psi(t,\alpha)\varepsilon(t)]^{-1},\qquad t>0.$$

Moreover, operator $S_\psi(t)$ has analytic continuation on the sector $\Sigma_{\theta-\frac{\pi}{2}}$ and for $z\in\Sigma_{\theta-\frac{\pi}{2}}$, $\eta\in(\frac{\pi}{2},\theta)$,

$$S_\psi(z)=\frac{1}{2\pi i}\int_{\Gamma_{r,\eta}}e^{\lambda\varepsilon(z)}R(\lambda;A)d\lambda.$$

(iv) For $s, t > 0$, one has

$$S_\psi(\varepsilon^{-1}(t+s)) = S_\psi(\varepsilon^{-1}(t))S_\psi(\varepsilon^{-1}(s)).$$

(v) For $t > 0$, one has

$$D^\alpha_\psi S_\psi(t) = AS_\psi(t).$$

Proof. From Theorem 2.1(i), one obtains for $S(t) \in B(X)$ and there exists $C_1 > 0$,

$$\left\|S(t)\right\| \leq C_1, \quad t > 0.$$

Therefore, for $S_\psi(t) \in B(X)$, there exists $C_1 > 0$, such that,

$$\left\|S_\psi(t)\right\| = \left\|S(\varepsilon(t))\right\| \leq C_1, \quad t > 0.$$

This proves (i). Next, to prove (ii) from Theorem 2.1 (ii), for any $S(t) \in B(X; D(A))$, there exists $C_2 > 0$, such that

$$\left\|AS(t)x\right\| \leq C_2 t^{-1}, \quad t > 0.$$

This implies the following,

$$\left\|AS_\psi(t)x\right\| = \left\|AS(\varepsilon(t))x\right\| \leq C_2[\varepsilon(t)]^{-1}, \quad t > 0,$$

for $S_\psi(t) \in B(X; D(A))$.

To prove (iii) observe that the function $t \mapsto S(t)$ is differentiable on $(0, \infty)$, and $S'(t) = AS(t)$. Consequently, for $t > 0$,

$$\begin{aligned}
S'_\psi(t) &= \frac{d}{dt}S(\varepsilon(t)) \\
&= \frac{1}{\psi(t,\alpha)}S'(\varepsilon(t)) \\
&= \frac{1}{\psi(t,\alpha)}AS(\varepsilon(t)) \\
&= \frac{1}{\psi(t,\alpha)}AS_\psi(t).
\end{aligned}$$

Next, from Theorem 2.1 (iii), for $n = 1$, one has the following,

$$\left\|S'(t)\right\| \leq M_n t^{-1}.$$

Therefore, for $t > 0$,

$$\begin{aligned}
\left\|S'_\psi(t)\right\| &= \left\|\frac{1}{\psi(t,\alpha)}AS_\psi(t)\right\| \\
&= \frac{1}{\psi(t,\alpha)}\left\|AS_\psi(t)\right\| \\
&\leq \frac{1}{\psi(t,\alpha)}M_n[\varepsilon(t)]^{-1} \\
&= M_n[\psi(t,\alpha)\varepsilon(t)]^{-1}.
\end{aligned}$$

Next, to show that it has analytic continuation $S_\psi(z)$ to the sector $\Sigma_{\theta-\frac{\pi}{2}}$, suppose $z \in S_{\eta-\frac{\pi}{2}}$, and $\lambda = |\lambda|e^{\pm\eta i}, |\lambda| \geq r$, then

$$\varepsilon(z)\lambda = \varepsilon(z)|\lambda|e^{\pm\eta i} = |\varepsilon(z)|e^{i\arg(\varepsilon(z))}|\lambda|e^{\pm\eta i} = |\varepsilon(z)||\lambda|e^{i(arg(\varepsilon(z)))\pm\eta},$$

where $\frac{\pi}{2} < \arg(\varepsilon(z)) + \eta < \frac{3\pi}{2}$ and $-\frac{3\pi}{2} < \arg(\varepsilon(z)) + \eta < -\frac{\pi}{2}$. Then, $\mathrm{Re}(\lambda z < 0)$,

$$\begin{aligned}
\left\|S_\psi(z)\right\| &= \left\|\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda\varepsilon(z)}R(\lambda;A)d\lambda\right\| \\
&= \left\|\frac{1}{2\pi i}\int_{\Gamma_{|\varepsilon(z)|^{-1},\eta}} e^{\lambda\varepsilon(z)}R(\lambda;A)d\lambda\right\| \\
&\leq \frac{1}{2\pi}\int_{\Gamma_{|\varepsilon(z)|^{-1},\eta}} |e^{\lambda\varepsilon(z)}|\left\|R(\lambda;A)\right\||d\lambda| \\
&\leq \frac{2M}{2\pi}\int_{|\varepsilon(z)|^{-1}}^{\infty} \frac{e^{|\lambda||\varepsilon(z)|\cos(\arg(\varepsilon(z))+\arg(\lambda))}}{|\lambda|}d|\lambda| \\
&\quad + \frac{M}{2\pi}\int_{-\eta}^{\eta} e^{|\varepsilon(z)||\varepsilon(z)|^{-1}\cos(\arg(\varepsilon(z))+\arg(\lambda))}d\{\arg(\lambda)\}.
\end{aligned}$$

Let $u = |\varepsilon(z)||\lambda| \implies du = |\varepsilon(z)|d|\lambda|$, where $|\lambda| = |\varepsilon(z)|^{-1}$ which implies $u = 1$. Therefore,

$$\left\|S_\psi(z)\right\| \leq \frac{M}{\pi}\int_1^{\infty} e^{\cos(\arg(\varepsilon(z))+\eta)}du + \frac{M}{2\pi}\int_{-\eta}^{\eta} e^{\cos(\arg(\varepsilon(z))+\eta)}d\eta.$$

This implies the boundedness of $\left\|S_\psi(z)\right\|$. Note that the value of $\varepsilon(z)$ is not unique. Thus, if we choose the principle value of $\varepsilon(z)$, then for $z \in \Sigma_{\theta-\frac{\pi}{2}}, \eta \in (\frac{\pi}{2}, \theta)$,

$$z \mapsto S_\psi(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda\varepsilon(z)}R(\lambda;A),$$

can be considered as an analytic continuation of $S_\psi(t)$ to the sector $\Sigma_{\theta-\frac{\pi}{2}}$. Since union of the sector $\Sigma_{\eta-\frac{\pi}{2}}$ is $\Sigma_{\theta-\frac{\pi}{2}}$, the function is also analytic on $\Sigma_{\theta-\frac{\pi}{2}}$.

Then, to prove (iv), from Theorem 2.1(iv), for $x \in X$ and $t, s > 0$,

$$S(t)S(s) = S(t+s).$$

Consequently, since $S_\psi(t) = S(\varepsilon(t))$, and let $t' = \varepsilon^{-1}(t+s)$,

$$\begin{aligned}
S_\psi(\varepsilon^{-1}(t+s)) &= S_\psi(t') \\
&= S(\varepsilon(t')) \\
&= S(\varepsilon(\varepsilon^{-1}(t+s))) \\
&= S(t+s) \\
&= S(t)S(s) \\
&= S(\varepsilon(\varepsilon^{-1}(t)))S(\varepsilon(\varepsilon^{-1}(s))) \\
&= S_\psi(\varepsilon^{-1}(t))S_\psi(\varepsilon^{-1}(s)).
\end{aligned}$$

Lastly, to prove (v), one gets for $t > 0$ in (iii),

$$S'_\psi(t) = \frac{1}{\psi(t,\alpha)} A S_\psi(t),$$

where,

$$D^\alpha_\psi S_\psi(t) = \psi(t,\alpha) S'_\psi(t) = A S_\psi(t).$$

□

Based on Theorem 3.1 (iii), we retrieve that $S_\psi(t)$ is analytic. Additionally, Theorem 3.1 (v) implies that $S_\psi(t)$ is the solution operator of the homogeneous part of problem (1.2). Therefore, $S_\psi(t)$ is called an analytic general conformable semigroup to the problem (1.2).

Theorem 3.2. *Let A be a sectorial operator and $S_\psi(t)$ is an analytic general conformable semigroup defined in equation* (3.1)*, $D(A)$ is a domain of sectorial operator A, then the following statements hold.*

(i) If $x \in \overline{D(A)}$, then $\lim_{t\to 0^+} S_\psi(t)x = x$,

(ii) For all $x \in X$ and $t \geq 0$,

$$\int_0^t \varepsilon'(\tau) S_\psi(\tau) x d\tau \in D(A),$$

$$A \int_0^t \varepsilon'(\tau) S_\psi(\tau) x d\tau = S_\psi(t)x - x.$$

Moreover, if $\tau \mapsto \varepsilon(\tau) A S_\psi(\tau)x$ is integrable on $(0,\epsilon)$ for $\epsilon > 0$, then for $t \geq 0$,

$$S_\psi(t)x - x = \int_0^t \varepsilon'(\tau) A S_\psi(\tau) x d\tau.$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then

$$\lim_{t\to 0^+} \frac{S_\psi(t)x - x}{t} = Ax.$$

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then

$$\lim_{t\to 0^+} A S_\psi(t)x = Ax.$$

Proof. To prove (i), from Theorem (2.2) (i), one has, $x \in \overline{D(A)}$, then $\lim_{t\to 0^+} S(t)x = x$. Consequently,

$$\lim_{t\to 0^+} S_\psi(t)x = \lim_{t\to 0^+} S(\varepsilon(t))x = \lim_{\varepsilon(t)\to 0^+} S(\varepsilon(t))x = x.$$

Next, to prove (ii), from Theorem 2.2 (ii), one has the following, for all $x \in D(A)$ and $t > 0$,

$$\int_0^t S(\tau) x d\tau \in D(A), \tag{3.2}$$

$$\int_0^t A S(\tau) x d\tau = S(t)x - x. \tag{3.3}$$

Consequently, let

$$\int_0^t \varepsilon'(\tau)S_\psi(\tau)xd\tau = \int_0^t \varepsilon'(\tau)S(\varepsilon(\tau))xd\tau = \int_0^{\varepsilon(t)} S(r)xdr.$$

From (3.2),

$$\int_0^t \varepsilon'(\tau)S_\psi(\tau)xd\tau = \int_0^{\varepsilon(t)} S(r)xdr \in D(A).$$

Then, from (3.3), one has,

$$A\int_0^t \varepsilon'(\tau)S_\psi(\tau)xd\tau = A\int_0^{\varepsilon(t)} S(r)xdr = S(\varepsilon(t))x - x = S_\psi(t)x - x.$$

Then, to prove (iii), from Theorem 2.2(iii), one has for $x \in D(A)$ and $Ax \in \overline{D(A)}$,

$$\lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax.$$

This implies,

$$\lim_{t\to 0^+} \frac{S_\psi(t)x - x}{t} = \lim_{t\to 0^+} \frac{S(\varepsilon(t))x - x}{\varepsilon(t)} = \lim_{\varepsilon(t)\to 0^+} \frac{S(\varepsilon(t))x - x}{\varepsilon(t)} = Ax.$$

Lastly, to prove (iv), from Theorem 3.1(ii), one has for $x \in D(A)$,

$$AS_\psi(t)x = S_\psi(t)Ax,$$

and from Theorem 3.2 (i), if $Ax \in \overline{D(A)}$, then

$$\lim_{t\to 0^+} AS_\psi(t)x = \lim_{t\to 0^+} S_\psi(t)Ax = Ax.$$

□

Theorem 3.3. *Let $A : D(A) \subseteq X \to X$ be a sectorial linear operator. For $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) > 0$,

$$R(\lambda; A) = \int_0^\infty e^{-\lambda\varepsilon(t)}\varepsilon'(t)S_\psi(t)dt. \tag{3.4}$$

Proof. Note that $S_\psi(t) = S(\varepsilon(t))$, one has

$$\int_0^\infty e^{-\lambda\varepsilon(t)}\varepsilon'(t)S_\psi(t)dt = \int_0^\infty e^{-\lambda\varepsilon(t)}\varepsilon'(t)S(\varepsilon(t))dt.$$

Then, by letting $\tau = \varepsilon(t)$ and from Theorem 2.3, one gets

$$\int_0^\infty e^{-\lambda\varepsilon(t)}\varepsilon'(t)S(\varepsilon(t))dt = \int_0^\infty e^{-\lambda\tau}S(\tau)d\tau = R(\lambda; A). \tag{3.5}$$

□

3.2. **Fractional Power of Sectorial Operator.** The following theorem provides the properties of semigroup associated with general conformable fractional derivative when a sectorial operator, A has fractional power β.

Theorem 3.4. *Let A be a sectorial linear operator generating the analytic general conformable semigroup, $S_\psi(t)$, then the following statements hold.*

(i) For $t > 0$ and $\beta \geq 0$, $S_\psi(t) : X \to D(A^\beta)$;

(ii) For $x \in D(A^\beta)$, $S_\psi(t)A^\beta x = A^\beta S_\psi(t)x$;

(iii) For $t > 0, A^\beta S_\psi(t)$ is bounded and

$$\left\|A^\beta S_\psi(t)\right\| \leq M_\beta(\varepsilon(t))^{-\beta};$$

(iv) For $0 < \beta \leq 1$ and $x \in D(A^\beta)$,

$$\left\|S_\psi(t)x - x\right\| \leq C_\beta(\varepsilon(t))^\beta \left\|A^\beta x\right\|.$$

Proof. Firstly, to prove (i), from Theorem 2.4 (i) and (ii), since

$$S_\psi(t) = S(\varepsilon(t)),$$

then (i) and (ii) are easy to show. Next, to prove (iii), from Theorem 2.4 (iii), one has

$$\left\|A^\beta S_\psi(t)\right\| = \left\|A^\beta S(\varepsilon(t))\right\| \leq M_\beta(\varepsilon(t))^{-\beta}.$$

Lastly, to prove (iv), from Theorem 2.4 (iv), one has

$$\left\|S_\psi(t)x - x\right\| = \left\|S(\varepsilon(t))x - x\right\| \leq C_\beta(\varepsilon(t))^\beta \left\|A^\beta x\right\|.$$

□

3.3. **Inhomogeneous General Conformable Fractional Cauchy Problem.** This section provides the solution of GCFCP.

Let us define Banach space, $L_\psi^{\alpha,p}((0,T];X)$ by

$$L_\psi^{\alpha,p}((0,T];X) = \{f : (0,T] \mapsto X : \int_0^T \left\|f(t)\right\|^p \varepsilon'(t)dt < +\infty\},$$

where $0 < \alpha < 1, p \geq 1$, with its norm

$$\left\|f\right\|_\psi = \int_0^T \left\|f(t)\right\|^p \varepsilon'(t)dt.$$

The following theorem shows that $u(t)$ is the solution if $f \in L_\psi^{\alpha,p}((0,T];X)$.

Theorem 3.5. *Let $u_0 \in X$ and $f \in L_\psi^{\alpha,1}((0,T];X)$. If $u : [0,T] \to X$ is a solution to the problem in equation 1.2, then*

$$u(t) = S_\psi(t)u_0 + \int_0^t \varepsilon'(p)S_\psi(\varepsilon^{-1}(\varepsilon(t) - \varepsilon(p)))f(p)dp. \tag{3.6}$$

Proof. Since $S_\psi(t)$ is the analytic conformable semigroup generated by sectorial operator A and $u(t)$ is the solution to the problem in equation (1.2), then □

$$v(\tau) = S_\psi\left(\varepsilon^{-1}(\varepsilon(t) - \varepsilon(\tau))\right) u(\tau). \tag{3.7}$$

Since $S_\psi(t) = S\left(\varepsilon(t)\right)$, then equation (3.7) becomes,

$$v(\tau) = S\left(\varepsilon(t) - \varepsilon(\tau)\right) u(\tau). \tag{3.8}$$

From Theorem 2.5 and by differentiating equation (3.8), one has

$$\begin{aligned} v'(\tau) &= -\varepsilon'(\tau) S'\left(\varepsilon(t) - \varepsilon(\tau)\right) u(\tau) + S\left(\varepsilon(t) - \varepsilon(\tau)\right) u'(\tau) \\ &= -\varepsilon'(\tau) S'\left(\varepsilon(t) - \varepsilon(\tau)\right) u(\tau) + S\left(\varepsilon(t) - \varepsilon(\tau)\right) \left[\varepsilon'(\tau) A u(\tau) + \varepsilon'(\tau) f(\tau)\right] \\ &= \varepsilon'(\tau) S\left(\varepsilon(t) - \varepsilon(\tau)\right) f(\tau). \end{aligned}$$

If $f \in L_\psi^{\alpha,1}((0,T];X)$, then $\varepsilon'(\tau)S\left(\varepsilon(t) - \varepsilon(\tau)\right) f(\tau)$ is integrable. By integrating both sides from 0 to t,

$$\begin{aligned} \int_0^t v'(\tau) d\tau &= \int_0^t \varepsilon'(\tau) S\left(\varepsilon(t) - \varepsilon(\tau)\right) f(\tau) d\tau \\ v(t) - v(0) &= \int_0^t \varepsilon'(\tau) S\left(\varepsilon(t) - \varepsilon(\tau)\right) f(\tau) d\tau \\ u(t) &= S(\varepsilon(t)) u_0 + \int_0^t \varepsilon'(\tau) S\left(\varepsilon(t) - \varepsilon(\tau)\right) f(\tau) d\tau. \end{aligned}$$

Therefore, it becomes

$$u(t) = S_\psi(t) u_0 + \int_0^t \varepsilon'(\tau) S_\psi\left(\varepsilon^{-1}(\varepsilon(t) - \varepsilon(\tau))\right) f(\tau) d\tau. \tag{3.9}$$

4. Conclusion

A non-homogeneous general conformable fractional Cauchy problem is investigated in this study. The problem is solved by using general conformable Laplace transform to obtain the solution operator of general conformable fractional Cauchy problem. This solution operator is then governed the analytic general conformable semigroup by employing the properties of sectorial operator. As a result, the proved local properties of the analytic general conformable semigroup are crucial to establish the existence and uniqueness of the solution of general conformable fractional Cauchy problem. Thus, these findings can be applied in solving mathematical models describing some real application phenomena such as spatio-temporal model of epidemic disease [45] and time-dependent model of cancer cell invasion [46].

Acknowledgments: This work was supported by Universiti Teknologi Malaysia, *Potential Academic Staff*, (ref no)PY/2024/02277/ (cost center no) Q.J130000.2754.04K17.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A New Definition of Fractional Derivative, J. Comput. Appl. Math. 264 (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002.

[2] A. Kilbas, H.M. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, (2006).

[3] T.C. Sideris, Ordinary Differential Equations and Dynamical Systems, Atlantis Press, 2013. https://doi.org/10.2991/978-94-6239-021-8.

[4] M. Meiler, R. Cordero–Soto, S.K. Suslov, Solution of the Cauchy Problem for a Time-Dependent Schrödinger Equation, J. Math. Phys. 49 (2008), 072102. https://doi.org/10.1063/1.2938698.

[5] T. Reeve, B. Tomas Johansson, The Method of Fundamental Solutions for a Time-Dependent Two-Dimensional Cauchy Heat Conduction Problem, Eng. Anal. Bound. Elem. 37 (2013), 569–578. https://doi.org/10.1016/j.enganabound.2012.12.008.

[6] L. Byszewski, Existence and Uniqueness of a Classical Solution to a Functional-differential Abstract Nonlocal Cauchy Problem, Int. J. Stoch. Anal. 12 (1998), 91–97. https://doi.org/10.1155/s1048953399000088.

[7] A. Raheem, D. Bahuguna, Existence and Uniqueness of a Solution for a Fractional Differential Equation by Rothe's Method, J. Nonlinear Evol. Equ. Appl. 4 (2013), 43–54.

[8] J. Peng, K. Li, A Novel Characteristic of Solution Operator for the Fractional Abstract Cauchy Problem, J. Math. Anal. Appl. 385 (2012), 786–796. https://doi.org/10.1016/j.jmaa.2011.07.009.

[9] R. Wang, D. Chen, T. Xiao, Abstract Fractional Cauchy Problems with Almost Sectorial Operators, J. Differ. Equ. 252 (2012), 202–235. https://doi.org/10.1016/j.jde.2011.08.048.

[10] Y. Li, H. Sun, Z. Feng, Fractional Abstract Cauchy Problem with Order $\alpha \in (1,2)$, Dyn. Partial. Differ. Equ. 13 (2016), 155–177. https://doi.org/10.4310/dpde.2016.v13.n2.a4.

[11] H. Fan, J. Mu, Initial Value Problem for Fractional Evolution Equations, Adv. Differ. Equ. 2012 (2012), 49. https://doi.org/10.1186/1687-1847-2012-49.

[12] K. Li, J. Peng, J. Gao, Nonlocal Fractional Semilinear Differential Equations in Separable Banach Spaces, Electron. J. Differ. Equ. 2013 (2013), 7.

[13] P. Chen, Y. Li, Existence of Mild Solutions for Fractional Evolution Equations with Mixed Monotone Nonlocal Conditions, Z. Angew. Math. Phys. 65 (2013), 711–728. https://doi.org/10.1007/s00033-013-0351-z.

[14] B.H. Guswanto, T. Suzuki, Existence and Uniqueness of Mild Solutions for Fractional Semilinear Differential Equations, Electron. J. Differ. Equ. 2015 (2015), 168.

[15] B.H. Guswanto, Fractional Nonlinear Evolution Equations With Sectorial Linear Operators, J. Fract. Calc. Appl. 10 (2019), 213–227.

[16] B.H. Guswanto, Nonlocal Reaction-Diffusion Model With Subdiffusive Kinetics, J. Fract. Calc. Appl. 13 (2022), 198–211.

[17] C. Li, D. Qian, Y. Chen, On Riemann-liouville and Caputo Derivatives, Discret. Dyn. Nat. Soc. 2011 (2011), 562494. https://doi.org/10.1155/2011/562494.

[18] D. Tverdyi, R. Parovik, Application of the Fractional Riccati Equation for Mathematical Modeling of Dynamic Processes with Saturation and Memory Effect, Fractal Fract. 6 (2022), 163. https://doi.org/10.3390/fractalfract6030163.

[19] O.T. BİRGANİ, S. CHANDOK, N. DEDOVİC, S. RADENOVİC, A Note on Some Recent Results of the Conformable Fractional Derivative, Adv. Theory Nonlinear Anal. Appl. 3 (2019), 11–17. https://doi.org/10.31197/atnaa.482525.

[20] D. Zhao, M. Luo, General Conformable Fractional Derivative and Its Physical Interpretation, Calcolo 54 (2017), 903–917. https://doi.org/10.1007/s10092-017-0213-8.

[21] L. Rabhi, M. Al Horani, R. Khalil, Inhomogeneous Conformable Abstract Cauchy Problem, Open Math. 19 (2021), 690–705. https://doi.org/10.1515/math-2021-0064.

[22] K. Hilal, A. Kajouni, N. Chefnaj, Existence of Solution for a Conformable Fractional Cauchy Problem with Nonlocal Condition, Int. J. Differ. Equ. 2022 (2022), 6468278. https://doi.org/10.1155/2022/6468278.

[23] S. Al-Sharif, R. Al-Shorman, New Results on the Generator of Conformable Semigroups, Progr. Fract. Differ. Appl. 8 (2022), 77–89. https://doi.org/10.18576/pfda/080105.

[24] B.H. Guswanto, S. Maryani, N. Istikaanah, Analytic Conformable Semigroups and Regularity of Solutions to Conformable Fractional Cauchy Problem, Int. J. Anal. Appl. 23 (2025), 117. https://doi.org/10.28924/2291-8639-23-2025-117.

[25] A.T. Deresse, Analytical Solution of One-Dimensional Nonlinear Conformable Fractional Telegraph Equation by Reduced Differential Transform Method, Adv. Math. Phys. 2022 (2022), 7192231. https://doi.org/10.1155/2022/7192231.

[26] Y. Alhojilan, H.M. Ahmed, New Results Concerning Approximate Controllability of Conformable Fractional Noninstantaneous Impulsive Stochastic Evolution Equations via Poisson Jumps, Mathematics 11 (2023), 1093. https://doi.org/10.3390/math11051093.

[27] B.H. Guswanto, A. Marfungah, D.O. Yuniarto, M.I. Jihad, S. Maryani, N. Istikaanah, et al. From Continuous Time Random Walks to Multidimensional Conformable Diffusion Equation, IAENG Int. J. Appl. Math. 54 (2024), 1445–1458.

[28] U.N. Katugampola, A New Fractional Derivative with Classical Properties, arXiv:1410.6535 (2014). http://arxiv.org/abs/1410.6535v2.

[29] P.M. Guzman, G. Langton, L.M.L.M. Bittencurt, J. Medina, J.E.N. Valdes, A New Definition of a Fractional Derivative of Local Type, J. Math. Anal. 9 (2018), 88–98.

[30] S. Al-Sharif, A. Malkawi, Modification of Conformable Fractional Derivative With Classical Properties, Ital. J. Pure Appl. Math. 44 (2020), 30–39.

[31] A. Kajouni, A. Chafiki, K. Hilal, M. Oukessou, A New Conformable Fractional Derivative and Applications, Int. J. Differ. Equ. 2021 (2021), 6245435. https://doi.org/10.1155/2021/6245435.

[32] P. Guzman, L. Lugo, J. Nápoles Valdés, M. Vivas-Cortez, On a New Generalized Integral Operator and Certain Operating Properties, Axioms 9 (2020), 69. https://doi.org/10.3390/axioms9020069.

[33] O. Kahouli, A. Jmal, O. Naifar, A.M. Nagy, A. Ben Makhlouf, New Result for the Analysis of Katugampola Fractional-Order Systems—application to Identification Problems, Mathematics 10 (2022), 1814. https://doi.org/10.3390/math10111814.

[34] D.R. Anderson, D.J. Ulness, Properties of the Katugampola Fractional Derivative with Potential Application in Quantum Mechanics, J. Math. Phys. 56 (2015), 063502. https://doi.org/10.1063/1.4922018.

[35] F. Martínez, P.O. Mohammed, J.E.N. Valdés, Non-conformable Fractional Laplace Transform, Kragujev. J. Math. 46 (2022), 341–354. https://doi.org/10.46793/kgjmat2203.341m.

[36] A. Fleitas, J.A. Méndez-Bermúdez, J. Nápoles Valdés, J.S. Almira, On Fractional Liénard-Type Systems, Rev. Mex. Física 65 (2019), 618–625.

[37] A.M. Moter, Fractional Differential Equations and Fractional Semigroups of Operators, Int. J. Innov. Sci. Res. Rev. 6 (2024), 6250–6253.

[38] M. Al-Khaleel, S. Al-Sharif, A. AlJarrah, Atomic Solution for Certain Gardner Equation, Symmetry 15 (2023), 440. https://doi.org/10.3390/sym15020440.

[39] N. Benkhettou, A. Salim, J.E. Lazreg, M. Benchohra, A Study on Some New Conformable Fractional Differential Equations With Retardation and Anticipation, Mem. Differ. Equ. Math. Phys. 92 (2024), 41–58.

[40] A. Jhangeer, W.A. Faridi, M. Alshehri, Soliton Wave Profiles and Dynamical Analysis of Fractional Ivancevic Option Pricing Model, Sci. Rep. 14 (2024), 23804. https://doi.org/10.1038/s41598-024-74770-1.

[41] A. Aitbrahim, J.E. Ghordaf, A. El Hajaji, K. Hilal, J.E. Valdes, A Comparative Analysis of Conformable, Non-Conformable, Riemann-Liouville, and Caputo Fractional Derivatives, Eur. J. Pure Appl. Math. 17 (2024), 1842–1854. https://doi.org/10.29020/nybg.ejpam.v17i3.5237.

[42] D. Anderson, E. Camrud, D. Ulness, On the Nature of the Conformable Derivative and Its Applications to Physics, J. Fract. Calc. Appl. 10 (2019), 92–135.

[43] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, 2012.

[44] A. Ansari, The Generalized Laplace Transform and Fractional Differential Equations of Distributed Orders, Differ. Equ. Control Process. 3 (2012), 128–137.

[45] A.V.R. Amaral, J.A. González, P. Moraga, Spatio-temporal Modeling of Infectious Diseases by Integrating Compartment and Point Process Models, Stoch. Environ. Res. Risk Assess. 37 (2022), 1519–1533. https://doi.org/10.1007/s00477-022-02354-4.

[46] A. Padder, L. Almutairi, S. Qureshi, A. Soomro, A. Afroz, E. Hincal, A. Tassaddiq, Dynamical Analysis of Generalized Tumor Model with Caputo Fractional-Order Derivative, Fractal Fract. 7 (2023), 258. https://doi.org/10.3390/fractalfract7030258.