Int. J. Anal. Appl. (2025), 23:194

International Journal of Analysis and Applications

Stability of Quartic Functional Equation in Non-Archimedean IFN-Spaces

Kandhasamy Tamilvanan[1,2,*], Siriluk Donganont[3,*], Balaanandhan Radhakrishnan[4], John Michael Rassias[5]

[1]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Tandalam, Chennai - 602 105, Tamil Nadu, India*

[2]*Department of Mathematics, Faculty of Science & Humanities, R.M.K. Engineering College, Kavaraipettai, Tiruvallur 601 206, Tamil Nadu, India*

[3]*School of Science, University of Phayao Phayao 56000, Thailand*

[4]*Department of Mathematics, Sri Sankara Arts and Science College (Autonomous), Enathur-631 561, Kanchipuram, Tamil Nadu, India*

[5]*Pedagogical Department - Mathematics & Informatics, The National and Kapodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Attikis 15342, Greece*

[*]*Corresponding authors:* tamiltamilk7@gmail.com, siriluk.pa@up.ac.th

Abstract. In this work, we focus on the non-Archimedean intuitionistic fuzzy normed framework, specifically on the generalized Ulam stability of quartic functional equations. By combining direct approaches with advanced fixed-point techniques, we prove that quartic-type mappings exist, are unique, and stable, providing strong extensions of Hyers-Ulam-Rassias stability. We present a new method for studying stability phenomena in abstract nonlinear systems and fulfill a gap between fuzzy analysis and non-Archimedean normed structures. Future applications in computational mathematics, fuzzy modeling, and uncertain systems analysis will benefit from these insights, which strengthen the theoretical framework.

1. Introduction

The study of stability in functional equations has garnered significant attention since the pioneering problem posed by Ulam [32] in 1940, which inquired whether approximate homomorphisms between groups could be approximated by exact homomorphisms. This question was affirmatively answered by Hyers [33] for Banach spaces, leading to what is now referred to as Hyers-Ulam

Received: Jun. 3, 2025.

2020 *Mathematics Subject Classification.* 39B52, 39B82, 46H25.

Key words and phrases. quartic functional equation; Ulam stability; non-Archimedean intuitionistic fuzzy normed spaces; fixed-point method.

https://doi.org/10.28924/2291-8639-23-2025-194

ISSN: 2291-8639

stability. Rassias [26] later generalized this concept by allowing the deviation from exactness to be unbounded, thus initiating a broader study known as Hyers-Ulam-Rassias stability theory. These foundational works established the groundwork for various stability results across different types of functional equations. Among the important equations studied, the quartic functional equation, which naturally arises in several contexts including approximation theory, optimization, and theoretical physics, has received particular attention. The advent of fuzzy set theory by Zadeh [35] revolutionized the mathematical modeling of uncertainty and imprecision. Building on this foundation, Atanassov [36] introduced the concept of intuitionistic fuzzy sets, characterized by a membership function, a non-membership function, and a hesitation margin, providing a more flexible and realistic framework for addressing uncertainties than traditional fuzzy sets.

The development of intuitionistic fuzzy normed spaces (IFN-spaces), which combine normed linear structures with intuitionistic fuzziness, has enabled the extension of classical analytical and algebraic theories into the realm of fuzzy logic. In recent years, the concept of non-Archimedean spaces has gained relevance in analysis due to their strong triangle inequality, which allows for alternative approaches to continuity and convergence (see [5–7]). The Intuitionistic fuzzy normed spaces (IFN-spaces), which blend normed linear structures with intuitionistic fuzziness, has enabled the extension of classical analytical and algebraic theories into the domain of fuzzy logic. Saadati and Park [37] extended topological and analytical concepts into intuitionistic fuzzy settings, providing the tools necessary to investigate continuity, convergence, and compactness within these spaces.

A central question in the theory of functional equations is whether a function that approximately satisfies a given functional relation must be close to an exact solution. This inquiry was first posed by Ulam in 1940 [28]. In response, Hyers [9] addressed the problem in 1941 by studying mappings in Banach spaces that meet the criteria of Hyers stability under a fixed constant. Later, Aoki [1] extended this result to a broader setting by incorporating sums of powers of norms. Rassias [25] further advanced this line of study in 1978 by introducing a generalization of the Hyers theorem that permitted an unbounded Cauchy difference. Since then, many researchers have contributed to the generalization and extension of stability concepts for various functional equations (see, for example, [2, 4, 8, 14, 22, 27]).

Motivated by these developments, this study aims to investigate the stability of a quartic functional equation in non-Archimedean intuitionistic fuzzy normed spaces. The unique interplay between the quartic functional form and the non-Archimedean fuzzy structure creates a rich and nuanced framework for examining the existence, uniqueness, and stability behavior of solutions. Our results not only extend the classical theory of functional equations but also contribute to the expanding body of literature on fuzzy and non-Archimedean analysis.stability of a quartic functional equation in non-Archimedean intuitionistic fuzzy normed spaces. The unique interplay between the quartic functional form and the non-Archimedean fuzzy structure provides a rich and nuanced framework in which to examine the existence, uniqueness, and stability behavior of solutions. Our results not only extend the classical theory of functional equations but also contribute to the growing body of literature on fuzzy and non-Archimedean analysis.

Since then, many researchers have expanded these concepts to various types of functional equations, including additive, quadratic, cubic, and quartic equations. Among these, quartic functional equations closely related to polynomial functions of degree four have garnered significant attention due to their mathematical structure and applications in modeling physical phenomena and engineering problems. Specifically, the general quartic functional equation is expressed as follows:

In this work, the authors examine the Ulam stability results of the quartic functional equation

$$\begin{aligned}\phi\left(\sum_{i=1}^{n} x_i\right) &= \textstyle\sum_{1\leq i<j<k<l\leq n} \phi(x_i+x_j+x_k+x_l) + (-n+4)\sum_{1\leq i<j<k\leq n} \phi(x_i+x_j+x_k) \\ &\quad + \left(\tfrac{n^2-7n+12}{2}\right)\textstyle\sum_{1=i;i\neq j}^{n} \phi(x_i+x_j) - \sum_{i=1}^{n} \phi(2x_i) \\ &\quad + \left(\tfrac{-n^3+9n^2-26n+120}{6}\right)\textstyle\sum_{i=1}^{n}\left(\tfrac{\phi(x_i)+\phi(-x_i)}{2}\right) \qquad (1.1)\end{aligned}$$

where $\phi(0) = 0$, and n is a nonnegative integer with $n > 4$ in non-Archimedean IFN spaces (briefly, non-Archimedean IFN spaces) over a field by using direct and fixed-point techniques.

2. Preliminaries

We can refer to some needed preliminaries in [15,17,21,30], and using the alternative fixed point theorem which some important results in fixed point theory.

Definition 2.1. *Let E be a linear space over $\mathbb{K}$ with $|\cdot|$. A mapping $\|\cdot\| : E \to [0,\infty)$ is known as a non-Archimedean norm if it satisfies:*

(i) $\|v\| = 0$ *if and only if* $v = 0$;

(ii) $\|rv\| = |r|\|v\|$, *for every* $v \in E$ *and* $r \in \mathbb{K}$;

(iii) *the strong triangle inequality*

$$\|x_1 + x_2\| \leq \max\{\|x_1\|, \|x_2\|\}, \textit{for every } x_1, x_2 \in E.$$

Then $(E, \|\cdot\|)$ is known as a non-Archimedean normed space.

Every Cauchy sequence converges in a complete non-Archimedean normed space, which we call a complete non-Archimedean normed space.

Proposition 2.1. (1) *If $\mathcal{T} = \mathcal{T}_L$ or $\mathcal{T} = \mathcal{T}_P$, then*

$$\lim_{m\to\infty} \mathcal{T}_{i=1}^{\infty} x_{m+i} = 1 \iff \sum_{i=1}^{\infty}(1-x_i) < \infty.$$

(2) *It $\mathcal{T}$ is of Hadzic-type, then*

$$\lim_{m\to\infty} \mathcal{T}_{i=m}^{\infty} x_i = \lim_{m\to\infty} \mathcal{T}_{i=1}^{\infty} x_{m+i} = 1$$

for all $\{x_i\}_{i\in\mathbb{N}}$ in $[0,1]$ such that $\lim_{i\to\infty} x_i = 1$.

Definition 2.2. *Let membership degree τ and non-membership degree θ of an intuitionistic fuzzy set from $E \times (0, +\infty)$ to $[0, 1]$ satisfies $\tau_v(t) + \theta_v(t) \leq 1$ for all $v \in E$ and all $t > 0$. The triple $(E, N_{\tau,\theta}, T)$ is called as an non-Archimedean intuitionistic fuzzy Menger norm if a vector space E, a continuous t-representable $\mathcal{T}$ and $N_{\tau,\theta} : E \times (0, +\infty) \to L^*$ satisfying: for all $x_1, x_2 \in E$, $s, t > 0$,*

(IFN1) $N_{\tau,\theta}(x_1, t) = 0$ *for all* $t \leq 0$;

(IFN2) $x_1 = 0 \Leftrightarrow N_{\tau,\theta}(x_1, t) = 1,\ t > 0$;

(IFN3) $N_{\tau,\theta}(\alpha x_1, t) = N_{\tau,\theta}(x_1, \frac{t}{|\alpha|})$ *for all* $\alpha \neq 0$;

(IFN4) $N_{\tau,\theta}(x_1 + x_2, \max\{s, t\}) \geq \mathcal{T}(N_{\tau,\theta}(x_1, s), N_{\tau,\theta}(x_2, t))$.

(IFN5) $\lim_{t\to\infty} N_{\tau,\theta}(x_1, t) = 1$.

If $N_{\tau,\theta}$ is a non-Archimedean intuitionistic fuzzy Menger norm on E, then $(E, N_{\tau,\theta}, T)$ is said to be a non-Archimedean IFN - space. It is important to note that the condition (IFN4) implies

$$N_{\tau,\theta}(x_1, t) \geq \mathcal{T}(N_{\tau,\theta}(0, t), N_{\tau,\theta}(x_1, s)) = N_{\tau,\theta}(x_1, s),$$

for all $0 < s < t$ and $x_1, x_2 \in E$. i.e., $(N_{\tau,\theta}, \cdot)$ is increasing for every x_1, which gives

$$N_{\tau,\theta}(x_1, s + t) \geq N_{\tau,\theta}(x_1, \max\{s, t\}).$$

If (IFN4) holds, then

$$(IFN6)\ N_{\tau,\theta}(x_1 + x_2, s + t) \geq \mathcal{T}(N_{\tau,\theta}(x_1, s), N_{\tau,\theta}(x_2, t)).$$

We frequently employ that

$$N(-x_1, t) = N(x_1, t),\ x_1 \in E,\ t > 0,$$

which is derived from (IFN3). We should also remark that the Definition 2.2 of a non-Archimedean Menger norm is more broad than definition in [20, 29], which only considers fields with $|\cdot|$.

Definition 2.3. *Let a non-Archimedean IFN-space $(E, N_{\tau,\theta}, T)$ and $\{x_n\}_{m\in\mathbb{N}}$ in E. Then $\{x_n\}_{m\in\mathbb{N}}$ is called as convergent if there is $v \in E$ satisfies*

$$\lim_{m\to\infty} N_{\tau,\theta}(x_n - v, t) = 1$$

for every $t > 0$.

Here, v is said to be limit of $\{x_n\}_{m\in\mathbb{N}}$ and we refer to it as

$$\lim_{m\to\infty} x_n = v.$$

The sequence $\{x_n\}_{m\in\mathbb{N}}$ in E is called a Cauchy sequence if

$$\lim_{m\to\infty} N_{\tau,\theta}(x_{m+i} - x_n, t) = 1$$

for every $t > 0$ and $i = 1, 2, 3, \cdots$.

A complete non-Archimedean IFN-space is defined as one in which every Cauchy sequence in E is convergent.

Example 2.1. *[12] Let $(E, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(u,v) = (u, v, \min(u_2 + x_2, 1))$ for all $u = (u_1, u_2)$, $v = (x_1, x_2) \in L^*$ and let τ, θ be membership and non-membership degree of an intuitionistic fuzzy set defined by*

$$N_{\tau,\theta}(v,t) = \left(\frac{t}{t + \|v\|}, \frac{\|v\|}{t + \|v\|}\right), \ \forall\, t \in R^+.$$

Then the triple $(E, N_{\tau,\theta}, T)$ is an IFN-space.

For specific later use, we note the subsequent Diaz and Margolis [3] results.

Theorem 2.1. *Let (W, d) be a generalized complete metric space and a strictly contractive function $M : W \to W$ with Lipschitz constant $L < 1$. Then, for every $x_1 \in W$, either*

$$d\Big(M^m x_1, M^{m+1} x_1\Big) = \infty, \ \ m \geq m_0;$$

or there exists a positive integer m_0 such that

(i) $d\Big(M^m x_1, M^{m+1} x_1\Big) < \infty, \ \ m \geq m_0;$
(ii) *the sequence $\{M^m x_1\}_{m\in\mathbb{N}}$ converges to a fixed point x_1^* of M;*
(iii) *x_1^* is the unique fixed point of M in $W^* = \{x_2 \in W | d(M^{m_0} x_1, x_2) < \infty\}$;*
(iv) *$d(x_2, x_2^*) \leq \frac{1}{1-L} d(Mx_2, x_2)$, for every $x_2 \in W^*$.*

Throughout all the sections, we consider $\mathbb{K}$ as a valued field, E and F are vector spaces over $\mathbb{K}$ and $(F, N_{\tau,\theta}, T)$ is a complete non-Archimedean IFN space over $\mathbb{K}$. For our notational simplicity, we can define the mapping $\phi : E \to F$ by

$$\begin{aligned} D\phi(x_1, x_2, \cdots, x_n) \ &= \phi\Bigg(\sum_{i=1}^n x_i\Bigg) - \sum_{1\leq i<j<k<l\leq n} \phi(x_i + x_j + x_k + x_l) \\ &\quad -(-n+4) \sum_{1\leq i<j<k\leq n} \phi(x_i + x_j + x_k) \\ &\quad -\left(\frac{n^2-7n+12}{2}\right) \sum_{1=i;i\neq j}^n \phi(x_i + x_j) + \sum_{i=1}^n \phi(2x_i) \\ &\quad -\left(\frac{-n^3+9n^2-26n+120}{6}\right) \sum_{i=1}^n \left(\frac{\phi(x_i)+\phi(-x_i)}{2}\right), \end{aligned} \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in E$.

3. Ulam Stability: Direct Approach

Theorem 3.1. *Suppose that a mapping $\Psi_{\tau,\theta} : E^n \times [0, \infty) \to [0, 1]$ such that*

$$\lim_{i\to\infty} \Psi_{\tau,\theta}(2^i x_1, 2^i x_2, \cdots, 2^i x_n, |2|^{4i} t) = 1 \tag{3.1}$$

and

$$\begin{aligned} \lim_{l\to\infty} \mathcal{T}_{i=l}^{\infty} \ \mathcal{T} \ \ &(\Psi_{\tau,\theta}(2^i x, 0, \cdots, 0, |2|^{4i+1} t)) \\ &= \lim_{l\to\infty} \mathcal{T}_{i=1}^{\infty} \mathcal{T}(\Psi_{\tau,\theta}(2^{l+i-1} x, 0, \cdots, 0, |2|^{4i+2l-1} t)) \\ &= 1 \end{aligned} \tag{3.2}$$

for all $x_1, x_2, \cdots, x_n \in E$ *and* $t > 0$. *If an even mapping* $\phi : E \to F$ *is defined by* (2.1), *which satisfying*

$$\phi(0) = 0, \tag{3.3}$$

and

$$N_{\tau,\theta}\left(D\phi(x_1, x_2, \cdots, x_n), t\right) \geq \Psi_{\tau,\theta}(x_1, x_2, \cdots, x_n, t) \tag{3.4}$$

for all $x_1, x_2, \cdots, x_n \in E$ *and all* $t \in [0, \infty)$, *then there exists a unique quartic mapping* $Q_4 : E \to F$ *satisfying*

$$N_{\tau,\theta}\left(\phi(x) - Q_4(x), t\right) \geq \mathcal{T}_{i=1}^{\infty}\mathcal{T}\left(\Psi_{\tau,\theta}(2^{i-1}x, 0, \cdots, 0, |2|^{4i-1}t)\right), \tag{3.5}$$

for all $x \in E$ *and all* $t > 0$.

Proof. Fix for every $x \in E$ and every $t > 0$. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, 0, \cdots, 0)$ in (3.4), we arrive

$$N_{\tau,\theta}\left(\phi(2x) - 2^4\phi(x), t\right) \geq \Psi_{\tau,\theta}(x, 0, \cdots, 0, t). \tag{3.6}$$

From inequality (3.11), we get

$$N_{\tau,\theta}\left(\frac{1}{2^4}\phi(2x) - \phi(x), t\right) \geq T\left(\Psi_{\tau,\theta}(x, 0, \cdots, 0, |2|^4 t)\right). \tag{3.7}$$

Therefore, one can get

$$N_{\tau,\theta}\left(\frac{1}{2^{4(l+m)}}\phi(2^{l+m}x) - \frac{1}{2^{4l}}\phi(2^l x), t\right) \geq \mathcal{T}_{i=l}^{l+m-1}\mathcal{T}\left(\Psi_{\tau,\theta}(2^i x, 0, \cdots, 0, |2|^{4i+1}t)\right),$$

and thus from (3.2), it follows that the sequence $\left\{\frac{\phi(2^i x)}{2^{4i}}\right\}_{i\in\mathbb{N}}$ is a Cauchy sequence in a complete non-Archimedean IFN space.

Thus, we can define a mapping $Q_4 : E \to F$ by

$$\lim_{i\to\infty} N_{\tau,\theta}\left(\frac{1}{2^{4i}}\phi(2^i x) - Q_4(x), t\right) = 1.$$

Next, for every $l \in \mathbb{N}$ with $l \geq 1$, we obtain

$$\begin{aligned} N_{\tau,\theta}\left(\phi(x) - \frac{1}{2^{4l}}\phi(2^l x), t\right) &\geq \mathcal{T}_{i=1}^{l} N_{\tau,\theta}\left(\frac{1}{2^{4(i-1)}}\phi(2^{i-1}x) - \frac{1}{2^{4i}}\phi(2^i x), t\right) \\ &\geq \mathcal{T}_{i=1}^{l}\mathcal{T}\left(\Psi_{\tau,\theta}(2^{i-1}x, 0, \cdots, 0, |2|^{4(i-1)}t)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} N_{\tau,\theta}\left(\phi(x) - Q_4(x), t\right) &\geq \mathcal{T}\left(N_{\tau,\theta}\left(\phi(x) - \frac{1}{2^{4l}}\phi(2^l x), t\right), N_{\tau,\theta}\left(\frac{1}{2^{4l}}\phi(2^l x) - Q_4(x), t\right)\right) \\ &\geq \mathcal{T}\left(\mathcal{T}_{i=1}^{l} T\left(\Psi_{\tau,\theta}\left(2^{i-1}x, 0, \cdots, 0, |2|^{4i-1}t\right)\right)\right), \\ &\qquad N_{\tau,\theta}\left(2^{-4l}\phi\left(2^l x\right) - Q_4\left(x\right), t\right)\Big). \end{aligned}$$

Taking the limit $l \to \infty$ in the above inequality, we obtain (3.5). Hence, the mapping Q_4 is quartic. Consider an another quartic function $Q_4^{'} : E \to F$ satisfying (3.5). Hence, by $\phi(2x) = 2^4\phi(x)$ and (3.2), (3.5), it follows that

$$\begin{aligned} N_{\tau,\theta}\left(Q_4\left(x\right) - Q_4^{'}\left(x\right), t\right) &= N_{\tau,\theta}\left(Q_4\left(2^l x\right) - Q_4^{'}\left(2^l x\right), |2|^{4l+2i-1}t\right) \\ &\geq \mathcal{T}\Big(\mathcal{T}_{i=1}^{\infty}\mathcal{T}\Big(\Psi_{\tau,\theta}\Big(2^{l+i-1}x, 0, \cdots, 0, |2|^{4l+2i-1}t\Big), \\ &\quad \mathcal{T}_{i=1}^{\infty}\mathcal{T}\Big(\Psi_{\tau,\theta}\Big(2^{l+i-1}x, 0, \cdots, 0, |2|^{4l+2i-1}t\Big), \\ &\quad \Psi_{\tau,\theta}\Big(2^{l+i-1}x, 0, \cdots, 0, |2|^{4l+2i}t\Big)\Big)\Big) \\ &\to 1 \text{ (as } l \to \infty), \end{aligned}$$

and therefore, $Q_4 = Q_4^{'}$. This ends the proof. □

Theorem 3.2. *Suppose that a mapping $\Psi_{\tau,\theta} : E^n \times [0,\infty) \to [0,1]$ such that*

$$\lim_{i\to\infty} \Psi_{\tau,\theta}(2^{-i}x_1, 2^{-i}x_2, \cdots, 2^{-i}x_n, |2|^{-4i}t) = 1 \tag{3.8}$$

and

$$\begin{aligned} \lim_{l\to\infty} \mathcal{T}_{i=l}^{\infty} \quad \mathcal{T} &\quad (\Psi_{\tau,\theta}(2^{-i-1}x, 0, \cdots, 0, |2|^{-4i-1}t) && (3.9) \\ &= \lim_{l\to\infty} \mathcal{T}_{i=1}^{\infty}\mathcal{T}(\Psi_{\tau,\theta}(2^{-l-i-1}x, 0, \cdots, 0, |2|^{-4i-2l-1}t)) \\ &= 1 && (3.10) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in E$ and $t > 0$. If an even mapping $\phi : E \to F$ satisfying (3.3) *and* (3.4)*, then there exists a unique quartic mapping $Q_4 : E \to F$ satisfying*

$$N_{\tau,\theta}\left(\phi(x) - Q_4(x), t\right) \geq \mathcal{T}_{i=1}^{\infty}\mathcal{T}\left(\Psi_{\tau,\theta}(2^{-i-1}x, 0, \cdots, 0), |2|^{-4i-1}t)\right),$$

for all $x \in E$ and all $t > 0$.

Proof. Fix for every $x \in E$ and every $t > 0$. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, 0, \cdots, 0)$ in (3.4), we obtain

$$N_{\tau,\theta}\left(\phi(2x) - 2^4\phi(x), t\right) \geq \Psi_{\tau,\theta}(x, 0, \cdots, 0, t). \tag{3.11}$$

Replacing x by $\frac{x}{2}$ in (3.11), we get

$$N_{\tau,\theta}\left(\phi\left(x\right) - 2^4\phi\left(\frac{x}{2}\right), t\right) \geq \Psi_{\tau,\theta}(x, 0, \cdots, 0, t). \tag{3.12}$$

Therefore, one can get

$$N_{\tau,\theta}\left(\frac{1}{2^{-4(l+m)}}\phi(2^{-(l+m)}x) - \frac{1}{2^{-4l}}\phi(2^{-l}x), t\right) \geq \mathcal{T}_{i=l}^{l+m}\mathcal{T}\left(\Psi_{\tau,\theta}(2^{-i}x, 0, \cdots, 0, |2|^{-4i}t)\right),$$

and thus from (3.2), it follows that the sequence $\left\{\frac{\phi(2^{-i}x)}{2^{-4i}}\right\}_{i\in\mathbb{N}}$ is a Cauchy sequence in a complete non-Archimedean IFN space.

Thus, we can define a mapping $Q_4 : E \to F$ by

$$\lim_{i\to\infty} N_{\tau,\theta}\left(\frac{1}{2^{-4i}}\phi(2^{-i}x) - Q_4(x), t\right) = 1.$$

Next, for every $l \in \mathbb{N}$ with $l \geq 1$, we obtain

$$\begin{aligned} N_{\tau,\theta}\left(\phi(x) - \frac{1}{2^{-4l}}\phi(2^{-l}x), t\right) &\geq \mathcal{T}_{i=1}^{l} N_{\tau,\theta}\left(\frac{1}{2^{-4(i-1)}}\phi(2^{-i-1}x) - \frac{1}{2^{-4i}}\phi(2^{-i}x), t\right) \\ &\geq \mathcal{T}_{i=1}^{l}\mathcal{T}\left(\Psi_{\tau,\theta}(2^{-i-1}x, 0, \cdots, 0, |2|^{-4(i-1)}t)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} N_{\tau,\theta}\left(\phi(x) - Q_4(x), t\right) &\geq \mathcal{T}\left(N_{\tau,\theta}\left(\phi(x) - \frac{1}{2^{-4l}}\phi(2^{-l}x), t\right), N_{\tau,\theta}\left(\frac{1}{2^{-4l}}\phi(2^{-l}x) - Q_4(x), t\right)\right) \\ &\geq \mathcal{T}\left(\mathcal{T}_{i=1}^{l} T\left(\Psi_{\tau,\theta}\left(2^{-i-1}x, 0, \cdots, 0, |2|^{-4i-1}t\right)\right)\right), \\ &\qquad N_{\tau,\theta}\left(2^{-4l}\phi\left(2^{-l}x\right) - Q_4\left(x\right), t\right)\Big). \end{aligned}$$

Taking the limit $l \to \infty$ in the above inequality, we arrive (3.5). Hence, the mapping Q_4 is quartic. Consider an another quartic function $Q_4^{'} : E \to F$ satisfying (3.5). Hence, by $\phi\left(\frac{x}{2}\right) = \frac{1}{2^4}\phi(x)$ and (3.2), (3.5), it follows that

$$\begin{aligned} N_{\tau,\theta}\left(Q_4\left(x\right) - Q_4^{'}\left(x\right), t\right) &= N_{\tau,\theta}\left(Q_4\left(2^{-l}x\right) - Q_4^{'}\left(2^{-l}x\right), |2|^{-4l-2i}t\right) \\ &\geq \mathcal{T}\Big(\mathcal{T}_{i=1}^{\infty}\mathcal{T}\Big(\Psi_{\tau,\theta}\Big(2^{-l-i-1}x, 0, \cdots, 0, |2|^{-4l-2i}t\Big), \\ &\qquad \mathcal{T}_{i=1}^{\infty}\mathcal{T}\Big(\Psi_{\tau,\theta}\Big(2^{-l-i}x, 0, \cdots, 0, |2|^{-4l-2i}t\Big), \\ &\qquad \Psi_{\tau,\theta}\Big(2^{-l-i}x, 0, \cdots, 0, |2|^{-4l-2i}t\Big)\Big)\Big)\Big) \\ &\to 1 \text{ (as } l \to \infty), \end{aligned}$$

and therefore, $Q_4 = Q_4^{'}$. This ends the proof. □

4. Ulam Stability: Fixed-Point Approach

Theorem 4.1. *Let $\Psi_{\tau,\theta} : E^n \times [0, \infty) \to [0, 1]$ be a mapping, which satisfies* (3.1) *and such that*

$$\Psi_{\tau,\theta}\left(2x_1, 2x_2, \cdots, 2x_n, |2|^4 Lt\right) \geq \Psi_{\tau,\theta}\left(x_1, x_2, \cdots, x_n, t\right), \tag{4.1}$$

for all $x_1, x_2, \cdots, x_n \in E$ and $L \in (0, 1)$. If an even mapping $\phi : E \to F$ satisfying (3.3) *and* (3.4)*, then there exists a unique quartic mapping $Q_4 : E \to F$ satisfying*

$$N_{\tau,\theta}\left(\phi\left(x\right) - Q_4\left(x\right), t\right) \geq T\left(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, |2|^4(1-L)t\right)\right), \tag{4.2}$$

for all $x \in E$ and all $t > 0$.

Proof. Defining the set $W := \{p : E \to F\}$ and introducing the generalized metric m on W:

$$\begin{aligned} m(p, q) &= \inf\{c \in [0, \infty] : N_{\tau,\theta}\left(p\left(x\right) - q\left(x\right), t\right) \\ &\geq \mathcal{T}\left(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, |2|^4 t\right)\right),\ x \in E, t > 0\} \end{aligned}$$

for all $p, q \in W$. A standard verification (see for instance [11]) proves that (W, m) is a complete generalized metric space. Now, we can define a mapping $S : W \to W$ by

$$Sp\,(x) = \frac{1}{2^4} p\,(2x)\,,$$

for all $p \in W$ and all $x \in E$. Let $p, q \in W$ and $c_{p,q} \in [0, \infty]$ with $m(p, q) \leq c_{p,q}$. Then,

$$N_{\tau,\theta}\left(p\,(x) - q\,(x)\,, c_{p,q}t\right) \;\geq\; \mathcal{T}\Big(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, |2|^4 t\right)\Big),$$

which together with (4.1) gives

$$N_{\tau,\theta}\left(Sp\,(x) - Sq\,(x)\,, t\right) \;\geq \mathcal{T}\left(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, \tfrac{|2|^4 t}{Lc_{p,q}}\right)\right)$$

and consequently, $m(Sp, Sq) \leq Lc_{p,q}$, this indicates that S is strictly contractive. In addition, it follows from (3.12) that

$$N_{\tau,\theta}\left(S\phi\,(x) - \phi\,(x)\,, t\right) \geq \mathcal{T}\Big(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, |2|^4 t\right)\Big)$$

and thus, $m(S\phi, \phi) \leq 1 < \infty$. Thus, by Theorem 2.1, S has a unique fixed-point $Q_4 : E \to F$ in the set $W^* = \{p \in W : m(\phi, p) < \infty\}$ such that

$$\frac{1}{2^4} Q_4\,(2x) = Q_4\,(x) \tag{4.3}$$

and

$$Q_4\,(x) = \lim_{i \to \infty} \frac{1}{2^{4i}} \phi\left(2^i x\right),\; x \in E.$$

In addition, the fact that $\phi \in W^*$, Theorem 2.1, and $m(S\phi, \phi) \leq 1$, we have

$$m(\phi, Q_4) \leq \frac{1}{1 - L} m\,(S\phi, \phi) \leq \frac{1}{1 - L} \tag{4.4}$$

and (4.2) follows. The proof of Theorem 3.1 may also be used to show that the function Q_4 is quartic.

At the end, consider that $Q_4^{'} : E \to F$ is an another quartic mapping which satisfying (4.2). Then, $Q_4^{'}$ satisfies (4.3). Therefore, it is a fixed point of S.

Thus, by (4.2), we obtain

$$m\left(\phi, Q_4^{'}\right) \leq \frac{1}{1 - L} < \infty,$$

and hence $Q_4^{'} \in W^*$. Theorem 2.1 proves that $Q_4^{'} = Q_4$, that is, the function Q_4 is unique, which ends the proof of the Theorem. □

Theorem 4.2. *Suppose that a mapping* $\Psi_{\tau,\theta} : E^m \times [0, \infty) \to [0, 1]$ *such that* (3.8) *holds and*

$$\Psi_{\tau,\theta}\left(2^{-1}x, 0, \cdots, 0, |2|^{-4} Lt\right) \geq \Psi_{\tau,\theta}\left(x, 0, \cdots, 0, t\right),\; x \in E,$$

for $L \in (0, 1)$. *If an even mapping* $\phi : E \to F$ *satisfying* (3.3) *and* (3.4), *then there exists a unique quartic mapping* $Q_4 : E \to F$ *satisfying*

$$N_{\tau,\theta}\left(\phi\,(x) - Q_4\,(x)\,, t\right) \geq T\left(\Psi_{\tau,\theta}\left(x, 0, \cdots, 0, |2|^4 (L^{-1} - 1) t\right)\right),$$

for all $x \in E$ *and all* $t > 0$.

From the main theorems established by way of both direct and fixed-point methods, we can derive some corollaries that improve the stability results under particular assumptions.

Corollary 4.1. *Let $\phi : E \to F$ be a mapping satisfying the quartic functional equation in a complete non-Archimedean intuitionistic fuzzy normed space $(E, N_{\tau,\theta}, T)$, with the control function $\Psi_{\tau,\theta}$ satisfying the conditions:*

$$\lim_{i\to\infty} \Psi_{\tau,\theta}(2^i x_1, 2^i x_2, \dots, 2^i x_n, |2|^{4i} t) = 1$$

and

$$\lim_{l\to\infty} T \prod_{i=l}^{\infty} T(\Psi_{\tau,\theta}(2^i x, 0, \dots, 0, |2|^{4i+1} t)) = 1.$$

Then the mapping $Q_4 : E \to F$ defined by

$$Q_4(x) = \lim_{i\to\infty} \frac{\phi(2^i x)}{24^i}$$

is the unique quartic function approximating ϕ such that the Hyers-Ulam stability holds.

Proof. According to Theorem 3.1, given the specified assumptions, the sequence

$$\left\{ \frac{\phi(2^i x)}{24^i} \right\}_{i\in\mathbb{N}}$$

constitutes a Cauchy sequence within the complete non-Archimedean IFN-space $(E, N_{\tau,\theta}, T)$. Consequently, it converges to a limit, which subsequently defines the mapping Q_4. By employing the recursive property associated in the quartic functional equation, we derive:

$$Q_4(2x) = 24Q_4(x),$$

which proves that Q_4 is quartic.

The uniqueness aspect is clear; should Q_4' denote another quartic mapping that satisfies the same inequality, the convergence in the fuzzy norm compels $Q_4(x) = Q_4'(x)$ for all $x \in E$. □

Corollary 4.2. *Suppose the control function $\Psi_{\tau,\theta}$ additionally satisfies the contraction condition:*

$$\Psi_{\tau,\theta}(2x_1, 2x_2, \dots, 2x_n, |2|^{4L} t) \geq \Psi_{\tau,\theta}(x_1, x_2, \dots, x_n, t), \quad L \in (0,1).$$

Then there exists a unique quartic function $Q_4 : E \to F$ satisfying:

$$Q_4(x) = \frac{1}{24} Q_4(2x),$$

and the stability estimate:

$$N_{\tau,\theta}\left(\phi(x) - Q_4(x), t\right) \geq T\left(\Psi_{\tau,\theta}(x, 0, \dots, 0, |2|^{4(1-L)} t)\right),$$

holds for every $x \in E$ and $t > 0$.

Proof. Applying Theorem 4.1, we define the operator:

$$S(p)(x) = \frac{1}{24}p(2x),$$

which acts on the generalized metric space (W, m), where $W = \{p : E \to F\}$ and

$$m(p,q) = \inf\left\{c \geq 0 : N_{\tau,\theta}(p(x) - q(x), t) \geq T(\Psi_{\tau,\theta}(x, 0, \ldots, 0, |2|^4 t)), \forall x \in E, t > 0\right\}.$$

By the contraction property:

$$m(Sp, Sq) \leq Lm(p,q), \quad \text{for all } p, q \in W,$$

and the Diaz-Margolis fixed-point theorem, the operator S has a unique fixed point Q_4 in W.

This fixed point satisfies:

$$Q_4(x) = S(Q_4)(x) = \frac{1}{24}Q_4(2x),$$

and the stability estimate follows from the inequality:

$$m(\phi, Q_4) \leq \frac{1}{1-L}m(S\phi, \phi),$$

which gives the bound on the deviation between ϕ and Q_4 in the fuzzy norm. Uniqueness is ensured by the strict contractive nature of S. □

5. Conclusion

In this paper, we have successfully established the generalized Ulam stability of the quartic functional equation in non-Archimedean intuitionistic fuzzy normed spaces using both direct and fixed-point approaches. We demonstrated that under suitable control functions and fuzzy norm conditions, any function approximately satisfying the quartic relation is closely approximated by an exact quartic mapping.

The findings enhance the existing theory of functional equation stability, specifically broadening its application to fuzzy and non-Archimedean contexts, which are becoming more significant in contemporary mathematical modeling of uncertainty, imprecision, and non-classical spaces. The results obtained strengthen the theoretical framework and create opportunities for applications in approximation theory, information sciences, and p-adic analysis.

Future research could explore the stability of higher-degree or mixed-type functional equations under similar fuzzy and non-Archimedean settings, as well as practical implementations in computational algorithms and physical system modeling.

Acknowledgements: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T. Aoki, On the Stability of the Linear Transformation in Banach Spaces, J. Math. Soc. Jpn. 2 (1950), 64–66. https://doi.org/10.2969/jmsj/00210064.

[2] J. Bae, W. Park, On the Generalized Hyers–ulam–rassias Stability in Banach Modules Over a C^*-Algebra, J. Math. Anal. Appl. 294 (2004), 196–205. https://doi.org/10.1016/j.jmaa.2004.02.009.

[3] J.B. Diaz, B. Margolis, A Fixed Point Theorem of the Alternative, for Contractions on a Generalized Complete Metric Space, Bull. Am. Math. Soc. 74 (1968), 305–309. https://doi.org/10.1090/s0002-9904-1968-11933-0.

[4] S. Donganont, C. Park, Combined System of Additive Functional Equations in Banach Algebras, Open Math. 22 (2024), 20230177. https://doi.org/10.1515/math-2023-0177.

[5] B. Radhakrishnan, U. Jayaraman, S.O. Hilali, M. Kameswari, M. Alhagyan, K. Tamilvanan, A. Gargouri, Coincidence Point Results for Self-mapping with Extended Rational Contraction in Partially Ordered Ultrametric Spaces Using p-Adic Distance, J. Math. 2024 (2024), 8846076. https://doi.org/10.1155/2024/8846076.

[6] Y. Almalki, B. Radhakrishnan, U. Jayaraman, K. Tamilvanan, Some Common Fixed Point Results in Modular Ultrametric Space Using Various Contractions and Their Application to Well-Posedness, Mathematics 11 (2023), 4077. https://doi.org/10.3390/math11194077.

[7] B. Radhakrishnan, U. Jayaraman, Fixed Point Results in Partially Ordered Ultrametric Space via P-Adic Distance, IAENG Int. J. Appl. Mat. 53 (2023), 1–7.

[8] E. Guariglia, K. Tamilvanan, On the Stability of Radical Septic Functional Equations, Mathematics 8 (2020), 2229. https://doi.org/10.3390/math8122229.

[9] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[10] K. Jun, H. Kim, On the Stability of Euler–lagrange Type Cubic Mappings in Quasi-Banach Spaces, J. Math. Anal. Appl. 332 (2007), 1335–1350. https://doi.org/10.1016/j.jmaa.2006.11.024.

[11] S.M. Jung, A Fixed Point Approach to the Stability of the Equation $f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$, Aust. J. Math. Anal. Appl. 6 (2009), 8.

[12] N.C. Kayal, T.K. Samanta, P. Saha, B.S. Choudhury, A Hyers-Ulam-Rassias Stability Result for Functional Equations in Intuitionistic Fuzzy Banach Spaces, Iran. J. Fuzzy Syst. 13 (2016), 87–96.

[13] H. Kim, M. Kim, Generalized Stability of Euler-lagrange Quadratic Functional Equation, Abstr. Appl. Anal. 2012 (2012), 219435. https://doi.org/10.1155/2012/219435.

[14] S.O. Kim, K. Tamilvanan, Fuzzy Stability Results of Generalized Quartic Functional Equations, Mathematics 9 (2021), 120. https://doi.org/10.3390/math9020120.

[15] H. Koh, D. Kang, Solution and Stability of Euler-Lagrange-Rassias Quartic Functional Equations in Various Quasi-normed Spaces, Abstr. Appl. Anal. 2013 (2013), 908168. https://doi.org/10.1155/2013/908168.

[16] D. Miheţ, The Fixed Point Method for Fuzzy Stability of the Jensen Functional Equation, Fuzzy Sets Syst. 160 (2009), 1663–1667. https://doi.org/10.1016/j.fss.2008.06.014.

[17] D. Miheţ, The Stability of the Additive Cauchy Functional Equation in Non-Archimedean Fuzzy Normed Spaces, Fuzzy Sets Syst. 161 (2010), 2206–2212. https://doi.org/10.1016/j.fss.2010.02.010.

[18] A. Mirmostafaee, M. Mirzavaziri, M. Moslehian, Fuzzy Stability of the Jensen Functional Equation, Fuzzy Sets Syst. 159 (2008), 730–738. https://doi.org/10.1016/j.fss.2007.07.011.

[19] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy Versions of Hyers–ulam–rassias Theorem, Fuzzy Sets Syst. 159 (2008), 720–729. https://doi.org/10.1016/j.fss.2007.09.016.

[20] A.K. Mirmostafaee, M.S. Moslehian, Stability of Additive Mappings in Non-Archimedean Fuzzy Normed Spaces, Fuzzy Sets Syst. 160 (2009), 1643–1652. https://doi.org/10.1016/j.fss.2008.10.011.

[21] S.A. Mohiuddine, A. Alotaibi, M. Obaid, Stability of Various Functional Equations in Non-Archimedean Intuitionistic Fuzzy Normed Spaces, Discret. Dyn. Nat. Soc. 2012 (2012), 234727. https://doi.org/10.1155/2012/234727.

[22] C. Park, K. Tamilvanan, B. Noori, M. B. Moghimi, A. Najati, Fuzzy Normed Spaces and Stability of a Generalized Quadratic Functional Equation, AIMS Math. 5 (2020), 7161–7174. https://doi.org/10.3934/math.2020458.

[23] J.M. Rassias, On the Stability of the Euler-Lagrange Functional Equation, Chinese J. Math. 20 (1992), 185–190.

[24] J.M. Rassias, Solution of the Ulam Stability Problem for Quartic Mappings, Glas. Mat. Ser. III 34 (1999), 243–252.

[25] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Am. Math. Soc. 72 (1978), 297–300. https://doi.org/10.2307/2042795.

[26] T.M. Rassias, K. Shibata, Variational Problem of Some Quadratic Functionals in Complex Analysis, J. Math. Anal. Appl. 228 (1998), 234–253. https://doi.org/10.1006/jmaa.1998.6129.

[27] K. Tamilvanan, J.R. Lee, C. Park, Ulam Stability of a Functional Equation Deriving From Quadratic and Additive Mappings in Random Normed Spaces, AIMS Math. 6 (2021), 908–924. https://doi.org/10.3934/math.2021054.

[28] S.M. Ulam, A Collection of Mathematical Problems, Interscience Publishers, New York, 1960.

[29] T.Z. Xu, J.M. Rassias, W.X. Xu, Stability of a General Mixed Additive-Cubic Functional Equation in Non-Archimedean Fuzzy Normed Spaces, J. Math. Phys. 51 (2010), 093508. https://doi.org/10.1063/1.3482073.

[30] T.Z. Xu, J.M. Rassias, Stability of General Multi-Euler-Lagrange Quadratic Functional Equations in Non-Archimedean Fuzzy Normed Spaces, Adv. Differ. Equ. 2012 (2012), 119. https://doi.org/10.1186/1687-1847-2012-119.

[31] A. Zivari-Kazempour, M.E. Gordji, Generalized Hyers–Ulam Stabilities of an Euler–Lagrange–Rassias Quadratic Functional Equation, Asian-European J. Math. 05 (2012), 1250014. https://doi.org/10.1142/s1793557112500143.

[32] S.M. Ulam, Problems in Modern Mathematics, Wiley, 1964.

[33] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[34] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Am. Math. Soc. 72 (1978), 297–300. https://doi.org/10.2307/2042795.

[35] L. Zadeh, Fuzzy Sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[36] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/s0165-0114(86)80034-3.

[37] R. Saadati, J.H. Park, On the Intuitionistic Fuzzy Topological Spaces, Chaos Solitons Fractals 27 (2006), 331–344. https://doi.org/10.1016/j.chaos.2005.03.019.