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On the Boundedness of Hardy Operators on λ-Central Amalgam-Morrey Spaces

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Abstract. In this paper, we introduce the idea of λ-central Amalgam-BMO spaces and λ-central Amalgam-Morrey spaces. We obtain the boundedness of the Hardy operators λ-central Amalgam-Morrey spaces.

1. Introduction

The boundedness of operators on function spaces is an important class of problem gaining lots of attention around the globe. Among many other Hardy-type operators are classical operators, the exploration of which began with Hardy's proposal of the one-dimensional Hardy operator [12]. Over time, researchers extended the definition of Hardy-type operators to higher dimensions. Christ and Grafakos [5] introduced the n-dimensional Hardy operator. In [8], authors made contributions by introducing the n-dimensional fractional Hardy operators and studying their commutators. Fractional Hardy operators are the generalization of classical Hardy operators, allowing for more flexibility in the types of functions they can handle. In [9], continuing their investigation by studying Hardy operators with rough kernels authors studied central BMO estimates for commutators of n-dimensional rough Hardy operators.

Furthermore, the research expanded to explore the boundedness of various Hardy-type operators and their commutators on different function spaces see [17–19]. The study of Morrey spaces started with the work of Morrey [15] who firstly introduce these spaces while studing the local

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behavior of a certain kind of PDEs. Later on the λ-central BMO spaces and Morrey spaces were introduced in [1]. In [10,14], authors extended this study by generalizing the central Morrey space to include variable exponent. Particularly in [10], Fu et al. obtained the boundedness of ingular integral operators with rough kernal on central Morrey spaces with variable exponent.

On the other hand, the λ-central bounded mean oscillation spaces, Morrey type spaces and related function spaces have interesting applications in studying boundedness of operators including singular integral operators; see for example [11,27,28]. In [16], the boundedness of Hardy-type operators with rough kernels and their commutators on central Morrey spaces with variable exponents were proved. The boundedness of the multilinear fractional integral operators and their commutators on the central Morrey spaces with variable exponent can be seen in [33]. For more results in function spaces see [20–26,29–32].

Let $0 < p \leq \infty$. By L^p we denote the space of all measurable functions f such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}.$$

In 1926, Wiener [34] introduced the idea of amalgam spaces. Let $p, q \in (0, \infty)$, then Amalgam space $(L^p, L^q)(\mathbb{R})$ is defined as

$$(L^p, L^q)(\mathbb{R}) := \left\{ f \in L^p_{\text{loc}} : \left[\sum_{m \in \mathbb{Z}} \left(\int_m^{m+1} |f(x)|^p \, dx\right)^{q/p}\right]^{1/q} < \infty \right\}.$$

Many writers have studied amalgam spaces or some of its applications [3,6,7]. The fact that amalgam spaces provide information regarding the local L^p and global L^q features of the functions, whereas L^p spaces do not, is a key distinction when comparing them to L^p spaces. By a weight ω, we mean a locally integrable function on $\mathbb{R}^n$ which takes values almost everywhere in $(0, \infty)$. Let $1 \leqslant p < \infty$, then weighted Lebesgue spaces $L^p_\omega(\mathbb{R}^n)$ the space of all functions such that the norm $\|f\|_{L^p_\omega(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(y)|^p \omega(y) dy\right)^{1/p}$ is finite.

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $0 < t < \infty$ and v, ω are two weights. Then the weighted Amalgam space $\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)$ is the space of all measurable functions g such that

$$\|f\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} := \left\| \left(\frac{1}{\omega(B(\cdot, t))} \int_{B(\cdot, t)} |f(y)|^p \, \omega(y) dy \right)^{1/p} \right\|_{L^q_v(\mathbb{R}^n)} < \infty,$$

where $B(y, r) := \{x \in \mathbb{R}^n : |y - x| < r\}$.

Let f be a locally integrable function on $\mathbb{R}^n$. The Hardy operators are defined as

$$\mathcal{H}f(z) := \frac{1}{|z|^n} \int_{|x| < |z|} f(x) dx, \quad \mathcal{H}^* f(z) := \int_{|x| \geq |z|} \frac{f(x)}{|x|^n} dx,$$

where $z \in \mathbb{R}^n \setminus \{0\}$.

Motivated by these papers, we will define the idea of λ-central Amalgam-BMO spaces and λ-central Amalgam-Morrey spaces. Our main goal is to obtain the boundedness of the Hardy operators on λ-central Amalgam-Morrey spaces. Let's break down the outline: In section 2, we provide essential background information, including standard notations and Lemmas related to variable Lebesgue spaces. Then we give the definitions of λ-central Amalgam-BMO spaces and λ-central Amalgam-Morrey spaces. These serve as a foundation for the subsequent sections. Last section contains main results in which we will establish the boundedness of the Hardy operators on λ-central Amalgam-Morrey spaces.

Furthermore, we use the symbols $|A|$ and χ_A to represent the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$, respectively. The expression $f \approx g$ signifies the existence of positive constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$. In this paper C denotes the constant and its value can vary from line to line.

2. Preliminaries

Let $\ell \in \mathbb{Z}$, $R_\ell : B_\ell \setminus B_{\ell-1}$ where $B_\ell : \left\{x \in \mathbb{R}^n \ : \ |x| \leq 2^\ell\right\}$. $\chi_\ell := \chi_{R_\ell}$, where χ_{R_ℓ} is the characteristic function of R_ℓ.

Let $1 \leq p, q < \infty$. The Hölder's inequality for weighted Amalgam space [13] is given as:

$$\int_{\mathbb{R}^n} |f(z)g(z)|dz \leq C \left\|f\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \|g\|_{\left(L^{p'}_{\omega'}, L^{q'}_{v'}\right)_t(\mathbb{R}^n)'}$$

where $\frac{1}{p} = \frac{1}{p'} = \frac{1}{q} = \frac{1}{q'} = 1, v' = v^{1-p'}, \omega' = v^{1-q'}$.

Lemma 2.1. *Assume that $0 < t < \infty$, $1 < p < \infty$, $1 \leq q \leq \infty$ and w, v are two weights. A characteristic function on $B(y_0, s_0)$ fulfills*

$$\left\|\chi_{B(y_0,s_0)}\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \leq C s_0^{n/q}.$$

Proof. Let $x \in \mathbb{R}^n$, we get

$$\begin{aligned}
\left\|\chi_{B(y_0,s_0)}\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} &= \left\{\int_{\mathbb{R}^n} \left[\frac{\left\|\chi_{B(y_0,s_0)}\chi_{B(x,t)}\right\|_{L^p_\omega}}{\left\|\chi_{B(x,t)}\right\|_{L^p_\omega}}\right]^q v(x)dx\right\}^{\frac{1}{q}} \\
&= \frac{1}{\left\|\chi_{B(\overrightarrow{0},t)}\right\|_{L^p_\omega}} \left[\int_{\mathbb{R}^n} \left\|\chi_{B\left(\overrightarrow{0},s_0\right)}\chi_{B(x-y_0,t)}\right\|^q_{L^p_\omega} v(x)dx\right]^{\frac{1}{q}} \\
&= \frac{1}{\left\|\chi_{B(\overrightarrow{0},t)}\right\|_{L^p_\omega}} \left[\int_{\mathbb{R}^n} \left\|\chi_{B\left(\overrightarrow{0},s_0\right)}\chi_{B(x,t)}\right\|^q_{L^p_\omega} v(x)dx\right]^{\frac{1}{q}} \\
&= \left\|\chi_{B\left(\overrightarrow{0},s_0\right)}\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}.
\end{aligned}$$

This means that, assuming nothing is lost in general, $B_1 := B(\overrightarrow{0}, 1)$ and $B_2 := B\left(s_0, \overrightarrow{0}\right)$ with $1 < s_0 < \infty$ are used. The geometric property tells us that there exists $M \in \mathbb{N}$ such that $B\left(\overrightarrow{0}, s_0\right) \subset$

$\bigcup_{i=1}^{M} B\left(x_i, 1\right)$, and that $M \sim |B_2|^n$ and $\{x_1, \ldots, x_M\}$.

$$\begin{aligned}\left\|\chi_{B\left(\vec{0}, s_0\right)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)} &= \left\|\chi_{B_2}\right\|^{\frac{1}{q}}_{\left(L_\omega^{p/q}, L_v^1\right)_t(\mathbb{R}^n)} \\ &\leq \left\|\sum_{i=1}^{M} \chi_{B(x_i,1)}\right\|^{\frac{1}{q}}_{\left(L_\omega^{p/q}, L_v^1\right)_t(\mathbb{R}^n)} \lesssim \left(\sum_{i=1}^{M} \left\|\chi_{B(x_i,1)}\right\|_{\left(L_\omega^{p/q}, L_v^1\right)_t(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\ &\sim |B_2|^{\frac{1}{q}} \left\|\chi_{B(\vec{0},1)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)} \sim s_0^{n/q}.\end{aligned}$$

This completes the proof. □

Remark 2.1. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $0 < t < \infty$, v, w are two weights and $\ell \in \mathbb{Z}$, a characteristic function on R_ℓ satisfies*

$$\left\|\chi_{R_\ell}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)} \leq \left\|\chi_{B_\ell}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)} \leq C 2^{\ell n/q}.$$

Now we define the λ-central Amalgam-Morrey space and λ-central Amalgam-BMO as follows.

Definition 2.1. *Assume that $0 < t < \infty$, $1 < p < \infty$, $1 \leq q \leq \infty$ and w, v are two weights. The λ-central Amalgam-Morrey space $\dot{\mathcal{B}}^\lambda_{\left(L_\omega^p, L_v^q\right)_t}(\mathbb{R}^n)$ for which the norm*

$$\|f\|_{\dot{\mathcal{B}}^\lambda_{\left(L_\omega^p, L_v^q\right)_t}(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{\left\|f \chi_{B(0,R)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)}}{|B(0,R)|^\lambda \left\|\chi_{B(0,R)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)}} \right),$$

is finite.

Definition 2.2. *Assume that $0 < t < \infty$, $1 < p < \infty$, $1 \leq q \leq \infty$ and w, v are two weights. The grand λ-central* BMO *space* $\mathrm{CBMO}^\lambda_{\left(L_\omega^p, L_v^q\right)_t}(\mathbb{R}^n)$ *is defined by the norm*

$$\|f\|_{\mathrm{CBMO}^\lambda_{\left(L_\omega^p, L_v^q\right)_t}(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{\left\|(f - f_B) \chi_{B(0,R)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)}}{|B(0,R)|^\lambda \left\|\chi_{B(0,R)}\right\|_{\left(L_\omega^p, L_v^q\right)_t(\mathbb{R}^n)}} \right),$$

here $f_B = \frac{1}{|B(0,R)|} \int_{B(0,R)} f(x)dx$.

3. Main results

We will now articulate and establish our primary outcomes, specifically the boundedness of the Hardy operators and their adjoint operators on λ-central Amalgam-Morrey spaces. For simplicity, let $2^k B$ represent the ball B enlarged by a factor of 2^k in terms of its radius,

$$C_k = 2^k B \setminus 2^{k-1} B,$$

for $k \in \mathbb{Z}$.

Theorem 3.1. *Assume that $0 < t < \infty$, $1 < p < \infty$, $1 \leq q \leq \infty$ and w, v are two weights and $\lambda \in \mathbb{R}$ with $\lambda + 1/q > 0$. Then*

$$\|\mathcal{H}(f)\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)},$$

and

$$\|\mathcal{H}^*(f)\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)} \leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)}.$$

Proof. By using Hölder's inequality and the definition of Hardy operator $\mathcal{H}$, we have

$$\begin{aligned}
|\mathcal{H}f(z).\chi_B(z)| &\leq \frac{1}{|z|^n}\left|\int\limits_{|x|\leq|z|} f(x)dx\right| \cdot \chi_B(z) \\
&\leq C\sum_{j=-\infty}^{0} \frac{1}{|z|^n}\left|\int\limits_{|x|\leq|z|} f(x)dx\right| \cdot \chi_{C_j}(z) \\
&\leq C\sum_{j=-\infty}^{0} |2^jB|^{-1}\left|\int\limits_{2^jB} f(x)dx\right| \cdot \chi_{C_j}(z) \\
&\leq C\sum_{j=-\infty}^{0} |2^jB|^{-1}\|f\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^{p'}_{\omega'}, L^{q'}_{v'}\right)_t(\mathbb{R}^n)} \cdot \chi_{C_j}(z).
\end{aligned}$$

By taking the $\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)$ norm yields

$$\left\|(\mathcal{H}f\chi_B)\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{0} |2^jB|^{-1}\|f\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^{p'}_{\omega'}, L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{C_j}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}.$$

Hence we get

$$\begin{aligned}
&\left\|(\mathcal{H}f\chi_B)\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{0} |2^jB|^{-1+\lambda}\frac{\|f\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}}{|2^jB|^{\lambda}\|\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}}\|\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^{p'}_{\omega'}, L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{C_j}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{0} |2^jB|^{\lambda}\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)}\|\chi_{C_j}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \\
&\leq C\sum_{j=-\infty}^{0} |2^jB|^{\lambda}\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}.
\end{aligned}$$

By using the fact that $\lambda + 1/q > 0$, thus we have

$$\left\|(\mathcal{H}f\chi_B)\right\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{0} |2^jB|^{\lambda}\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega, L^q_v\right)_t}(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^p_\omega, L^q_v\right)_t(\mathbb{R}^n)}$$

$$\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}\frac{|2^jB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^jB}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}}{\|\chi_B\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}$$
$$\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}\left(\frac{|2^jB|}{|B|}\right)^{\lambda+1/q}|B|^{\lambda}\|\chi_B\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}$$
$$\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}2^{nj(\lambda+1/q)}|B|^{\lambda}\|\chi_B\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}.$$

Thus we have

$$\begin{aligned}\|\mathcal{H}(f)\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)} &\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}2^{nj(\lambda+1/p)}\\ &\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{\left(L^p_\omega,L^q_v\right)_t}(\mathbb{R}^n)}.\end{aligned}$$

Next we will prove the boundedness of $\mathcal{H}^*$ on λ central Amalgam-Morrey spaces. By using Hölder's inequality and the definition of Hardy operator $\mathcal{H}^*$, we have

$$\begin{aligned}|\mathcal{H}^*f(z).\chi_B(z)| &\leq \left|\int_{|x|>|z|}\frac{1}{|x|^n}f(x)dx\right|\cdot\chi_B(z)\\ &\leq C\sum_{j=-\infty}^{0}\left|\int_{|x|>|z|}\frac{1}{|x|^n}f(x)dx\right|\cdot\chi_{C_j}(z)\\ &\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}\left|\int_{C_i}\frac{1}{|x|^n}f(x)dx\right|\cdot\chi_{C_j}(z)\\ &\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1}\left|\int_{2^iB}f(x)dx\right|\cdot\chi_{C_j}(z)\\ &\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1}\|f\chi_{2^iB}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^iB}\|_{\left(L^{p'}_{\omega'},L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\cdot\chi_{C_j}(z).\end{aligned}$$

By taking $\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)$ norm, we get

$$\begin{aligned}&\|(\mathcal{H}^*f\chi_B)\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\\ &\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1}\|f\chi_{2^iB}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^iB}\|_{\left(L^{p'}_{\omega'},L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{C_j}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\\ &\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1}\|f\chi_{2^iB}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\|\chi_{2^iB}\|_{\left(L^{p'}_{\omega'},L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{\left(L^p_\omega,L^q_v\right)_t(\mathbb{R}^n)}\end{aligned}$$

$$\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1}\|f\chi_{2^iB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}\|\chi_{2^iB}\|_{\left(L^{p'}_{\omega'},L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}$$

$$\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{-1+\lambda}\frac{\|f\chi_{2^iB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}}{|2^iB|^{\lambda}\|\chi_{2^iB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}}\|\chi_{2^iB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}\|\chi_{2^iB}\|_{\left(L^{p'}_{\omega'},L^{q'}_{v'}\right)_t(\mathbb{R}^n)}\|\chi_{2^jB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}$$

$$\leq C\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{\lambda}\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\|\chi_{2^jB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}.$$

Here using the fact that $\lambda+1/p>0$, we have

$$\begin{aligned}
\|(\mathcal{H}^*f\chi_B)\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)} &\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}|2^iB|^{\lambda}\|\chi_{2^jB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}\sum_{i=j}^{\infty}2^{(i-j)n\lambda}\frac{|2^jB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^jB}\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}}{\|\chi_B\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}2^{n\lambda}2^{jn/q}|B|^{\lambda}\|\chi_B\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}\\
&\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\sum_{j=-\infty}^{0}|B|^{\lambda}\|\chi_B\|_{(L^p_\omega,L^q_v)_t(\mathbb{R}^n)}.
\end{aligned}$$

So we have

$$\|\mathcal{H}^*(f)\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)}\leq C\|f\|_{\dot{\mathcal{B}}^{\lambda}_{(L^p_\omega,L^q_v)_t}(\mathbb{R}^n)},$$

which completes the proof.

□

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